HIROSHIMA MATH. J.
8 (1978), 151–164

Some General Properties of Behavior Spaces of Harmonic Semiexact Differentials on an Open Riemann Surface

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(Received September 19, 1977)

Introduction

Beside the original definition due to Kusunoki [3], there are several different ways to define semiexact canonical differentials (see Kusunoki [4, 5], Mori [9]; cf. also Ahlfors-Sario [2], Mizumoto [8] and Yoshida [13]). Above all, the following characterization of semiexact canonical differentials also by Kusunoki ([4]) is remarkable: Let R be an open Riemann surface and φ a meromorphic semiexact differential on R. Then φ is a semiexact canonical differential if and only if there is a canonical region R' on R such that (i) the real part du of φ is exact and square integrable on $R-R'$, and (ii) for any square integrable real harmonic semiexact differential ω on $R-R'$ the mixed Dirichlet integral $(du, \omega^*)_{R-R'}$ of du and ω^* over $R-R'$ is equal to the contour integral $\int_{\partial R'} u\omega$.

A similar characterization is obtained for harmonic differentials with Γ_χ-behavior in the sense of Yoshida ([13], in particular, pp. 186–187). Since, as is well known (cf. [5], [9]), semiexact canonical differentials correspond to one of the special extreme cases, the case $\Gamma_\chi=\Gamma_{hm}$ (the space of real harmonic measures on R), the results in [13] is certainly a generalization of Kusunoki's characterization. On the other hand, we considered in [11] spaces of (complex) harmonic semiexact differentials with certain simple properties and called them behavior spaces. We also showed that we can use such a behavior space Λ_0 to describe a more general boundary behavior, Λ_0-behavior, of analytic (meromorphic) differentials.

The aim of the present article is to show some properties of behavior spaces. It is easy to see that we can apply the very same definition of Λ_0-behavior not only to analytic differentials but also to C^1-differentials (defined near the ideal boundary of R). See Definition 3. We shall generalize some of Kusunoki's characterizations of semiexact canonical differentials to the case of C^1-differentials with Λ_0-behavior. Then we shall introduce an equivalence relation among behavior spaces on R. We can easily see that Λ_0- and $\tilde{\Lambda}_0$-behaviors are the same if and only if Λ_0 is equivalent to $\tilde{\Lambda}_0$. In other words, Λ_0-behavior is determined by the equivalence class of Λ_0. As an immediate consequence of this, we know that generalized singularities introduced in [12] are also divided into equivalence classes.

We shall also consider transformations of behavior spaces and show that every transformation changes a behavior space into another behavior space. Furthermore, we prove that two behavior spaces are equivalent if and only if any one is the image of the other under some transformation.

Such consideration actually offers some advantages to us. For instance, we can choose the most suitable behavior space(s) among the equivalence class(es) in accordance with the nature of individual problem which we are concerned with.

1. Let R be an open Riemann surface of genus g ($\leq \infty$), J the set $\{1, 2,\dots, g\}$. Take a fixed canonical exhaustion $\mathscr{R}=\{R_n\}_{n=1}^{\infty}$ of R. We denote by g_n the genus of R_n and set $J_n=\{1, 2,\dots, g_n\}$. Let $\Xi(R)=\{A_j, B_j\}_{j\in J}$ be a canonical homology basis of R modulo dividing cycles such that (i) $\{A_j, B_j\}_{j\in J_n}$ is a canonical homology basis of R_n modulo its border, and (ii) $A_j, B_j \subset R-\bar{R}_n$ for every $j \in J-J_n$ (cf. [2], p. 72). For convenience' sake we set $R_0=\phi$ and $J_0=\phi$.

For a Lebesgue measurable complex differential λ on R we denote by $\bar{\lambda}$ the complex conjugate of λ and by λ^* the conjugate differential of λ. Let $\Lambda=\Lambda(R)$ be the *real* Hilbert space of square integrable *complex* differentials on R with the inner product $<\lambda_1, \lambda_2>=\mathrm{Re}\iint_R \lambda_1 \wedge \bar{\lambda}_2^*$, $\lambda_1, \lambda_2 \in \Lambda$. The norm of $\lambda \in \Lambda$ is given by $\|\lambda\|=\sqrt{<\lambda, \lambda>}$. We set $\Lambda_h=\Lambda_h(R)=\{\lambda \in \Lambda | \lambda$ is harmonic on $R\}$ and $\Lambda_{hse}=\Lambda_{hse}(R)=\{\lambda \in \Lambda_h | \lambda$ is semiexact$\}$. We also set $\Lambda_{e0}^{(1)}=\Lambda_{e0}^{(1)}(R)=\{\lambda \in \Lambda | {}^{\exists}f \in C^2(R), {}^{\exists}f_n \in C_0^2(R)$ such that $df=\lambda$ and $\|df-df_n\| \to 0, n \to \infty\}$. Finally let $\Lambda_c^1=\Lambda_c^1(R)$ be the totality of closed C^1-differentials on R and set $\Lambda_{c0}^1=\Lambda_{c0}^1(R)=\{\lambda \in \Lambda_c^1 | \lambda=0$ outside a compact set on $R\}$.

Let L be a straight line in the complex plane $\mathbf{C}$ which passes through the coordinate origin. For brevity, we shall refer to such an L as a line in $\mathbf{C}$. We denote by $\bar{L}$ the complex conjugate of L: $\bar{L}=\{z \in \mathbf{C} | \bar{z} \in L\}$. For $z_1, z_2 \in \mathbf{C}$ we write $z_1 \equiv z_2 \bmod L$ to express that z_1-z_2 belongs to L.

DEFINITION 1 ([7], [11]). *A (closed) subspace $\Lambda_0=\Lambda_0(R, \mathscr{L})$ of Λ_{hse} is called a behavior space associated with $\mathscr{L}=\{L_j\}_{j\in J}$, a family of lines in* $\mathbf{C}$, *if*

(i) $i\Lambda_0^* = \Lambda_0^{\perp}$ [*i.e.*, $\Lambda_h = \Lambda_0 \oplus i\Lambda_0^*$],

(ii) $\int_{A_j} \lambda_0 \equiv \int_{B_j} \lambda_0 \equiv 0 \bmod L_j$, $j \in J$, *for every* $\lambda_0 \in \Lambda_0$,

where $\Lambda_0^{\perp}$ denotes the orthogonal complement of Λ_0 in Λ_h.

