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# *A Note on the Polynomial Grade and the Valuative Dimension*

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It is well known that the grade of a finite module over a noetherian local ring is equal to or less than the Krull dimension of the module. The main purpose of this note is to generalize this inequality to the case where the ring is not necessarily noetherian.

The notion of *polynomial grades*, which generalizes the classical one of grades, was introduced by Northcott [9]. On the other hand, in order to develop the Krull dimension theory in polynomial rings, the concept of *valuative dimensions* was initiated by Jafford [7]. We shall first define *the polynomial height* of a prime ideal and study some basic properties. We shall also show a relation between the polynomial grade and the valuative dimension of a finite module. We can see that the valuative dimension of a ring $A$ coincides with the supremum of the polynomial heights of prime ideals in $A$ (Theorem 1). The main result is Theorem 2: The polynomial grade of a finite module is equal to or less than the valuative dimension of the module.

Throughout this paper all rings are assumed to be commutative with identity and all modules are assumed to be unitary. If $A$ is such a ring and $m$ is a positive integer, then $A^{(m)}$ will stand for the polynomial ring $A[X_1,\dots, X_m]$ in $m$ indeterminates over $A$, and $A^{(0)}$ will do for $A$. If $\mathfrak{a}$ is an ideal of $A$, then $\mathfrak{a}^{(m)}$ will denote the ideal $\mathfrak{a}[X_1,\dots, X_m]$ of $A^{(m)}$ and $\mathfrak{a}^{(0)}$ the ideal $\mathfrak{a}$. Moreover if $M$ is an $A$-module, then $M^{(m)}$ will denote the $A^{(m)}$-module $M\otimes_A A^{(m)}$ and $M^{(0)}$ will be understood similarly. We shall also denote by $V(\mathfrak{a})$ the set of prime ideals of $A$ which contain $\mathfrak{a}$ and denote by $\mathrm{Ann}(M)$ the set of annihilators of $M$. We shall write $\mathrm{ht}(\mathfrak{p})$ for the height of the prime ideal $\mathfrak{p}$ of $A$.

Since the sequence $\{\mathrm{ht}(\mathfrak{p}^{(m)})\}$ $(m=0, 1,\dots)$ is an increasing one, we can consider its limit.

Definition. Let $\mathfrak{p}$ be a prime ideal of a ring $A$. We shall denote by $\mathrm{Ht}(\mathfrak{p})$ the limit of the sequence $\{\mathrm{ht}(\mathfrak{p}^{(m)})\}$ $(m=0, 1,\dots)$ and call it *the polynomial height of* $\mathfrak{p}$.

First we shall give a number of elementary properties of the polynomial height of $\mathfrak{p}$.

PROPOSITION 1. *For a prime ideal* $\mathfrak{p}$ *of a ring* $A$, *we have the following statements*:

(1) $\mathrm{ht}(\mathfrak{p}) \leqq \mathrm{Ht}(\mathfrak{p})$.

(2) *If* $m$ *is a non-negative integer, then* $\mathrm{Ht}(\mathfrak{p}) = \mathrm{Ht}(\mathfrak{p}^{(m)})$.

(3) *If* $\mathfrak{q}$ *is a prime ideal of* $A$ *such that* $\mathfrak{q} \subseteq \mathfrak{p}$, *then* $\mathrm{Ht}(\mathfrak{q}) \leqq \mathrm{Ht}(\mathfrak{p})$.

(4) *If* $\mathfrak{a}$ *is an ideal of* $A$ *such that* $\mathfrak{a} \subseteq \mathfrak{p}$, *then* $\mathrm{Ht}(\mathfrak{p}) \geqq \mathrm{Ht}(\mathfrak{p}/\mathfrak{a})$.

(5) *Let* $S$ *be a multiplicative subset of* $A$. *If* $\mathfrak{p} \cap S = \phi$, *then* $\mathrm{Ht}(\mathfrak{p}) = \mathrm{Ht}(\mathfrak{p}A_S)$.

(6) *If* $m$ *is a non-negative integer, then* $\mathrm{Ht}(\mathfrak{p}) = \mathrm{Ht}(\mathfrak{p}^{(m)}A^{(m)}_{\mathfrak{p}^{(m)}})$.

(7) *If* $A$ *is a noetherian ring, then* $\mathrm{ht}(\mathfrak{p}) = \mathrm{Ht}(\mathfrak{p})$.

