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Some Boundary Value Problems for the Hamilton-Jacobi Equation

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1. Introduction

This paper is devoted to the study of some initial value-boundary value problems for the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + F\left(\frac{\partial u}{\partial x}\right) = 0 \qquad (t \geq 0,\ x \geq 0). \tag{1}$$

It will be shown that (1) is governed by contraction semigroups on certain closed subsets of the space of bounded uniformly continuous functions on the half-line $\mathbf{R}^+ = [0, \infty)$ via the Crandall-Liggett generation theorem [10] for nonlinear semigroups.

Preparatory to these results, results concerning existence of periodic solutions, positive solutions, and even solutions for the Cauchy problem for (1) with $x \in \mathbf{R} = (-\infty, \infty)$ will be obtained.

The Cauchy problem for the Hamilton-Jacobi equation has been studied from a semigroup point by Aizawa [2, 3], Burch [7], and Tamburro [15, 16]. The results and techniques in [7] are refined and further developed in this paper to gain information about some boundary value problems for (1). Results concerning these problems complement the recent work of Feltus [11]. Feltus studied existence and uniqueness for (1) with Dirichlet boundary condition at the origin, but he didn't establish continuous dependence results. Some earlier results on boundary value problems for (1) were obtained by Aizawa and Kikuchi [1, 4] and Benton [5, 6].

Before stating the main result we introduce some notation. J denotes either $\mathbf{R} = (-\infty, \infty)$ or $\mathbf{R}^+ = [0, \infty)$. For $1 \leq p \leq \infty$, $L^p(J)$ denotes the usual real Lebesgue space with norm $\|\cdot\|_p$. $W^{n,p}(J)$ denotes the Sobolev space of all $f \in L^p(J)$ such that the jth derivative of f belongs to $L^p(J)$ for $j \leq n$. BUC(J) denotes the bounded uniformly continuous real functions on J. For $Y(\mathbf{R})$ any space of functions on $\mathbf{R}$ we *define*

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$$Y_e(\mathbf{R}) = \{u \in Y(\mathbf{R})\colon u(x) = u(-x) \quad \text{for all } x \in \mathbf{R}\},$$

$$Y_\pi(\mathbf{R}) = \{u \in Y_e(\mathbf{R})\colon u(0) = 0,\ u(x) \geq 0 \quad \text{for all } x \in \mathbf{R}\},$$

$$Y_p(\mathbf{R}) = \{u \in Y(\mathbf{R})\colon u(x + p) = u(x) \quad \text{for all } x \in \mathbf{R}\},$$

where p is some nonzero real number. Thus the subscript e stands for "even", while the subscript π stands for "positive etc." and the subscript p stands for "periodic with period p".

For $k>0$ we *define*

$$B(k) = \{u \in \mathrm{BUC}(\mathbf{R})\colon \|u\|_\infty \leq k; \quad \text{for all} \quad x, y \in \mathbf{R},$$
$$|u(x + y) - u(x)| \leq k|y| \quad \text{and} \quad u(x + y) - 2u(x) + u(x - y) \leq ky^2\}.$$

The main result of this paper is the following theorem.

THEOREM I. *Let $F \in C^2(\mathbf{R})$, $F(0)=0$, $F''(x) \geq 0$ and $F(-x)=F(x)$ for all $x \in \mathbf{R}$. Consider*

$$\frac{\partial u}{\partial t} + F\left(\frac{\partial u}{\partial x}\right) = 0 \qquad a.e. \text{ for } \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}^+, \tag{1}$$

$$u(0, x) = u_0(x) \qquad \text{for} \quad x \in \mathbf{R}^+, \tag{2}$$

$$\frac{\partial u}{\partial x}(t, 0) = 0 \qquad \text{for} \quad t \in \mathbf{R}^+, \tag{3}$$

$$u(t, 0) = 0 \qquad \text{for} \quad t \in \mathbf{R}^+, \tag{4}$$

$$u(t, x) \geq 0 \qquad \text{for} \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}^+. \tag{5}$$

Then:

(i) (1)–(3) *is governed by a contraction semigroup on* $\mathrm{BUC}(\mathbf{R}^+)$.

(ii) (1)–(5) *is governed by a contraction semigroup on* $\mathrm{BUC}_\pi(\mathbf{R}^+)$.

Preliminary to Theorem I we establish the following result concerning the Cauchy problem.