We denote by $\mathscr{B}$ the set of all behavior spaces on R.

DEFINITION 2 ([11]). *Let L be a line in* $\mathbf{C}$. *Two behavior spaces Λ_0 and*

Λ_0' associated with $\mathscr{L}=\{L_j\}_{j\in J}$ and $\mathscr{L}'=\{L_j'\}_{j\in J}$ respectively are called dual to each other with respect to L (or L-dual) if

1°) $<\lambda_0, \bar{\lambda}_0'^*> + i<\lambda_0, i\bar{\lambda}_0'^*> \equiv 0 \bmod L$ *for any pair* $(\lambda_0, \lambda_0')\in\Lambda_0\times\Lambda_0'$,

2°) $L_j\circ L_j' = \{z\in\mathbf{C}|z = z_jz_j',\ z_j\in L_j,\ z_j'\in L_j'\} = L$ *for every* $j\in J$

are satisfied.

For the sake of simplicity, we shall henceforth consider mainly the case $L=\mathbf{R}$, the real axis. Then it is obvious that a behavior space Λ_0 and its complex conjugate $\bar{\Lambda}_0=\{\lambda\in\Lambda_h|\bar{\lambda}\in\Lambda_0\}$ are mutually $\mathbf{R}$-dual (cf. [7], [11]). Conversely we have

PROPOSITION 1. *Let Λ_0 and Λ_0' be two behavior spaces which are $\mathbf{R}$-dual to one another. Then $\Lambda_0'=\bar{\Lambda}_0$.*

PROOF. Since $L=\mathbf{R}$, condition 1°) in Definition 2 means $<\lambda_0, i\bar{\lambda}_0'^*>=0$ for any pair $(\lambda_0, \lambda_0')\in\Lambda_0\times\Lambda_0'$. Therefore $\Lambda_0\subset i\bar{\Lambda}_0'^{*\perp}=\bar{\Lambda}_0'$. Similarly we have $\Lambda_0'\subset\bar{\Lambda}_0$. Hence $\Lambda_0'=\bar{\Lambda}_0$, which is to be proved.

COROLLARY. *For any behavior space Λ_0 there exists a unique behavior space Λ_0' $(=\bar{\Lambda}_0)$ which is dual to Λ_0 with respect to $\mathbf{R}$.*

More generally we have

PROPOSITION 1′. *Let $L_\theta=\{z\in\mathbf{C}\,|\,z=te^{i\theta},\ t\in\mathbf{R}\}$ be a line in $\mathbf{C}$, $\theta\in[0,\pi)$. Then two behavior spaces Λ_0 and Λ_0' are L_θ-dual to each other if and only if $\Lambda_0'=e^{i\theta}\bar{\Lambda}_0$.*

The following proposition will be proved in sec. 4.

PROPOSITION 2. *In Definition 2 condition 1°) implies condition 2°).*

2. Fix an $R_n\in\mathscr{R}$, $n\geq 1$ and set $V=R-\bar{R}_n$ where $\bar{R}_n$ is the closure of R_n. We denote by V_k $(k=1, 2,\dots, \kappa_n)$ the components of V. Let φ, ψ be semiexact C^1-differentials on $\bar{V}$. Then, because of semiexactness of φ, there is a single-valued C^2-function Φ on $\bar{V}-\bigcup_{j\in J-J_n}(A_j\cup B_j)$ such that $d\Phi=\varphi$ on $\bar{V}$. The function Φ consists of κ_n functions Φ_k which are separately defined on $\bar{V}_k-\bigcup_{j\in J-J_n}(A_j\cup B_j)\cap\bar{V}_k$, $k=1, 2,\dots, \kappa_n$. Furthermore, each Φ_k is determined only up to an additive constant. Nevertheless the quantity

$$\int_{\partial R_m}\Phi\psi = \sum_{k=1}^{\kappa_n}\int_{\partial R_m}\Phi_k\psi \qquad (m\geq n)$$

is well defined, since ψ is also semiexact. We shall call such Φ a primitive function of φ on $\bar{V}$.

For later use we shall state the following lemma without proof (cf. [1], [11]).

LEMMA. *Let φ, ψ be semiexact C^1-differentials on $\bar{V}=R-R_n$ $(n\geq 0)$. Then for every $m>n$ we have*

$$<\varphi, i\psi^*>_{R_m-R_n}$$
$$= -\operatorname{Im}\int_{\partial R_m}\Phi\bar{\psi}+\operatorname{Im}\int_{\partial R_n}\Phi\bar{\psi}+\operatorname{Im}\sum_{j\in J_m-J_n}\left(\int_{A_j}\varphi\int_{B_j}\bar{\psi}-\int_{B_j}\varphi\int_{A_j}\bar{\psi}\right),$$

Φ being a primitive function of φ on $\bar{V}$.

3. Let $\Lambda_0=\Lambda_0(R, \mathscr{L})$ be a behavior space on R associated with $\mathscr{L}=\{L_j\}_{j\in J}$. We shall first prove the following

THEOREM 1.*) *Let $R_n\in\mathscr{R}$ and $V=R-\bar{R}_n$. Let φ be a semiexact C^1-differential on $\bar{V}$ such that $\|\varphi\|_V<\infty$ and $\int_{A_j}\varphi\equiv\int_{B_j}\varphi\equiv 0 \mod L_j$ for all $j\in J-J_n$. Let Φ be a primitive function of φ on $\bar{V}$. Then the following three conditions are equivalent to one another.*

(I) *There exist $\lambda_0\in\Lambda_0$, $\lambda_{e0}\in\Lambda_{e0}^{(1)}$ such that $\varphi=\lambda_0+\lambda_{e0}$ on V.*

(II) $<\varphi, i\omega^*>_V = -\operatorname{Im}\int_{\partial V}\Phi\bar{\omega}$ *for any $\omega\in\Lambda_0$.*

(III) $\lim_{m\to\infty}\operatorname{Im}\int_{\partial R_m}\Phi\bar{\omega}=0$ *for every $\omega\in\Lambda_0$.*

PROOF. Before carrying out the proof, we recall that $\Lambda_0\subset\Lambda_{hse}$.
First we shall show that (II)$\Leftrightarrow$(III). Let $m>n$. Then by Lemma we have

$$<\varphi, i\omega^*>_{V\cap R_m}$$
$$= -\operatorname{Im}\int_{\partial V}\Phi\bar{\omega}-\operatorname{Im}\int_{\partial R_m}\Phi\bar{\omega}+\operatorname{Im}\sum_{j\in J_m-J_n}\left(\int_{A_j}\varphi\int_{B_j}\bar{\omega}-\int_{B_j}\varphi\int_{A_j}\bar{\omega}\right).$$

The last term on the right vanishes, since

$$\int_{\substack{A_j\\B_j}}\varphi\equiv 0,\qquad \int_{\substack{A_j\\B_j}}\omega\equiv 0 \qquad \mod L_j,\quad j\in J-J_n.$$

By letting m tend to infinity, we obtain the equivalence (II)$\Leftrightarrow$(III).