(8) $\mathrm{ht}(\mathfrak{p}) = 0$ *if and only if* $\mathrm{Ht}(\mathfrak{p}) = 0$.

(9) *Let* $f$ *be an element of* $\mathfrak{p}$. *If* $f$ *is not contained in any minimal prime ideal of* $A$, *then* $\mathrm{Ht}(\mathfrak{p}/(f)) \leqq \mathrm{Ht}(\mathfrak{p}) - 1$.

PROOF. The assertions (1), (2), (3) and (4) are immediate consequences of the definition and (5) follows from the fact that $\mathrm{ht}(\mathfrak{p}^{(n)}) = \mathrm{ht}((\mathfrak{p}A_S)^{(n)})$ for a non-negative integer $n$. The assertion (6) follows from (2) and (5). Now the assertion (7) is well known (cf. (14.A) of [8]). Next, if $\mathfrak{p}$ is a minimal prime ideal of $A$, then $\mathfrak{p}^{(n)}$ is a minimal prime ideal of $A^{(n)}$ and conversely if $\mathfrak{P}$ is a minimal prime ideal of $A^{(n)}$, then $\mathfrak{P} = \mathfrak{p}^{(n)}$ where $\mathfrak{p}$ is a minimal prime ideal of $A$ (cf. (30.3) of [4]). The assertion (8) follows from these facts. We also see that if an element $f$ of $\mathfrak{p}$ is not contained in any minimal prime ideal of $A$, then $f$ is not contained in any minimal prime ideal of $A^{(n)}$. Hence $\mathrm{ht}((\mathfrak{p}/(f))^{(n)}) = \mathrm{ht}(\mathfrak{p}^{(n)}/fA^{(n)}) \leqq \mathrm{ht}(\mathfrak{p}^{(n)}) - 1$, which implies $\mathrm{Ht}(\mathfrak{p}/(f)) \leqq \mathrm{Ht}(\mathfrak{p}) - 1$. This proves (9).

*q.e.d.*

If $A$ is an integral domain, then *the valuative dimension* of $A$, denoted by $\dim_v A$, is defined to be $\mathrm{Sup}\{\dim V \mid V$ is a valuation overring of $A\}$, and generally the valuative dimension of a ring $A$ is defined to be $\mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\dim_v(A/\mathfrak{p})\}$ (see [7]). In order to show the following Proposition 2 and Theorem 1, we need some of the previous results on the height and the valuative dimension that can be summarized as follows:

(a) [3, Theorem 1] If $\mathfrak{P}$ is a prime ideal of $A^{(n)}$ with $\mathfrak{P} \cap A = \mathfrak{p}$, then $\mathrm{ht}(\mathfrak{P}) = \mathrm{ht}(\mathfrak{p}^{(n)}) + \mathrm{ht}(\mathfrak{P}/\mathfrak{p}^{(n)}) \leqq \mathrm{ht}(\mathfrak{p}^{(n)}) + n$.

(b) [2, Cor. 2.10] If $A$ is a finite dimensional ring and if $\{m_\alpha\}_{\alpha \in \Gamma}$ is the set of maximal ideals of $A$, then $\dim A^{(n)} = \mathrm{Sup}_{\alpha \in \Gamma}\{\dim A_{\mathfrak{m}_\alpha}^{(n)}\} = n + \mathrm{Sup}_{\alpha \in \Gamma}\{\mathrm{ht}(m_\alpha^{(n)})\}$.

(c) [1, Theorem 6] If $A$ is an integral domain, then $\dim_v A = n$ if and only if $\dim A^{(n)} = 2n$.

(d) [1, Theorem 6] If $A$ is an integral domain and if $\dim_v A = n$, then $\dim A^{(k)} = k + n$ for $k \geqq n - 1$.

LEMMA 1. *Let* $\mathfrak{p}$ *be a prime ideal of* $A$ *and* $m$ *be a non-negative integer.*

*If* $\mathrm{ht}(\mathfrak{p}^{(m)}) \leqq m$, *then there exists an integer* $k$ *such that* $\mathrm{ht}(\mathfrak{p}^{(k)}) = k$ *and* $0 \leqq k \leqq m$.