PROPOSITION I. *Let $F \in C^2(\mathbf{R})$, $F(0)=0$, and $F''(x) \geq 0$ for all $x \in \mathbf{R}$.*

(a) (1)–(2) *is governed by a contraction semigroup on* $\mathrm{BUC}_p(\mathbf{R})$, *for each* $p \in \mathbf{R}$.

(b) *If also $F(x)=F(-x)$ for all $x \in \mathbf{R}$, then*:

(i) (1)–(2) *is governed by a contraction semigroup on* $\mathrm{BUC}_e(\mathbf{R})$.

(ii) (1)–(2) *is governed by a contraction semigroup on* $\mathrm{BUC}_\pi(\mathbf{R})$.

It is advantageous to solve mixed problems by the theory of nonlinear semigroups. For instance, semigroup approximation theory shows that the solution u of (1)–(3) (or (1)–(5)) depends continuously on both u_0 and F. Perturbation theory allows one to add a term of the form $G(u)$ to the right-hand side of (1). For propaganda advocating semigroup methods, see for instance Crandall [9] or Goldstein [12].

The proof of Theorem I proceeds by considering the pure initial value problem for (1), (2) in $(t, x) \in \mathbf{R}^+ \times \mathbf{R}$, and then showing that the associated semigroup on BUC(**R**) leaves certain closed subsets invariant. These closed subsets will have the boundary conditions (3) (and (3)–(5)) built into their definitions.

The relevant results from Burch [7] concerning this Cauchy problem are collected in Section 2. In Section 3 the generator of the semigroup is studied and its resolvent is shown to leave certain subsets invariant. This necessitates a study of the equations

$$u + \lambda F\left(\frac{\partial u}{\partial x}\right) = h, \qquad u + \lambda F\left(\frac{\partial u}{\partial x}\right) - \varepsilon \frac{\partial^2 u}{\partial x^2} = h.$$

In Section 4 the main theorem will be proved. We show that (1)–(3) is governed by a contraction semigroup on $B_e(k) = B(k) \cap \mathrm{BUC}_e(\mathbf{R})$, and (1)–(5) is governed by a semigroup on $B_\pi(k) = B(k) \cap \mathrm{BUC}_\pi(\mathbf{R})$. For the solution of (1)–(3), condition (3) is satisfied in a generalized sense, whereas for (1)–(5), conditions (3)–(5) are satisfied in the classical sense. Section 5 contains a discussion of related problems.

After our research was completed we learned of the interesting paper of Tomita [17]. Tomita treated the Dirichlet problem for (1) on the interval $0 \leq x \leq 1$ using Aizawa's results [2].

2. The semigroup associated with the Cauchy problem

The results stated in this section are proven in [7]. We assume $F \in C^2(\mathbf{R})$, $F'' \geq 0$, and $F(0) = 0$. Fix $k > 0$. Define A_k: $\mathscr{D}(A_k) \subset \mathrm{BUC}(\mathbf{R}) \to \mathrm{BUC}(\mathbf{R})$ by $\mathscr{D}(A_k) = \{u \in B(k)$: for some $\lambda_u > 0$, $u + \lambda_u F(u_x) \in B(k)\}$, $A_k u = F(u_x)$ for $u \in \mathscr{D}(A_k)$. Here u_x is the derivative of u. Below, $\mathscr{R}(I + \lambda A_k)$ denotes the range of $I + \lambda A_k$.

Proposition 2.1. [7, § 5]. *Let $k > 0$. A_k is accretive in* BUC(**R**), *and $\mathscr{D}(A_k) \subset \mathscr{R}(I + \lambda A_k)$ for each $\lambda > 0$. Thus $\bar{A}_k$ uniquely determines a contraction semigroup $\{T_k(t): t \in \mathbf{R}^+\}$ on $\overline{\mathscr{D}(\bar{A}_k)}$ via the Crandall-Liggett exponential formula*

$$T_k(t)f = \lim_{n \to \infty} \left(I + \frac{t}{n} A_k\right)^{-n} f, \quad t \in \mathbf{R}^+, f \in \overline{\mathscr{D}(\bar{A}_k)}.$$

Now define A_∞: $\mathscr{D}(A_\infty) \subset \mathrm{BUC}(\mathbf{R}) \to \mathrm{BUC}(\mathbf{R})$ by:
$\mathscr{D}(A_\infty) = \cup \{\mathscr{D}(A_k): k > 0\}$, and $A_\infty u = F(u_x)$ for $u \in \mathscr{D}(A_\infty)$. Let A be the closure of A_∞ in BUC(**R**).