Next we shall prove that (I) implies (III). Suppose that $\varphi=\lambda_0+\lambda_{e0}$ on V, $\lambda_0\in\Lambda_0$, $\lambda_{e0}\in\Lambda_{e0}^{(1)}$. Set $\psi=\lambda_0+\lambda_{e0}$ on R. Clearly ψ is semiexact. Let Ψ be a primitive function of ψ on $\bar{V}$. For any m $(>n)$ and $\omega\in\Lambda_0$ we have

$$\operatorname{Im}\int_{\partial R_m}\Phi\bar{\omega}=\operatorname{Im}\int_{\partial R_m}\Psi\bar{\omega}$$

*) cf. Theorem 4 in sec. 9.

$$= - <\psi, i\omega^*>_{R_m} + \mathrm{Im} \sum_{j\in J_m} \left(\int_{A_j} \psi \int_{B_j} \bar{\omega} - \int_{B_j} \psi \int_{A_j} \bar{\omega} \right).$$

Since $\psi \in \Lambda_0 + \Lambda_{e0}^{(1)}$ and $i\omega^* \in i\Lambda_0^* = \Lambda_0^{\perp}$, the term $<\psi, i\omega^*>_{R_m}$ tends to zero as $m \to \infty$. The period sum vanishes for every m, because $\int_{A_j, B_j} \psi = \int_{A_j, B_j} \lambda_0 \equiv 0$, $\int_{A_j, B_j} \omega \equiv 0 \mod L_j$, $j \in J$. Thus we have proved (I)$\Rightarrow$(III).

Finally we assume (III). Since $\int_{\partial V} \varphi = 0$, we can extend $\varphi|_V$ to a closed C^1-differential on the whole of R (cf. e.g., [10], [13]), which is denoted by $\hat{\varphi}$. Because of the semiexactness of φ, $\hat{\varphi} = d\hat{\Phi}$ can be assumed to be exact on $R - \bar{V} = R_n$ (see Remark below). Take an arbitrary $\omega \in \Lambda_0$. Then, since the A_j- and B_j-periods of φ and ω vanish mod L_j, $j \in J - J_n$, Lemma yields

$$<\hat{\varphi}, i\omega^*>_{R_m} = <\hat{\varphi}, i\omega^*>_{R_n} + <\varphi, i\omega^*>_{R_m - R_n}$$

$$= - \mathrm{Im} \int_{\partial R_n} \hat{\Phi}\bar{\omega} - \mathrm{Im} \int_{\partial(R_m - R_n)} \Phi\bar{\omega} + \mathrm{Im} \sum_{j \in J_m - J_n} \left(\int_{A_j} \varphi \int_{B_j} \bar{\omega} - \int_{B_j} \varphi \int_{A_j} \bar{\omega} \right)$$

$$= - \mathrm{Im} \int_{\partial R_m} \Phi\bar{\omega}.$$

On letting $m \to \infty$, we know that $\hat{\varphi}$ is orthogonal to $i\Lambda_0^* = \Lambda_0^{\perp}$. (Note that $\hat{\varphi}$ belongs to Λ.) Now the Dirichlet principle (cf. Lemma 5 in [11]; cf. also [2]) implies the existence of differentials $\lambda_0 \in \Lambda_0$, $\lambda_{e0} \in \Lambda_{e0}^{(1)}$ such that $\hat{\varphi} = \lambda_0 + \lambda_{e0}$ holds on R. This completes the proof of (III)$\Rightarrow$(I). q.e.d.

REMARK. We could dispence with the exactness of $\hat{\varphi}$ on R_n. Indeed, if $\hat{\varphi}$ is not exact on R_n, we take a regular analytic differential φ_0 with Λ_0-behavior (see [11], Theorem 2) such that $\int_{A_j, B_j} (\hat{\varphi} - \varphi_0) \equiv 0 \mod L_j$, $j \in J$. Applying a reasoning similar to that in the above proof for $\hat{\varphi} - \varphi_0$ instead of $\hat{\varphi}$, we know that $\varphi - \varphi_0$, and hence φ itself, has the property stated in (I).

4. In [11] we defined Λ_0-behavior for only analytic (meromorphic) differentials [defined near the ideal boundary of R]. Similarly, Γ_χ-behavior in Yoshida's sense ([13]) was defined for harmonic functions only. However, it is easy to see that the same definition can be applied to any C^1-differential (cf. [6]). Namely, we have

DEFINITION 3. *A C^1-differential φ defined near the ideal boundary of R is said to have Λ_0-behavior if there exist $\lambda_0 \in \Lambda_0$, $\lambda_{e0} \in \Lambda_{e0}^{(1)}$ and $R_n \in \mathscr{R}$ such that $\varphi = \lambda_0 + \lambda_{e0}$ on $R - \bar{R}_n$.*

A characterization of C^1-differentials with Λ_0-behavior is given by Theorem 1, which is considered a generalization of Kusunoki's results ([4]). See also [13]. Particularly we have

PROPOSITION 3 (cf. [13], p. 187). *Let φ be a semiexact C^1-differential on $\bar{V}=R-R_n$, $R_n \in \mathscr{R}$, such that $\int_{A_j}\varphi \equiv \int_{B_j}\varphi \equiv 0 \mod L_j$, $j \in J-J_n$. Suppose that φ has Λ_0-behavior. Then φ admits a representation $\varphi=\lambda_0+\lambda_{e0}$, $\lambda_0 \in \Lambda_0$, $\lambda_{e0} \in \Lambda_{e0}^{(1)}$ on the whole of V.*

Now the following proposition was proved in [11].