PROOF. We use induction on $m$. If $m=0$, then the assertion is clear. Therefore we may suppose that $m \geqq 1$ and assumed that the assertion holds for $m-1$. Suppose that $\mathrm{ht}(\mathfrak{p}^{(m)}) \leqq m$. The case $\mathrm{ht}(\mathfrak{p}^{(m)}) = m$ is trivial. Thus we may assume $\mathrm{ht}(\mathfrak{p}^{(m)}) \leqq m-1$. Since $\mathrm{ht}(\mathfrak{p}^{(m-)}) \leqq \mathrm{ht}(\mathfrak{p}^{(m)})$, we have $\mathrm{ht}(\mathfrak{p}^{(m-1)}) \leqq m-1$. By the inductive hypothesis we can prove our lemma. *q.e.d.*

PROPOSITION 2. *Let* $\mathfrak{p}$ *be a prime ideal of a ring* $A$ *and* $n$ *be a non-negative integer. Then* $\mathrm{Ht}(\mathfrak{p}) = n$ *if and only if* $\mathrm{ht}(\mathfrak{p}^{(n)}) = n$.

PROOF. We shall proceed by induction on $n$. We see that when $n=0$ the assertion follows from (8) of Proposition 1. Therefore it will be suppose that $n \geqq 1$ and that the assertion has been proved for all smaller values of the inductive variable. First we assume that $\mathrm{Ht}(\mathfrak{p}) = n$. Then it follows from our definition that there exists an integer $m$ such that $\mathrm{ht}(\mathfrak{p}^{(m)}) = n$ and $m > n$. Since the sequence $\{\mathrm{ht}(\mathfrak{p}^{(i)})\}$ $(i=0, 1, \ldots)$ is increasing, we see that $\mathrm{ht}(\mathfrak{p}^{(n)}) \leqq n$. If $\mathrm{ht}(\mathfrak{p}^{(n)}) \leqq n-1$, then it is seen that $\mathrm{ht}(\mathfrak{p}^{(n-1)}) \leqq \mathrm{ht}(\mathfrak{p}^{(n)}) \leqq n-1$, thus $\mathrm{ht}(\mathfrak{p}^{(n-1)}) \leqq n-1$. By Lemma 1, there exists an integer $k$ such that $0 \leqq k \leqq n-1$ and $\mathrm{ht}(\mathfrak{p}^{(k)}) = k$. Using the inductive hypothesis, we have $\mathrm{Ht}(\mathfrak{p}) = k$. This gives a contradiction. Thus we have $\mathrm{ht}(\mathfrak{p}^{(n)}) = n$. Next we assume that $\mathrm{ht}(\mathfrak{p}^{(n)}) = n$ and we shall show that $\mathrm{Ht}(\mathfrak{p}) = n$. It will suffice to prove that $\mathrm{ht}(\mathfrak{p}^{(m)}) = n$ for all large integers. Assume the contrary and we shall get a contradiction. There exists an integer $m$ such that $m > n$ and $\mathrm{ht}(\mathfrak{p}^{(m)}) > n$. Hence we have a sequence

$$\mathfrak{P}_0 \subsetneqq \mathfrak{P}_1 \subsetneqq \cdots \subsetneqq \mathfrak{P}_s$$

