HIROSHIMA MATH. J.
8 (1978), 63–69

A Note on Vector Fields up to Bordism

Toshio YOSHIDA
(Received August 20, 1977)

§1. Introduction

For a (differentiable) closed m-manifold M^m, let $\operatorname{Span} M^m$ denote the maximum number of linearly independent (tangent) vector fields on M^m, and $w_i M^m$ the i-th Stiefel-Whitney class of M^m. Then by [2, p. 39], we have the following:

(1.1) *If* $\operatorname{Span} M^m \geqq k$, *then* $w_i M^m = 0 \quad (i \geqq m - k + 1)$.

The converse of (1.1) is not true. The purpose of this note is to prove the following

THEOREM. *Let M^m be a closed m-manifold for which all Stiefel-Whitney numbers divisible by $w_m, \cdots, w_{m-k+1}$ are zero. If $k \leqq 6$, then there exists a closed m-manifold N^m such that N^m is unorientedly bordant to M^m and* $\operatorname{Span} N^m \geqq k$.

By R. E. Stong [3, p. 440], the following conjecture is proved for $k = 1, 2, 4$: Under the assumption of the theorem, M^m is unorientedly bordant to a manifold N^m which is fibered over the product $(S^1)^k$ of k-copies of the circle S^1. It is clear that the theorem holds if this conjecture is true.

The author wishes to express his hearty thanks to Professors M. Sugawara and T. Kobayashi for their valuable suggestions and discussions.

§2. Some manifolds having many vector fields

For a real (differentiable) n-plane bundle $\zeta \to V$ over a closed m-manifold V, we denote by $p: RP(\zeta) \to V$ the associated projective space bundle with fiber $RP(n-1)$ (the real projective $(n-1)$-space). Then $RP(\zeta)$ is a closed $(m+n-1)$-manifold and

(2.1) *the cohomology with Z_2 coefficients of $RP(\zeta)$ is the free module over the cohomology of V on $1, c, \cdots, c^{n-1}$, with the relation*

$$c^n = \sum_{i=1}^{n} p^*(w_i \zeta) c^{n-i},$$

where c is the first Stiefel-Whitney class of the canonical line bundle over $RP(\zeta)$ and $w_i \zeta$ is the i-th Stiefel-Whitney class of ζ.

LEMMA 2.2 [1, (23.3)]. *The total Stiefel-Whitney class of $RP(\zeta)$ is equal to*

$$(\textstyle\sum_{i=0}^{m} p^*(w_i V))(\sum_{i=0}^{n}(1+c)^{n-i} p^*(w_i\zeta)).$$

Now, let $\mathfrak{N}_* = \sum_m \mathfrak{N}_m$ be the unoriented bordism ring and let $[M] \in \mathfrak{N}_*$ denote the bordism class of a closed manifold M.

LEMMA 2.3 [3, Lemma 3.4]. *For any non-negative integers $n_1, \cdots, n_k$ ($n_1 > 0$, $k \geqq 2$), consider the projective space bundle*

$$X = RP(p_1^*\xi_{n_1} \oplus \cdots \oplus p_k^*\xi_{n_k}) \longrightarrow RP(n_1) \times \cdots \times RP(n_k),$$

where ξ_n is the canonical line bundle over the projective n-space $RP(n)$, and p_i: $RP(n_1) \times \cdots \times RP(n_k) \to RP(n_i)$ is the projection onto the i-th factor. Then the class $[X]$ of the closed m $(= n_1 + \cdots + n_k + k - 1)$-manifold X in $\mathfrak{N}_*$ is indecomposable if and only if

$$\binom{m-1}{n_1} + \cdots + \binom{m-1}{n_k} \equiv 1 \mod 2.$$

LEMMA 2.4 (cf. [3, Prop. 2.4]). *By using the canonical line bundle λ over the projective space bundle $RP(\xi_{n_1-1} \oplus 1)$ instead of ξ_{n_1} in the above lemma, consider the projective space bundle*

$$Y = RP(q_1^*\lambda \oplus q_2^*\xi_{n_2} \oplus \cdots \oplus q_k^*\xi_{n_k}) \longrightarrow RP(\xi_{n_1-1} \oplus 1) \times RP(n_2) \times \cdots \times RP(n_k),$$

where q_i is the projection of $RP(\xi_{n_1-1} \oplus 1) \times RP(n_2) \times \cdots \times RP(n_k)$ onto the i-th factor. Then the class $[Y]$ of the closed m-manifold Y in $\mathfrak{N}_*$ is indecomposable if and only if $[X]$ is so, where X is the one in the above lemma.