PROPOSITION 4. *Let φ be a regular analytic differential on R which has Λ_0-behavior. If there is a family of lines in* **C**, *$\hat{\mathscr{L}}=\{\hat{L}_j\}_{j\in J}$, such that $\int_{A_j}\varphi \equiv \int_{B_j}\varphi \equiv 0 \mod \hat{L}_j$ for every $j \in J$, then φ should be identically zero [$\hat{L}_j=L_j$ for all but a finite number of $j \in J$].*

It should be noted, however, that a similar theorem does not hold for harmonic differentials (and a fortiori C^1-differentials) with Λ_0-behavior. In fact, we have the following well known

PROPOSITION 5. *There exist harmonic semiexact differentials ω_{A_j}, ω_{B_j} on R such that*

(i) *for some dv', $dv'' \in \Lambda_{c0}^1$ and λ'_{e0}, $\lambda''_{e0} \in \Lambda_{e0}^{(1)}$ $\omega_{A_j}=dv'+\lambda'_{e0}$, $\omega_{B_j}=dv''+\lambda''_{e0}$ on R (in particular, ω_{A_j} and ω_{B_j} have Λ_0-behavior),*

(ii) $\int_{A_k}\omega_{A_j} = \int_{B_k}\omega_{B_j} = \delta_{jk}, \int_{B_k}\omega_{A_j} = \int_{A_k}\omega_{B_j} = 0,\ j,\ k \in J.$

PROOF. Omitted (cf., e.g., [2], [5]).

PROPOSITION 6. *Let Λ_0 be a behavior space associated with $\mathscr{L}=\{L_j\}_{j\in J}$ and z_j be (non-zero) complex numbers such that $z_j \equiv 0 \mod L_j$, $j \in J$. Then $z_j\omega_{A_j}$, $z_j\omega_{B_j} \in \Lambda_0$.*

PROOF. Take a sufficiently large integer $n\,(>0)$ and a primitive function Ω_{A_j} of ω_{A_j} on $R-R_n$. Since $z_j\omega_{A_j}$ has Λ_0-behavior, we have by Theorem 1

$$< z_j\omega_{A_j},\ i\lambda_0^* >_{R-R_n} = \operatorname{Im}\Big[z_j\int_{\partial R_n}\Omega_{A_j}\bar{\lambda}_0\Big]$$

for any $\lambda_0 \in \Lambda_0$. On the other hand, Lemma yields

$$< z_j\omega_{A_j},\ i\lambda_0^* >_{R_n} = \operatorname{Im}\Big[z_j\int_{B_j}\bar{\lambda}_0\Big] - \operatorname{Im}\Big[z_j\int_{\partial R_n}\Omega_{A_j}\bar{\lambda}_0\Big].$$

It follows that $<z_j\omega_{A_j},\ i\lambda_0^*> =0$ for any $\lambda_0 \in \Lambda_0$. Therefore $z_j\omega_{A_j} \in i\Lambda_0^{*\perp}=\Lambda_0$. Similarly we have $z_j\omega_{B_j} \in \Lambda_0$. q.e.d.

PROOF OF PROPOSITION 2. Let $L=L_\theta$. Let $\Lambda_0=\Lambda_0\,(R,\mathscr{L})$ and $\Lambda'_0=$

$\Lambda_0(R, \mathscr{L}')$ be behavior spaces associated with $\mathscr{L}=\{L_j\}_{j\in J}$ and $\mathscr{L}'=\{L'_j\}_{j\in J}$ respectively, and assume that they satisfy condition 1°). Then by Proposition 1' $\Lambda'_0=e^{i\theta}\bar{\Lambda}_0$. If z_j is a non-zero complex number such that $z_j\equiv 0 \mod L_j$, then $z_j\omega_{A_j}$ is an element of Λ_0 by Proposition 6. Hence $e^{i\theta}\bar{z}_j\bar{\omega}_{A_j}\in\Lambda'_0$ so that $0\neq\int_{A_j} e^{i\theta}\bar{z}_j\bar{\omega}_{A_j}=e^{i\theta}\bar{z}_j\equiv 0 \mod L'_j$. Therefore $L'_j=e^{i\theta}\bar{L}_j$, $j\in J$. This implies 2°).

q.e.d.

Theorem 1 also suggests that Λ_0-behavior actually defines boundary behavior of differentials. Namely, if Λ_0 and $\tilde{\Lambda}_0$ are two behavior spaces which coincide (yet in an ambiguous sense) near the ideal boundary, then Λ_0- and $\tilde{\Lambda}_0$-behaviors will be the same; a differential with Λ_0-behavior will have $\tilde{\Lambda}_0$-behavior and vice versa. Later we shall see that this is really true.

5. For our purposes, it will be convenient to introduce the following

DEFINITION 4. *Two behavior spaces Λ_0 and $\tilde{\Lambda}_0$ are said to be equivalent* ($\Lambda_0\sim\tilde{\Lambda}_0$) *if and only if conditions* (i), (ii) *below are fulfilled*:

(i) *every $\lambda_0\in\Lambda_0$ has $\tilde{\Lambda}_0$-behavior,*
(ii) *every $\tilde{\lambda}_0\in\tilde{\Lambda}_0$ has Λ_0-behavior.*

The relation $\sim$ obviously defines an equivalence relation in $\mathscr{B}$. Also, it is an immediate consequence of the definition that two behavior spaces define the same boundary behavior if and only if they are equivalent to each other. In other words, there is a one-to-one correspondence between $\mathscr{B}/\sim$ and the family $\mathscr{B}_0$ of boundary behaviors which are defined by means of behavior spaces.

Now let $\Lambda_0=\Lambda_0(R, \mathscr{L})\in\mathscr{B}$, $\mathscr{L}=\{L_j\}_{j\in J}$. Let J^* be a finite subset of J and $\mathscr{L}^*=\{L^*_j\}_{j\in J}$ a family of lines in $\mathbf{C}$. We set

$$\tilde{L}_j=\begin{cases} L_j, & j\in J-J^*,\\ L^*_j, & j\in J^*\end{cases}$$

and $\tilde{\mathscr{L}}=\{\tilde{L}_j\}_{j\in J}$. We then define

$$T^{J^*}_{\mathscr{L}^*}\Lambda_0=\left\{\lambda\in\Lambda_h \,\middle|\, \begin{array}{l}\lambda \text{ has } \Lambda_0\text{-behavior and}\\ \int_{A_j}\lambda\equiv\int_{B_j}\lambda\equiv 0 \mod \tilde{L}_j,\ j\in J\end{array}\right\}.$$