of $s+1$ prime ideals of $A^{(m)}$ with $s > n$ and $\mathfrak{P}_s = \mathfrak{p}^{(m)}$. Replacing $\mathfrak{P}_0$ by $(\mathfrak{P}_0 \cap A)^{(m)}$, if necessary, we may suppose that $\mathfrak{P}_0 = \mathfrak{p}_0{}^{(m)}$ where $\mathfrak{p}_0$ is a prime ideal of $A$. Put $\bar{A} = A/\mathfrak{p}_0$ and $\bar{\mathfrak{p}} = \mathfrak{p}/\mathfrak{p}_0$. Since $\bar{\mathfrak{p}}^{(m)}$ is isomorphic to $\mathfrak{p}^{(m)}/\mathfrak{p}_0{}^{(m)}$, it follows that $\mathrm{ht}(\bar{\mathfrak{p}}^{(m)}) > n$. On the other hand the prime ideal $\bar{\mathfrak{p}}^{(n)}$ is a homomorphic image of the prime ideal $\mathfrak{p}^{(n)}$, therefore $\mathrm{ht}(\bar{\mathfrak{p}}^{(n)}) \leqq \mathrm{ht}(\mathfrak{p}^{(n)})$. Accordingly $\mathrm{ht}(\bar{\mathfrak{p}}^{(n)}) \leqq n$. If $\mathrm{ht}(\bar{\mathfrak{p}}^{(n)}) \leqq n-1$, we have $\mathrm{ht}(\bar{\mathfrak{p}}^{(n-1)}) \leqq n-1$. Then, by Lemma 1, we can find an integer $k$ such that $\mathrm{ht}(\bar{\mathfrak{p}}^{(k)}) = k$ and $0 \leqq k \leqq n-1$. We can thus conclude that $\mathrm{Ht}(\bar{\mathfrak{p}}) = k$ by the inductive hypothesis. Since $\mathrm{ht}(\bar{\mathfrak{p}}^{(m)}) \leqq \mathrm{Ht}(\bar{\mathfrak{p}})$, this gives a contradiction. Hence we see that $\mathrm{ht}(\bar{\mathfrak{p}}^{(n)}) = n$. By considering the prime ideal $\bar{\mathfrak{p}}$ of the ring $\bar{A}$, we may assume that the ring $A$ is an integral domain. Put $S = A - p$. Then $(A^{(i)})_S$ is isomorphic to $A_{\mathfrak{p}}^{(i)}$ for a non-negative integer $i$. Since the height is not changed by any localization, we see that $\mathrm{ht}(\mathfrak{p}A_{\mathfrak{p}}^{(m)}) > n$ and $\mathrm{ht}(\mathfrak{p}A_{\mathfrak{p}}^{(n)}) = n$. Thus we may assume that $A$ is a quasi local domain with the maximal ideal $\mathfrak{p}$ and that $\mathrm{ht}(\mathfrak{p}^{(m)}) > n$ and $\mathrm{ht}(\mathfrak{p}^{(n)}) = n$. By (b) of the previous results, $\dim A^{(n)} = 2n$, and hence (c) shows that

$\dim_v A = n$. Therefore it follows from (d) that $\dim A^{(m)} = m+n$. Put $\mathfrak{P} = \mathfrak{p}^{(m)} + (X_1, \dots, X_m)$. Then $\mathfrak{P}$ is a prime ideal of $A^{(m)}$. Using (a), $\mathrm{ht}(\mathfrak{P}) = \mathrm{ht}(\mathfrak{p}^{(m)}) + \mathrm{ht}(\mathfrak{P}/\mathfrak{p}^{(m)}) = \mathrm{ht}(\mathfrak{p}^{(m)}) + m > n+m$, which leads to a contradiction. *q.e.d.*

LEMMA 2. *Let* $\mathfrak{p}$ *be a prime ideal of a ring* $A$ *with* $\mathrm{Ht}(\mathfrak{p}) = n$. *Then there exists a prime ideal* $\mathfrak{q}$ *such that* $\mathfrak{q} \subseteq \mathfrak{p}$ *and* $\mathrm{Ht}(\mathfrak{p}/\mathfrak{q}) = n$.

PROOF. Assume that $\mathrm{Ht}(\mathfrak{p}) = n$. By Proposition 2, we see $\mathrm{ht}(\mathfrak{p}^{(n)}) = n$. Then we can find a prime ideal $\mathfrak{q}$ of $A$ such that $\mathrm{ht}(\mathfrak{p}^{(n)}/\mathfrak{q}^{(n)}) = n$. Accordingly it follows from Proposition 2 that $\mathrm{Ht}(\mathfrak{p}/\mathfrak{q}) = n$. *q.e.d.*

THEOREM 1. *Let* $A$ *be a ring with* $\dim_v A < \infty$. *Then we have* $\dim_v A = \mathrm{Sup}\{\mathrm{Ht}(\mathfrak{p})\}$, *where* $\mathfrak{p}$ *runs over all prime ideals of* $A$.