PROOF. Let ξ' be the orthocomplement of ξ_{n_1-1} in the trivial bundle $RP(n_1 - 1) \times R^{n_1} \to RP(n_1 - 1)$, and consider the composition

$$\begin{aligned}\varphi = \pi i\colon RP(\xi_{n_1-1} \oplus 1) &\xrightarrow{i} RP(\xi_{n_1-1} \oplus \xi' \oplus 1) \\ &= RP(n_1 - 1) \times RP(n_1) \xrightarrow{\pi} RP(n_1)\end{aligned}$$

of the inclusion i and the projection π.

Since $\varphi^*\xi_{n_1} = \lambda$ by [3, p. 433], we have the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{q} & RP(\xi_{n_1-1} \oplus 1) \times RP(n_2) \times \cdots \times RP(n_k) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \varphi \times \mathrm{id}} \\ X & \xrightarrow{p} & RP(n_1) \times RP(n_2) \times \cdots \times RP(n_k) \end{array}$$

where p and q are the bundle projections and Φ is the bundle map defined naturally.

By Lemma 2.2, we see easily that the total Stiefel-Whitney classes of X and Y are given by

(2.5.1) $w(X) = (\prod_{i=1}^{k} p^*p_i^*(1 + \alpha_i)^{n_i+1})(\prod_{i=1}^{k}(1 + x + p^*p_i^*\alpha_i))$,

(2.5.2) $w(Y) = (q^*q_1^*r^*(1 + \alpha)^{n_1})(q^*q_1^*(1 + c + r^*\alpha))(q^*q_1^*(1 + c))$

$\cdot(\prod_{i=2}^{k} q^*q_i^*(1 + \alpha_i)^{n_i+1})(1 + y + q^*q_1^*c)(\prod_{i=2}^{k}(1 + y + q^*q_i^*\alpha_i))$,

where x and y are the first Stiefel-Whitney classes of the canonical line bundles over X and Y, respectively, $c=w_1\lambda$, $\alpha_i \in H^1(RP(n_i); Z_2)$ and $\alpha \in H^1(RP(n_1-1); Z_2)$ are the generators, and r: $RP(\xi_{n_1-1}\oplus 1)\to RP(n_1-1)$ is the bundle projection.

Therefore, the s-class $s_m(X)$ of X is given by

$$s_m(X) = \sum_{i=1}^{k}(n_i + 1)(p^*p_i^*\alpha_i)^m + \sum_{i=1}^{k}(x + p^*p_i^*\alpha_i)^m .$$

Thus we have the following equality since $k \geqq 2$:

$$s_m(X) = \sum_{i=1}^{k}(x + p^*p_i^*\alpha_i)^m .$$

Similarly, the s-class $s_m(Y)$ of Y is equal to

$$s_m(Y) = (y + q^*q_1^*c)^m + \sum_{i=2}^{k}(y + q^*q_i^*\alpha_i)^m .$$

As the canonical line bundle over Y is the induced bundle of that over X by Φ, we see that

$$\Phi^*x = y .$$

Also, we see that

$$\Phi^*p^*p_1^*\alpha_1 = q^*q_1^*c, \quad \Phi^*p^*p_i^*\alpha_i = q^*q_i^*\alpha_i \quad (2 \leqq i \leqq k),$$

and hence $\Phi^*s_m(X)=s_m(Y)$ by the above equalities.

By [4, p. 97], it is sufficient to show that $s_m(X)\neq 0$ if and only if $s_m(Y)\neq 0$. If $s_m(X)=0$, then $s_m(Y)=\Phi^*s_m(X)=0$. Conversely if $s_m(X)\neq 0$, then $s_m(X) =(p^*p_1^*\alpha_1^{n_1})\cdots(p^*p_k^*\alpha_k^{n_k})x^{k-1}$ and we see that

$$s_m(Y) = \Phi^*s_m(X) = (q^*q_1^*c^{n_1})(q^*q_2^*\alpha_2^{n_2})\cdots(q^*q_k^*\alpha_k^{n_k})y^{k-1}$$