Later we shall prove that every element of $T^{J^*}_{\mathscr{L}^*}\Lambda_0$ can be obtained from some element of Λ_0 by subtracting a suitable finite linear combination of ω_{A_j} and ω_{B_j} (Proposition 5) with complex coefficients. See Corollary 2 to Theorem 3. We shall call such a $T^{J^*}_{\mathscr{L}^*}$ a *transformation* determined by J^* and $\mathscr{L}^*$. In fact, we shall soon prove that $T^{J^*}_{\mathscr{L}^*}\Lambda_0$ belongs to $\mathscr{B}$ so long as Λ_0 does. We set

$$\mathscr{T}=\{T=T^{J^*}_{\mathscr{L}^*} \mid J^* \text{ is a finite subset of } J \text{ and } \mathscr{L}^* \text{ is a family of lines in } \mathbf{C}\}$$

and

$$\mathscr{B}[J^*, \mathscr{L}^*] = \{\Lambda_0 = \Lambda_0(R, \mathscr{L}) \in \mathscr{B} | L_j = L_j^*, j \in J^*\}.$$

We state a theorem whose proof is given in the next section.

THEROEM 2. *Every* $T_{\mathscr{L}^*}^{J^*}(\in \mathscr{T})$ *maps* $\mathscr{B}$ *onto* $\mathscr{B}[J^*, \mathscr{L}^*]$.

6. PROOF OF THEOREM 2. We set $\tilde{\Lambda}_0 = T_{\mathscr{L}^*}^{J^*}\Lambda_0$, $\Lambda_0 \in \mathscr{B}$. By the definition of $T_{\mathscr{L}^*}^{J^*}$ it is obvious that $\tilde{\Lambda}_0$ is contained in Λ_{hse} and that $\int_{A_j} \tilde{\lambda} \equiv \int_{B_j} \tilde{\lambda}_0 \equiv 0 \mod \tilde{L}_j$, $j \in J$, for every $\tilde{\lambda}_0 \in \tilde{\Lambda}_0$. Therefore we only need to show the equality $i\tilde{\Lambda}_0^* = \tilde{\Lambda}_0^\perp$. (Note that this implies the closedness of $\tilde{\Lambda}_0$.)

In the first place, let ω' and ω'' be any two elements of $\tilde{\Lambda}_0$. Then there are $\lambda_0', \lambda_0'' \in \Lambda_0$; $\lambda_{e0}', \lambda_{e0}'' \in \Lambda_{e0}^{(1)}$ and an $R_n \in \mathscr{R}$ such that

$$\omega' = \lambda_0' + \lambda_{e0}', \quad \omega'' = \lambda_0'' + \lambda_{e0}'' \quad \text{on} \quad R - \bar{R}_n.$$

Making use of Lemma twice, we have for $m > n$

$$\begin{aligned}
<\omega', i\omega''^*>_{R_m} &= -\mathrm{Im} \int_{\partial R_m} \Omega' \bar{\omega}'' + \mathrm{Im} \sum_{j \in J_m} \left(\int_{A_j} \omega' \int_{B_j} \bar{\omega}'' - \int_{B_j} \omega' \int_{A_j} \bar{\omega}'' \right) \\
&= <\lambda_0' + \lambda_{e0}', i(\lambda_0'' + \lambda_{e0}'')^*>_{R_m} \\
&\quad - \mathrm{Im} \sum_{J \in J_m} \left(\int_{A_j} \lambda_0' \int_{B_j} \bar{\lambda}_0'' - \int_{B_j} \lambda_0' \int_{A_j} \bar{\lambda}_0'' \right) + \mathrm{Im} \sum_{j \in J_m} \left(\int_{A_j} \omega' \int_{B_j} \bar{\omega}'' - \int_{B_j} \omega' \int_{A_j} \bar{\omega}'' \right) \\
&= <\lambda_0' + \lambda_{e0}', i(\lambda_0'' + \lambda_{e0}'')^*>_{R_m},
\end{aligned}$$

where Ω' is a primitive function of ω' on $R - R_n$. But the last term tends to zero as $m \to \infty$, for Λ_0 is a behavior space (hence $i\Lambda_0^* = \Lambda_0^\perp$) and any two of Λ_0, $\Lambda_{e0}^{(1)}$, $\Lambda_{e0}^{(1)*}$ are orthogonal to each other. Thus we have proved $i\tilde{\Lambda}_0^* \subset \tilde{\Lambda}_0^\perp$.

Assume, conversely, that $\lambda \in \Lambda_h$ is orthogonal to $\tilde{\Lambda}_0$. We have to show α) the semiexactness of $i\lambda^*$, β) $\int_{A_j} i\lambda^* \equiv \int_{B_j} i\lambda^* \equiv 0 \mod \tilde{L}_j$, $j \in J$ and γ) $i\lambda^*$ has Λ_0-behavior. To prove α), we set $\Lambda_{hm} = \Gamma_{hm} + i\Gamma_{hm}$, where Γ_{hm} is the space of *real* harmonic measures on R (see [2], p. 294). It is easily seen that $\Lambda_{hm} = \Lambda_{hse}^{*\perp}$ (cf. [11]). Due to this property, we have for any $du_{hm} \in \Lambda_{hm}$ and $\omega \in \Lambda_0$ ($\subset \Lambda_{hse}$),

$$\mathrm{Im} \int_{\partial R_m} u_{hm} \bar{\omega} = <du_{hm}, i\omega^*>_{R_m} \to 0 \qquad (m \to \infty).$$

Now Theorem 1 yields that du_{hm} has Λ_0-behavior. Since the period conditions $\int_{A_j} du_{hm} \equiv \int_{B_j} du_{hm} \equiv 0 \mod \tilde{L}_j$, $j \in J$, are trivially satisfied, we know that $du_{hm} \in \tilde{\Lambda}_0$ and hence $\Lambda_{hm} \subset \tilde{\Lambda}_0$. Consequently $\tilde{\Lambda}_0^{*\perp} \subset \Lambda_{hm}^{*\perp} = \Lambda_{hse}$ and this proves α).