PROOF. First we suppose that $A$ is a quasi local domain with the maximal ideal $\mathfrak{m}$. Then, by (3) of Proposition 1, $\mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\mathrm{Ht}(\mathfrak{p})\} = \mathrm{Ht}(\mathfrak{m})$ and hence it follows from Proposition 2 that $\mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\mathrm{Ht}(\mathfrak{p})\} = n$ if and only if $\mathrm{ht}(\mathfrak{m}^{(n)}) = n$. By (b) of previous results, this means that $\dim A^{(n)} = 2n$. Thus the assertion holds in this case by virtue of (c). Next we assume that $A$ is an integral domain. We can readily see that $\dim_v A = \mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\dim_v A_{\mathfrak{p}}\}$. Accordingly we have the following equality from (5) of Proposition 1 and the first case:

$$\begin{aligned} \dim_v A &= \mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\dim_v A_{\mathfrak{p}}\} = \mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\mathrm{Ht}(\mathfrak{p}A_{\mathfrak{p}})\} \\ &= \mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\mathrm{Ht}(\mathfrak{p})\}. \end{aligned}$$

This settles the case where $A$ is an integral domain.

Finally we proceed to the general case. The above argument, combining (4) of Proposition 1 and Lemma 2 with the definition of the valuative dimension, shows that

$$\begin{aligned} \dim_v A &= \mathrm{Sup}_{\mathfrak{q} \in \mathrm{Spec}(A)}\{\dim_v A/\mathfrak{q}\} \\ &= \mathrm{Sup}_{\mathfrak{q} \in \mathrm{Spec}(A)}\{\mathrm{Sup}_{\mathfrak{p}/\mathfrak{q} \in \mathrm{Spec}(A/\mathfrak{q})}\{\mathrm{Ht}(\mathfrak{p}/\mathfrak{q})\}\} \\ &= \mathrm{Sup}_{\mathfrak{p} \in \mathrm{Spec}(A)}\{\mathrm{Ht}(\mathfrak{p})\}. \end{aligned}$$

This completes the proof.

Let $\mathfrak{a}$ be an ideal of $A$ and $M$ an $A$-module. Then the upper bound of lengths of all $M$-sequences in $\mathfrak{a}$ will be called the classical grade of $\mathfrak{a}$ on $M$ and denoted by $gr_A\{\mathfrak{a}; M\}$. *The polynomial grade* of $\mathfrak{a}$ in $M$, denoted by $\mathrm{Gr}_A\{\mathfrak{a}; M\}$, is defined to be $\lim_{n \to \infty} \mathrm{gr}_{A^{(n)}}\{\mathfrak{a}^{(n)}; M^{(n)}\}$ (see [9]).

DEFINITION. Let $A$ be a ring and $M$ a non-zero $A$-module. Then *the valuative dimension of* $M$ is meant the valuative dimension of $A/\mathrm{Ann}(M)$ and it is denoted by $\dim_v M$.

We see that, by (7) of Proposition 1 and Theorem 1, if $A$ is a noetherian ring, then $\dim_v M = \dim M$.

LEMMA 3. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are proper ideals of a ring A such that* $V(\mathfrak{a}) = V(\mathfrak{b})$, *then* $\dim_v A/\mathfrak{a} = \dim_v A/\mathfrak{b}$.

PROOF. The assertion follows easily from the definition of valuative dimension. *q.e.d.*

Finally we shall show the main result of this paper, which suggests that the valuative dimension $\dim_v M$ of a module $M$ could be called *the polynomial dimension* and denoted by $\operatorname{Dim} M$.

THEOREM 2. *Let A be a quasi local ring with the maximal ideal* $\mathfrak{m}$ *and M a non-zero finite A-module. Then* $\mathrm{Gr}_A\{\mathfrak{m}; M\} \leqq \dim_v M$.

PROOF. If $\dim_v M$ is infinite, then there is nothing to prove. Therefore we may assume that $\dim_v M$ is finite. We use induction on $\dim_v M$. Put $\mathfrak{a} = \mathrm{Ann}(M)$, $\bar{A} = A/\mathfrak{a}$ and $\bar{\mathfrak{m}} = \mathfrak{m}/\mathfrak{a}$. Now suppose that $\dim_v M = 0$. Then $\dim_v \bar{A} = 0$. Hence, by Theorem 1, $\mathrm{Ht}(\bar{\mathfrak{m}}) = 0$, therefore $\mathrm{ht}(\bar{\mathfrak{m}}) = 0$. Thus the ideal $\mathfrak{m}$ is a minimal prime ideal over $\mathfrak{a}$. It follows from Exercise 4 of Chapter 6 in [9] that $\mathrm{Gr}_A\{\mathfrak{m}: M\} = 0$. Accordingly the assertion follows in this case. Next we assume that $\dim_v M \geqq 1$ and the inequality has been established for all non-negative integers smaller than $\dim_v M$. If $\mathrm{Gr}_A\{\mathfrak{m}; M\} = 0$, the assertion is clear. Thus we may suppose that $\mathrm{Gr}_A\{\mathfrak{m}; M\} \geqq 1$. By Theorem 8 of Chapter 5 in [9], there exists an $M^{(1)}$-regular element $f$ in $m^{(1)}$. Hence we obtain the following exact sequence of $A^{(1)}$-modules