by the above results. Since $c^2=(r^*\alpha)c$ by (2.1), this implies

$$s_m(Y) = (q^*q_1^*((r^*\alpha)^{n_1-1}c))(q^*q_2^*\alpha_2^{n_2})\cdots(q^*q_k^*\alpha_k^{n_k})y^{k-1} ,$$

which is non-zero. *q.e.d.*

By using the closed $m(=n_1+\cdots+n_k+k-1)$-manifolds

$$
(2.6)\quad
\begin{aligned}
RP(n_1, n_2, \cdots, n_k) &= RP(p_1^*\xi_{n_1} \oplus p_2^*\xi_{n_2} \oplus \cdots \oplus p_k^*\xi_{n_k}),\\
RP'(n_1, n_2, \cdots, n_k) &= RP(q_1^*\lambda \oplus q_2^*\xi_{n_2} \oplus \cdots \oplus q_k^*\xi_{n_k})
\end{aligned}
$$

in the above lemmas, define the m-manifold Q_m for any m which is not equal to 2^a-1, as follows:

$$
(2.7.1)\qquad Q_m = RP'(2^p, \overbrace{7, \cdots, 7}^{l}, 3, 1, 0),
$$

where $m=2^p(2q+1)-1$, $l=2^{p-2}q-1$, $p\geqq 2$ and $q\geqq 1$;

$$
(2.7.2)\quad
\begin{aligned}
&Q_{8l+9} = RP'(4, \overbrace{7, \cdots, 7}^{l}, 3, 0), \quad Q_{8l+5} = RP'(2, \overbrace{7, \cdots, 7}^{l}, 1, 0),\\
&Q_2 = RP(2), \quad Q_{8l+10} = RP(\overbrace{7, \cdots, 7}^{l+1}, 0, 0, 0),\\
&Q_{8l+4} = RP(\overbrace{7, \cdots, 7}^{l}, 1, 1, 0), \quad Q_{8l+6} = RP(\overbrace{7, \cdots, 7}^{l}, 3, 0, 0, 0),\\
&Q_{16l+16} = RP(\overbrace{7, \cdots, 7}^{2l+2}, 0), \quad Q_{16l+8} = RP(\overbrace{7, \cdots, 7}^{2l}, 3, 3, 0),
\end{aligned}
$$

where $l\geqq 0$.

Then, we see that the class $[Q_m]$ is indecomposable by Lemmas 2.3 and 2.4. Therefore, by the theorem of R. Thom (cf. [4, p. 96]), we have

$$
(2.8)\qquad \mathfrak{N}_* = Z_2[[Q_2], [Q_4], [Q_5], \cdots].
$$

LEMMA 2.9. (i) Span $Q_m \geqq 7(2^{p-2}q)-3+$Span $RP(2^p-1)$, *where* $m=2^p(2q+1)-1$, $p\geqq 2$ *and* $q\geqq 1$.

(ii) Span $Q_{8l+9}\geqq 7l+6$, Span $Q_{8l+5}\geqq 7l+2$, Span $Q_2=0$, Span $Q_{8l+10}\geqq 7l+7$, Span $Q_{8l+4}\geqq 7l+2$, Span $Q_{8l+6}\geqq 7l+3$, Span $Q_{16l+16}\geqq 14(l+1)$, Span $Q_{16l+8}\geqq 14l+6$, *where* $l\geqq 0$.

PROOF. It is well known that Span $RP(n)=n$ if $n=1$, 3 or 7, and the spans of X and Y are not smaller than those of the base spaces in Lemmas 2.3 and 2.4. Thus we see the lemma. *q.e.d.*

By this lemma, we obtain the following

LEMMA 2.10. *A manifold which is a product of some manifolds in* (2.7.1–2) *and whose span may be smaller than* 6 *is one of the following manifolds*:
(A) Q_2^j, $Q_2^{j-2}Q_4$, $Q_2^{j-3}Q_6$, $Q_2^{j-4}Q_4^2$, $Q_2^{j-5}Q_5^2$, $Q_2^{j-5}Q_4Q_6$ ($2j$-*manifolds*),
(B) $Q_2^{j-2}Q_5$, $Q_2^{j-4}Q_4Q_5$, $Q_2^{j-5}Q_5Q_6$ ($(2j+1)$-*manifolds*).