Next let ω_{A_j} be the differential constructed in Proposition 5: $\omega_{A_j} = dv' + \lambda_{e0}'$, $dv' \in \Lambda_{c0}^1$, $\lambda_{e0}' \in \Lambda_{e0}^{(1)}$. The differential $z_j \omega_{A_j}$ belongs to $\tilde{\Lambda}_0$ for every complex number z_j, $z_j \equiv 0 \mod \tilde{L}_j$ (cf. Proposition 6). Because $i\lambda^*$ has been known to be semiexact, we can apply Lemma to $i\lambda^*$ and $z_j\omega_{A_j}$ and obtain

$$0 = <\lambda, z_j\omega_{A_j}> = <\lambda^*, z_j dv'^*>$$

$$= -\operatorname{Re}\Big[\bar{z}_j \sum_{k\in J}\Big(\int_{A_k}\lambda^*\int_{B_k}\overline{dv'} - \int_{B_k}\lambda^*\int_{A_k}\overline{dv'}\Big)\Big] = \operatorname{Re}\Big(\bar{z}_j\int_{B_j}\lambda^*\Big).$$

Therefore we see that $\int_{B_j} i\lambda^* \equiv 0 \mod \tilde{L}_j$. Similarly we have $\int_{A_j} i\lambda^* \equiv 0 \mod \tilde{L}_j$. We have proved β).

Finally we shall prove γ). Take an arbitrary ω in Λ_0. By means of Proposition 5 we see that there are $\tilde{\omega} \in \tilde{\Lambda}_0$, $du \in \Lambda_{c0}^1$ and $\lambda_{e0} \in \Lambda_{e0}^{(1)}$ such that $\tilde{\omega} = \omega + du + \lambda_{e0}$ on R. Let Φ be a primitive function of $i\lambda^*$ on $R - R_n$, n being a sufficiently large integer. Then, by Lemma, for $m > n$ we have

$$\operatorname{Im}\int_{\partial R_m}\Phi\bar{\omega} = -<\lambda^*, \omega^*>_{R_m} + \operatorname{Re}\sum_{j\in J_m}\Big(\int_{A_j}\lambda^*\int_{B_j}\bar{\omega} - \int_{B_j}\lambda^*\int_{A_j}\bar{\omega}\Big)$$

$$= -<\lambda, \tilde{\omega} - du - \lambda_{e0}>_{R_m} + \operatorname{Re}\sum_{j\in J_m}\Big(\int_{A_j}\lambda^*\int_{B_j}\overline{(\tilde{\omega} - du)}$$

$$- \int_{B_j}\lambda^*\int_{A_j}\overline{(\tilde{\omega} - du)}\Big)$$

$$= <\lambda, du>_{R_m} - \operatorname{Re}\sum_{j\in J_m}\Big(\int_{A_j}\lambda^*\int_{B_j}\overline{du} - \int_{B_j}\lambda^*\int_{A_j}\overline{du}\Big) + \varepsilon_m,$$

where $\lim_{m\to\infty}\varepsilon_m = 0$. A further use of Lemma implies

$$\operatorname{Im}\int_{\partial R_m}\Phi\bar{\omega} = <\lambda, du>_{R_m} - <\lambda^*, du^*>_{R_m} + \varepsilon_m = \varepsilon_m.$$

Theorem 1 now allows us to conclude that $d\Phi = i\lambda^*$ has Λ_0-behavior. We have thus proved that $T_{\mathscr{L}^*}^{J^*}\mathscr{B} \subset \mathscr{B}[J^*, \mathscr{L}^*]$.

Finally let $\hat{\Lambda}_0 \in \mathscr{B}[J^*, \mathscr{L}^*]$. Then it is easy to see that $T_{\mathscr{L}}^{J^*}\hat{\Lambda}_0 \in \mathscr{B}$ and $T_{\mathscr{L}^*}^{J^*}(T_{\mathscr{L}}^{J^*}\hat{\Lambda}_0) = \hat{\Lambda}_0$. This completes the proof of Theorem 2.

COROLLARY. *If* $\Lambda_0 \in \mathscr{B}$ *and* $T \in \mathscr{T}$, *then* $T\Lambda_0 \sim \Lambda_0$.

7. We shall now define the product of two transformations. Suppose that $T_k = T_{\mathscr{L}_k^*}^{J_k^*}$ is a transformation determined by J_k^* and $\mathscr{L}_k^* = \{L_{jk}^*\}_{j\in J}$, $k = 1, 2$. Then the product $T_2 \circ T_1$ of T_1 and T_2 is defined as a transformation $T_{\mathscr{L}^*}^{J^*}$ determined by $J^* = J_1^* \cup J_2^*$ and $\mathscr{L}^* = \{L_j^*\}_{j\in J}$, where

$$L_j^* = \begin{cases} L_{j1}^* & \text{if } j \in J_1 - J_2, \\ L_{j2}^* & \text{otherwise.} \end{cases}$$

The transformation determined by $J_0^* = \phi$ and any $\mathscr{L}^*$ gives the identity (neutral element) with respect to the product. We note that the product defined above is non-commutative. Indeed, $T_2 \circ T_1 \neq T_1 \circ T_2$ if $J_1^* \cap J_2^* \neq \phi$ and $L_{j1}^* \neq L_{j2}^*$ for some $j \in J_1^* \cap J_2^*$, for example. Since the associative law $T_3 \circ (T_2 \circ T_1) = (T_3 \circ T_2) \circ T_1$, $T_k \in \mathscr{T}$, is obviously satisfied, $\mathscr{T}$ becomes a monoid which operates on B.

Now we shall prove

THEOREM 3. *Two behavior spaces Λ_0 and $\tilde{\Lambda}_0$ are equivalent if and only if $\tilde{\Lambda}_0 = T\Lambda_0$ for some $T \in \mathscr{T}$.*

PROOF. The if part is obvious (cf. Corollary to Theorem 2). We shall now prove the only if part.

Let $\Lambda_0 = \Lambda_0(R, \mathscr{L})$, $\mathscr{L} = \{L_j\}_{j \in J}$ and $\tilde{\Lambda}_0 = \Lambda_0(R, \tilde{\mathscr{L}})$, $\tilde{\mathscr{L}} = \{\tilde{L}_j\}_{j \in J}$. Let ω_{A_j} and ω_{B_j} $(j \in J)$ be the differentials constructed in Proposition 5. If ξ_j, η_j are non-zero complex numbers such that $\xi_j \equiv \eta_j \equiv 0 \mod L_j$, and $|\xi_j| < (2^j \|\omega_{A_j}\|)^{-1}$, $|\eta_j| < (2^j \|\omega_{B_j}\|)^{-1}$, $j \in J$, then the series

$$\omega = \sum_{j \in J} (\xi_j \omega_{A_j} + \eta_j \omega_{B_j})$$

is convergent and belongs to Λ_0. Furthermore, $\int_{A_j} \omega = \xi_j \neq 0$, $\int_{B_j} \omega = \eta_j \neq 0$ for every $j \in J$.