$$(*) \qquad 0 \longrightarrow M^{(1)} \xrightarrow{f} M^{(1)} \longrightarrow M_1 \longrightarrow 0$$

where the map $f$ is defined by $f(x) = fx$. We shall calculate the $\dim_v (M_1)_{\mathfrak{m}^{(1)}}$. Since $M^{(1)}$ and $M_1$ are finite $A^{(1)}$-modules and $\mathrm{Ann}(M^{(1)}) = \mathfrak{a}^{(1)}$, we have

$$\begin{aligned} V(\mathrm{Ann}(M_1)) &= \mathrm{Supp}(M_1) = \mathrm{Supp}(M^{(1)}/fM^{(1)}) \\ &= \mathrm{Supp}(M^{(1)}) \cap \mathrm{Supp}(A^{(1)}/(f)) \\ &= V(\mathfrak{a}^{(1)}) \cap V(f) = V(\mathfrak{a}^{(1)} + (f)). \end{aligned}$$

Thus $V((\mathrm{Ann}(M_1))_{\mathfrak{m}^{(1)}}) = V((\mathfrak{a}^{(1)} + (f))_{\mathfrak{m}^{(1)}})$. Therefore it follows from Lemma 3 that $\dim_v (A^{(1)}_{\mathfrak{m}^{(1)}}/(\mathrm{Ann}(M_1))_{\mathfrak{m}^{(1)}}) = \dim_v (A^{(1)}_{\mathfrak{m}^{(1)}}/(\mathfrak{a}^{(1)} + (f))_{\mathfrak{m}^{(1)}})$. Accordingly, since $\mathrm{Ann}((M_1)_{\mathfrak{m}^{(1)}}) = (\mathrm{Ann}(M_1))_{\mathfrak{m}^{(1)}}$, we see that

$$\begin{aligned}\dim_v (M_1)_{\mathfrak{m}^{(1)}} &= \dim_v (A^{(1)}_{\mathfrak{m}^{(1)}}/\mathrm{Ann}\,((M_1)_{\mathfrak{m}^{(1)}}))\\ &= \dim_v (A^{(1)}_{\mathfrak{m}^{(1)}}/(\mathrm{Ann}\,(M_1))_{\mathfrak{m}^{(1)}})\\ &= \dim_v (A^{(1)}_{\mathfrak{m}^{(1)}}/(\mathfrak{a}^{(1)} + (f))_{\mathfrak{m}^{(1)}})\\ &= \dim_v (\bar{A}^{(1)}/(\bar{f}))_{\bar{\mathfrak{m}}^{(1)}},\end{aligned}$$

where $\bar{f}$ is the homomorphic image of $f$ in $\bar{A}^{(1)}$. On the other hand, since $f$ is an $M^{(1)}$-regular element in $\mathfrak{m}^{(1)}$, we see from Exercise 4 of Chapter 6 in [9] that $f$ is not contained in any minimal prime ideal of $\mathfrak{a}^{(1)}$, and hence $\bar{f}/1$ is also not contained in any minimal prime ideal of $\bar{A}^{(1)}_{\bar{\mathfrak{m}}^{(1)}}$. By (6) and (9) of Proposition 1 and Theorem 1,

$$\begin{aligned}\dim_v (\bar{A}^{(1)}/(\bar{f}))_{\bar{\mathfrak{m}}^{(1)}} &= \mathrm{Ht}\,(\bar{\mathfrak{m}}^{(1)}\bar{A}^{(1)}_{\bar{\mathfrak{m}}^{(1)}}/(\bar{f}\bar{A}^{(1)})_{\bar{\mathfrak{m}}^{(1)}})\\ &\leqq \mathrm{Ht}\,(\bar{\mathfrak{m}}^{(1)}(\bar{A}^{(1)})_{\bar{\mathfrak{m}}^{(1)}}) - 1\\ &= \mathrm{Ht}\,(\bar{\mathfrak{m}}) - 1 = \dim_v \bar{A} - 1\\ &= \dim_v M - 1.\end{aligned}$$