The straightforward calculations by using (2.1) and Lemma 2.2 show the following tables on the Stiefel-Whitney numbers of manifolds in (A) and (B):

(2.11.1)

	Q_2^j	$Q_2^{j-2}Q_4$	$Q_2^{j-3}Q_6$	$Q_2^{j-4}Q_4^2$	$Q_2^{j-5}Q_5^2$	$Q_2^{j-5}Q_4Q_6$
w_{2j}	1	0	0	0	0	0
$w_{2j-2}w_2$		1	0	0	0	0
$w_{2j-3}w_3$			1	0	0	0
$w_{2j-4}w_4$				1	1	0
$w_{2j-5}w_5$				$j-4$	$j-5$	1

(2.11.2)

	$Q_2^{j-2}Q_5$	$Q_2^{j-4}Q_4Q_5$	$Q_2^{j-5}Q_5Q_6$
$w_{2j-1}w_2$	1	0	0
$w_{2j-3}w_4$		1	0
$w_{2j-4}w_5$			1

For example, the equality

$$w_{2j-5}(M_1M_2)w_5(M_1M_2) = (j-5)\mu, \qquad \text{for} \quad M_1 = Q_2^{j-5},\ M_2 = Q_5^2,$$

is shown as follows, where $\mu \in H^{2j}(M_1M_2; Z_2)$ is the mod 2 fundamental cohomology class of M_1M_2.

Since $Q_5 = RP'(2, 1, 0)$ by (2.7.2), we see that

$$w(Q_5) = (1+c+\alpha)(1+c)(1+y+c)(1+y+\alpha_2)(1+y)$$

by (2.5.2) where we use the notations c, α and α_2 instead of $q^*q_1^*c$, $q^*q_1^*r^*\alpha$ and $q^*q_2^*\alpha_2$, respectively, for the simplicity. According to (2.1), we have the two relations

$$c^2 = \alpha c, \quad y^3 = (\alpha_2 + c)y^2 + \alpha_2 cy\,.$$

Therefore, we see that $w(Q_5) = \sum_i w_iQ_5$ is given by

$$(*) \quad \begin{aligned} &w_0Q_5 = 1,\ w_1Q_5 = \alpha + \alpha_2 + c + y,\ w_2Q_5 = \alpha c + \alpha\alpha_2 + \alpha_2 c + \alpha y + y^2, \\ &w_3Q_5 = \alpha\alpha_2 c + \alpha y^2,\ w_iQ_5 = 0 \quad (i \geqq 4)\,. \end{aligned}$$

Thus we have $w_iM_2 = 0\ (i \geqq 7)$ and $(w_6M_2)(w_5M_2) = 0$ for the 10-manifold $M_2 = Q_5^2$. Also $Q_2 = RP(2)$ by (2.7.2) and hence $M_1 = Q_2^{j-5}$ is a $(2j-10)$-manifold. Therefore,

$w_{2j-5}(M_1M_2)w_5(M_1M_2)$

$$= (\textstyle\sum_{i=0}^{1} w_{2j-10-i}M_1 \times w_{5+i}M_2)(\sum_{k=0}^{5} w_{5-k}M_1 \times w_k M_2)$$

$$= w_{2j-10}M_1 \times (w_5 M_2)^2 + (w_{2j-11}M_1)(w_1 M_1) \times (w_6 M_2)(w_4 M_2).$$

Here, since $\dim Q_5 = 5$, we have the following by (*):

$$(w_5 M_2)^2 = (w_3 \times w_2 + w_2 \times w_3)^2 = 0,$$

$$(w_6 M_2)(w_4 M_2) = (w_3 \times w_3)(w_3 \times w_1 + w_2 \times w_2 + w_1 \times w_3)$$

$$= w_3 w_2 \times w_3 w_2 = (\alpha\alpha_2 c y^2) \times (\alpha\alpha_2 c y^2) \neq 0, \quad (w_i = w_i Q_5).$$

Also, it is easy to see that

$$(w_{2j-11}M_1)(w_1 M_1) = (j-5) w_2 Q_2 \times \cdots \times w_2 Q_2 .$$

Thus, we see that $w_{2j-5}(M_1 M_2) w_5(M_1 M_2) = (j-5)\mu$ as desired.