By our assumption ω has $\tilde{\Lambda}_0$-behavior and therefore there are differentials $\tilde{\lambda}_0 \in \tilde{\Lambda}_0$, $\tilde{\lambda}_{e0} \in \Lambda_{e0}^{(1)}$ such that $\omega = \tilde{\lambda}_0 + \tilde{\lambda}_{e0}$ outside some $\bar{R}_n$. Hence for $j \in J - J_n$ $\int_{A_j, B_j} \tilde{\lambda}_0 = \int_{A_j, B_j} \omega$ are non-zero complex numbers which are $\equiv 0 \mod L_j$ as well as $\equiv 0 \mod \tilde{L}_j$. Consequently we have $\tilde{L}_j = L_j$ for every $j \in J - J_n$.

Now the set $J^* = \{j \in J | L_j \neq \tilde{L}_j\}$ is a finite subset of J. If we set $T = T_{\tilde{\mathscr{L}}}^{J^*}$, then we can easily verify that $T\Lambda_0 = \tilde{\Lambda}_0$. In fact, the inclusion $T\Lambda_0 \supset \tilde{\Lambda}_0$ is obvious. To prove the converse inclusion relation, let λ be any element of $T\Lambda_0$. By the definition of $T\Lambda_0$, λ has Λ_0-behavior. Since Λ_0 is equivalent to $\tilde{\Lambda}_0$, we see that λ has $\tilde{\Lambda}_0$-behavior. Therefore there are $\tilde{\lambda}'_0 \in \tilde{\Lambda}_0$, $\tilde{\lambda}'_{e0} \in \Lambda_{e0}^{(1)}$ for which $\lambda = \tilde{\lambda}'_0 + \tilde{\lambda}'_{e0}$ outside some $\bar{R}_m$.

If we set $\lambda' = \lambda - \tilde{\lambda}'_0$, λ' is harmonic on R and is equal to $\tilde{\lambda}'_{e0}$ on $R - \bar{R}_m$. Furthermore, $\int_{A_j} \lambda' \equiv \int_{B_j} \lambda' \equiv 0 \mod \tilde{L}_j$ for $j \in J_m$, because $\int_{A_j} \lambda \equiv \int_{B_j} \lambda \equiv 0 \mod \tilde{L}_j$, $j \in J$. We can choose complex numbers x_j, y_j, $x_j \equiv y_j \equiv 0 \mod \tilde{L}_j$, $j \in J_m$, so that

$$\lambda'' = \lambda' - \sum_{j \in J_m} (x_j \omega_{A_j} + y_j \omega_{B_j})$$

has vanishing A_j- and B_j-periods, $j \in J_m$. Without loss of generality, we may assume that $\sum_{j \in J_m} (x_j \omega_{A_j} + y_j \omega_{B_j}) = \lambda'_{e0}$ on $R - \bar{R}_m$, where $\lambda'_{e0} \in \Lambda_{e0}^{(1)}$ (cf. Proposition 5). Then λ'' is a harmonic semiexact differential on R such that $\lambda'' = \lambda''_{e0}$ outside $\bar{R}_m$, $\lambda''_{e0} \in \Lambda_{e0}^{(1)}$. It follows that λ'' is identically zero on R. Since $x_j \omega_{A_j}$, $y_j \omega_{B_j} \in \tilde{\Lambda}_0$, we now conclude that $\lambda = \lambda' + \tilde{\lambda}'_0 = \sum_{j \in J_m} (x_j \omega_{A_j} + y_j \omega_{B_j}) + \tilde{\lambda}'_0$

belongs to $\tilde{\Lambda}_0$. q.e.d.

COROLLARY 1. *There is a one-to-one correspondence between* $\mathscr{B}_0$ *and* $\mathscr{B}/\mathscr{T}$.

COROLLARY 2. *Let* $\Lambda_0 \in \mathscr{B}$ *and* $T \in \mathscr{T}$. *Then for every* $\lambda \in T\Lambda_0$ *there exist* $\lambda_0 \in \Lambda_0$ *and* $x_j, y_j \in \mathbf{C}$ *such that*

(i) $x_j = y_j = 0$ *for all but a finite number of* $j \in J$,

(ii) $\lambda = \lambda_0 - \sum_{j \in J} (x_j \omega_{A_j} + y_j \omega_{B_j})$.

PROOF. Since $T\Lambda_0$ is equivalent to Λ_0, there is a transformation $T' \in \mathscr{T}$ such that $\Lambda_0 = T'(T\Lambda_0)$. Therefore every element λ of $T\Lambda_0$ can be written as

$$\lambda = \lambda_0 - \sum_{j \in J} (x_j \omega_{A_j} + y_j \omega_{B_j})$$

with $\lambda_0 \in \Lambda_0$ and $x_j, y_j \in \mathbf{C}$, where $x_j = y_j = 0$ except for a finite number of $j \in J$. q.e.d.

8. A similar argument as above shows the following

PROPOSITION 7. *Suppose that* $\Lambda_0, \Lambda_0' \in \mathscr{B}$ *are dual to each other with respect to a line* L *in* $\mathbf{C}$ *and* $T = T^{J^*}_{\mathscr{L}^*} \in \mathscr{T}$. *Let* T' *be another transformation determined by* J^* *and* $\mathscr{L}^{*\prime} = \{L_j^{*\prime} | L_j^{*\prime}$ *is a line in* $\mathbf{C}$ *such that* $L_j^{*\prime} \circ L_j^* = L, j \in J\}$. *Then* $T\Lambda_0$ *and* $T'\Lambda_0'$ *are dual to each other with respect to the line* L.

For an open set $D \subset R$, let $\mathscr{A}(D)$ be the family of analytic differentials on D. Let P be a regular partition of the ideal boundary of R and set $(P)\mathscr{A}_{\mathscr{L},n} = \{\varphi \in \mathscr{A}(R - \bar{R}_n) | \varphi$ is (P)semiexact and $\int_{A_j} \varphi \equiv \int_{B_j} \varphi \equiv 0 \mod L_j, \ j \in J - J_n\}$. We identify two elements φ_1, φ_2 in $\cup_{n=1}^{\infty} (P)\mathscr{A}_{\mathscr{L},n}$ when the difference $\varphi_1 - \varphi_2$ has Λ_0-behavior. Each equivalence class is called a $(P)\Lambda_0$-singularity (see [12]).