Consequently we can conclude that $\dim_v (M_1)_{\mathfrak{m}^{(1)}} \leqq \dim_v M - 1$. Thus it follows from the inductive hypothesis applied to the $A^{(1)}_{\mathfrak{m}^{(1)}}$-module $(M_1)_{\mathfrak{m}^{(1)}}$ that $\mathrm{Gr}_{A^{(1)}_{\mathfrak{m}^{(1)}}}\{\mathfrak{m}^{(1)}A^{(1)}_{\mathfrak{m}^{(1)}};\ (M_1)_{\mathfrak{m}^{(1)}}\} \leqq \dim_v (M_1)_{\mathfrak{m}^{(1)}}$. By localizing the exact sequence $(*)$ at $\mathfrak{m}^{(1)}$, we have an exact sequence

$$0 \longrightarrow (M^{(1)})_{\mathfrak{m}^{(1)}} \xrightarrow{f} (M^{(1)})_{\mathfrak{m}^{(1)}} \longrightarrow (M_1)_{\mathfrak{m}^{(1)}} \longrightarrow 0.$$

Accordingly, by Theorem 15 of Chapter 5 in [9] and the fact that the polynomial grade does not change by any faithfully flat extension (cf. [4], Cor. 1 to Prop. 2, § 1 and [6], § 3), we have

$$\begin{aligned}&\mathrm{Gr}_{A^{(1)}_{\mathfrak{m}^{(1)}}}\{\mathfrak{m}^{(1)}A^{(1)}_{\mathfrak{m}^{(1)}};\ (M_1)_{\mathfrak{m}^{(1)}}\}\\ &= \mathrm{Gr}_{A^{(1)}_{\mathfrak{m}^{(1)}}}\{\mathfrak{m}^{(1)}A^{(1)}_{\mathfrak{m}^{(1)}};\ (M^{(1)})_{\mathfrak{m}^{(1)}}\} - 1\\ &= \mathrm{Gr}_A\{\mathfrak{m};\ M\} - 1.\end{aligned}$$

Substituting this for the above inequality, we see

$$\mathrm{Gr}_A\{\mathfrak{m};\ M\} - 1 \leqq \dim_v (M_1)_{\mathfrak{m}^{(1)}} \leqq \dim_v M - 1.$$

Therefore $\mathrm{Gr}_A\{\mathfrak{m};\ M\} \leqq \dim_v M$. This completes the proof.

We can derive the following well known

COROLLARY. *Let $A$ be a noetherian local ring with the maximal ideal* $\mathfrak{m}$. *Let $M$ be a non-zero finite $A$-module. Then depth* $M \leqq \dim M$.

PROOF. Since $A$ is a noetherian ring, we have depth $M = gr_A\{\mathfrak{m}; M\} = \mathrm{Gr}_A\{\mathfrak{m}; M\}$ and $\dim M = \dim_v M$. Thus the proof of our corollary follows from the theorem.

## References

[1] J. Arnold, On the dimension theory of overring of an integral domain, Trans. Amer. Math. Soc. **138** (1969), 313–326.

[2] J. Arnold and R. Gilmer, The dimension sequence of a commutative ring, Amer. J. Math. **96** (1974), 385–408.

[3] J. Brewer, P. Montgomery, E. Rutter and W. Heinzer, Krull dimension of polynomial rings, Conference on commutative algebra proceedings, Lecture Notes in Math. **311**, Springer-Verlag, (1972), 26–45.

[4] R. Gilmer, Multiplicative ideal theory, Marcel-Dekker, New York, 1972.

[5] M. Hochster, Grade-sensitive modules and perfect modules, Proc. London Math. Soc. (3) **29** (1974), 55–76.

[6] S. Itoh, Axiomatic characterizations of grade for commutative rings, to appear in Hiroshima Math. J.

[7] P. Jafford, Theorie de la dimension dans les anneaux de polynomes, Gauthier-Villars, Paris, 1960.

[8] H. Matumura, Commutative algebra, Benjamin Inc., New York, 1970.

[9] D. G. Northcott, Finite free resolutions, Cambridge Univ. Press, 1976.

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