Also, we use the following

LEMMA 2.12. *Consider the closed* 10-*manifold* $T_{10} = RP'(4, 3, 1)$ *in* (2.6). *Then* $[T_{10}]$ *is indecomposable,* $\operatorname{Span} T_{10} \geqq 7$ *and*

$$[T_{10}] = [Q_2 Q_4^2] + [Q_5^2] + [Q_4 Q_6] + [Q_{10}].$$

PROOF. The first is a consequence of Lemmas 2.3 and 2.4. The second is clear. By (2.8), we have

$$[T_{10}] = a_1[Q_2^5] + a_2[Q_2^3 Q_4] + a_3[Q_2^2 Q_6] + a_4[Q_2 Q_4^2]$$

$$+ a_5[Q_5^2] + a_6[Q_4 Q_6] + a_7[Q_2 Q_8] + a_8[Q_{10}],$$

for some $a_i (=0$ or $1)$. $\operatorname{Span} T_{10} \geqq 7$ implies $w_i T_{10} = 0$ for $i \geqq 4$. Therefore we see that $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 = a_6$ by (2.11.1). Also we see that $a_8 = 1$ since $[T_{10}]$ is indecomposable. Thus

$$[T_{10}] = a_4[Q_2 Q_4^2] + a_4[Q_5^2] + a_4[Q_4 Q_6] + a_7[Q_2 Q_8] + [Q_{10}].$$

By the similar calculations to show (2.11.1–2), we have

$$w_1^{10}(T_{10}) = 0, \quad w_1^{10}(Q_2 Q_4^2) = 0, \quad w_1^{10}(Q_5^2) = 0,$$

$$w_1^{10}(Q_4 Q_6) = 0, \quad w_1^{10}(Q_2 Q_8) \neq 0, \quad w_1^{10}(Q_{10}) = 0.$$

These imply that $a_7 = 0$. Also, we have

$$w_2^4(T_{10}) w_1^2(T_{10}) = 0, \quad w_2^4(Q_2 Q_4^2) w_1^2(Q_2 Q_4^2) \neq 0,$$

$$w_2^4(Q_5^2) w_1^2(Q_5^2) = 0, \quad w_2^4(Q_4 Q_6) w_1^2(Q_4 Q_6) = 0, \quad w_2^4(Q_{10}) w_1^2(Q_{10}) \neq 0.$$

These imply that $a_4 = 1$. Thus we have the lemma. *q.e.d.*

§3. Proof of the theorem in §1

We prove the theorem for $k=6$, and the proof for the case $k \leqq 5$ is similar. By (2.8) and Lemma 2.10, $[M^{2j+1}]$ can be expressed as

$$[M^{2j+1}] = a[Q_2^{j-2}Q_5] + b[Q_2^{j-4}Q_4Q_5] + c[Q_2^{j-5}Q_5Q_6] + [N^{2j+1}],$$

where a, b, c are 0 or 1, and N^{2j+1} is a sum of products of some manifolds in (2.7.1–2) such that $\operatorname{Span} N^{2j+1} \geqq 6$. By (2.11.2) and the assumption of the theorem, we have $a=b=c=0$.

Similarly, by (2.8), Lemma 2.10 and (2.11.1), we see that

$$[M^{2j}] = a([Q_2^{j-4}Q_4^2] + [Q_2^{j-5}Q_5^2] + [Q_2^{j-5}Q_4Q_6]) + [N^{2j}],$$

where a is 0 or 1, and $\operatorname{Span} N^{2j} \geqq 6$. Therefore, by Lemma 2.12,

$$[M^{2j}] = a([Q_2^{j-5}T_{10}] + [Q_2^{j-5}Q_{10}]) + [N^{2j}],$$

where $\operatorname{Span} T_{10} \geqq 7$ and $\operatorname{Span} Q_{10} \geqq 7$. *q.e.d.*

References

[1] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Erg. d. Math. **33**, Springer-Verlag, 1964.

[2] J. Milnor and J. Stasheff, *Characteristic Classes*, Ann. of Math. Studies **76**, Princeton Univ. Press, 1974.

[3] R. E. Stong, *On fibering of cobordism classes*, Trans. Amer. Math. Soc. **178** (1973), 431–447.

[4] ———, *Notes on Cobordism Theory*, Math. Notes, Princeton Univ. Press, 1968.

The Faculty of Integrated Arts and Sciences,
Hiroshima University