As an easy consequence of Definition 4 we have

PROPOSITION 8. *Let* P *be a regular partition of the ideal boundary of* R *and let* $\Lambda_0, \tilde{\Lambda}_0 \in \mathscr{B}$, $\Lambda_0 \sim \tilde{\Lambda}_0$. *Then a* $(P)\Lambda_0$*-singularity is a* $(P)\tilde{\Lambda}_0$*-singularity and vice versa.*

Theorem 3, Propositions 7 and 8 allow us to choose a most suitable behavior space among the equivalence class when we deal with a concrete problem concerning boundary behavior (of differentials). Replacing the given behavior space by a new one which is equivalent to the original, formulation of the problem may be sometimes considerably simplified. (One of such examples will be found in another paper.)

9. Finally we shall mention the case of semiexact canonical differentials. Let $\Lambda_K=\Lambda_K(R)=\Gamma_{hm}+i\Gamma_{hse}$, Γ_{hse} being the space of square integrable *real* harmonic semiexact differentials on R. We know that Λ_K is a behavior space which is (**R**-) dual to itself ([11]). A semiexact canonical differential is a meromorphic differential on R which has Λ_K-behavior ([3–5], [9] etc.).

Theorem 1 implies the following theorem due to Kusunoki:

THEOREM 4. ([4, 5]) *Let $\varphi=du+idu^*$ be a meromorphic semiexact differential on R. Suppose that for some $R_n\in\mathscr{R}$ du is exact on $V=R-\bar{R}_n$ and $\|du\|_V<\infty$. Then the following two assertions are equivalent*:

(I′) *φ is a semiexact canonical differential.*
(II′) *For any square integrable (real) harmonic semiexact differential τ_V on $\bar{V}$, $<du, \tau_V^*>_V=-\int_{\partial V}u\tau_V$.*

For the proof we only need to show the equivalence of (II′) and (II) in Theorem 1 under the assumption $\Lambda_0=\Lambda_K$.

In the first place, let $\omega=\sigma+i\tau\in\Lambda_K$. Then we have $<\varphi, i\omega^*>_V=-<du, \tau^*>_V+<du, \sigma>_V$ and $\mathrm{Im}\int_{\partial V}\Phi\bar{\omega}=\int_{\partial V}u^*\sigma-\int_{\partial V}u\tau$. (Note that du^* is semiexact.) Thus condition (II) is equivalent to

$$(*)\qquad <du, \tau^*>_V = -\int_{\partial V}u\tau \qquad \text{for every}\quad \tau\in\Gamma_{hse},$$

$$(**)\qquad <du, \sigma>_V = -\int_{\partial V}u^*\sigma \qquad \text{for every}\quad \sigma\in\Gamma_{hm}.$$

Assume (II). Let τ_V be a (real) harmonic semiexact differential on $\bar{V}$ such that $\|\tau_V\|_V<\infty$. Then there is a closed C^1-differential τ_R on R whose restriction to $\bar{V}$ is equal to τ_V (cf. [10], [13]; see also the proof of Theorem 1). By the Dirichlet principle ([2], [5], [11] etc.) there are $\tau_{hse}\in\Gamma_{hse}$, $\tau_{hm}\in\Gamma_{hm}$ and $\tau_{e0}\in\Gamma_{e0}^{(1)}$ such that $\tau_R=\tau_{hse}+\tau_{hm}^*+\tau_{e0}$, where $\Gamma_{e0}^{(1)}=\{\lambda\in\Lambda_{e0}^{(1)}|\lambda \text{ is real}\}$. Since $\tau_R|_V=\tau_V$ is semiexact (on $\bar{V}$), $\tau_{hm}^*\in\Gamma_{hm}^*\cap\Gamma_{hse}=\{0\}$ and hence $\tau_{hm}=0$.

Now let $\hat{u}$ be a (real valued) C^1-function on R such that $\hat{u}|_V=u$. Clearly $\|d\hat{u}\|<\infty$. For any $\varepsilon>0$ we can find a (real valued) function $f\in C_0^2(R)$ such that

$$|<d\hat{u}, \tau_{e0}^*-df^*>_{R_k}| \le \|d\hat{u}\|\cdot\|\tau_{e0}-df\| < \varepsilon$$

holds for every k. One can choose k so large that $<d\hat{u}, df^*>_{R_k}=-\int_{\partial R_k}\hat{u}df=0$, for the function f has compact support. Then we have

$$\left|<du, \tau_{e0}^*>_{V\cap R_k}+\int_{\partial V}u\tau_{e0}\right| = \left|\int_{\partial R_k}u\tau_{e0}\right| = |<d\hat{u}, \tau_{e0}^*>_{R_k}|$$

$$= |<d\hat{u}, \tau_{e0}^* - df^*>_{R_k}| < \varepsilon.$$

It follows that $<du, \tau_{e0}^*>_V = -\int_{\partial V} u\tau_{e0}$. Consequently we have $<du, \tau_V^*>_V = -\int_{\partial V} u\tau_V$. We have shown that (II) implies (II').

Conversely assume (II'). Then every $\tau \in \Gamma_{hse}$ satisfies (*), for τ is certainly square integrable and harmonic semiexact on $\bar{V}$. Next let $\sigma = ds \in \Gamma_{hm}$. Then by the definition of Γ_{hm} there are $\sigma_m = ds_m \in \Gamma_{hm}(\bar{R}_m)$ such that $\|\sigma_m - \sigma\|_{R_m} \to 0$ as $m \to \infty$. We note that under appropriate normalization $\{s_m\}$ is uniformly convergent to the function s on ∂V (cf. [2], p. 147). Also we have $|<du, \sigma - \sigma_m>_{V \cap R_m}| \leq \|du\|_V \cdot \|\sigma - \sigma_m\|_{R_m} \to 0$, $m \to \infty$. Since du^* is semiexact, it follows that

$$<du, \sigma>_V = \lim_{m\to\infty} <du, \sigma_m>_{V \cap R_m}$$

$$= \lim_{m\to\infty} \int_{\partial V} s_m du^* = \int_{\partial V} s du^*.$$

The last term (Stieltjes integral) can be integrated by parts (cf. [6]) and thus (**) follows. This completes the proof of Theorem 4.

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