PROPOSITION 2.2. [7, § 6]. *A is m-accretive and densely defined on* BUC(**R**). *A determines a contraction semigroup given by*

$$T(t)f = \lim_{n\to\infty}\left(I + \frac{t}{n}A\right)^{-n} f \qquad (f \in \mathrm{BUC}(\mathbf{R}),\ t \in \mathbf{R}^+).$$

3. Invariant subsets

In this section we show that the semigroup constructed in Section 2 leaves certain subsets of BUC(**R**) invariant. This will allow us to consider certain restrictions Q of the operator A and show that they satisfy the hypotheses of the Crandall-Liggett theorem [10] (i.e. $\overline{\mathscr{D}(Q)} \subset \mathscr{R}(I+\lambda Q)$ for $\lambda > 0$ sufficiently small, since a restriction Q of an accretive operator A is automatically accretive).

For the rest of the paper the function F: $\mathbf{R} \to \mathbf{R}$ satisfies $F \in C^2(\mathbf{R})$, $F(0)=0$, $F''(x) \geq 0$ and $F(-x) = F(x)$ for all $x \in \mathbf{R}$. Fix $p \in \mathbf{R}$. We next define three restrictions A_p^k, A_e^k, and A_π^k of A_k. Fix $k > 0$. Define

$$\mathscr{D}(A_p^k) = \{u \in \mathscr{D}(A_k) \colon u(x+p) = u(x) \quad \text{for all } x \in \mathbf{R}\},$$

$$\mathscr{D}(A_e^k) = \{u \in \mathscr{D}(A_k) \colon u(x) = u(-x) \quad \text{for all } x \in \mathbf{R}\},$$

$$\mathscr{D}(A_\pi^k) = \{u \in \mathscr{D}(A_e^k) \colon u(0) = 0,\ u(x) \geq 0 \quad \text{for all } x \in \mathbf{R}\}.$$

For $u \in \mathscr{D}(A_p^k)$ [resp. $u \in \mathscr{D}(A_e^k)$, $u \in \mathscr{D}(A_\pi^k)$], we define $A_p^k u = Au = F(u_x)$ [resp. $A_e^k u = Au = F(u_x)$, $A_\pi^k u = Au = F(u_x)$].

Thus A_e^k is the restriction of A_k to the even functions in its domain, A_π^k is the restriction of A_e^k to the nonnegative functions in its domain which vanish at the origin, and A_p^k is the restriction of A^k to the periodic functions with period p in its domain. The following lemma gives useful information about (the union over k of) $\mathscr{D}(A_e^k)$, $\mathscr{D}(A_\pi^k)$, and $\mathscr{D}(A_p^k)$.

LEMMA 3.1. (i) *If* $u \in \cup\{B(k)\colon k > 0\} \cap \{v \in \mathrm{BUC}(\mathbf{R})\colon v$ *is even, nonnegative, and* $v(0) = 0\}$, *then u is continuously differentiable at the origin and* $u_x(0) = 0$.

(ii) $C_e^3(\mathbf{R}) \cap W^{3,\infty}(\mathbf{R}) \subset \cup\ \{\mathscr{D}(A_e^k)\colon k > 0\}$.

(iii) $C_\pi^3(\mathbf{R}) \cap W^{3,\infty}(\mathbf{R}) \subset \cup\ \{\mathscr{D}(A_\pi^k)\colon k > 0\}$.

(iv) $C_p^3(\mathbf{R}) \cap W^{3,\infty}(\mathbf{R}) \subset \cup\ \{\mathscr{D}(A_p^k)\colon k > 0\}$.

PROOF. (i): Note that any function in $B(k)$ is continuous and piecewise C^1. The one-sided bound on the second centered difference quotient (in the definition of $B(k)$) implies that any jump in the first derivative of a function in $B(k)$ must be a downward jump. So let $u \in B(k)$ be even, nonnegative, and $u(0)$

$=0$. Suppose u is not C^1 at the origin. Then since $u_x(0^-)\leq 0\leq u_x(0^+)$, we must have $u_x(0^-)<u_x(0^+)$ and thus u_x has an upward jump at zero, a contradiction. This proves (i).

For (ii) and (iii), and (iv), it suffices to establish that

$$C^3(\mathbf{R}) \cap W^{3,\infty}(\mathbf{R}) \subset \cup \{\mathscr{D}(A_k): k > 0\}. \tag{6}$$

Let $u \in C^3(\mathbf{R}) \cap W^{3,\infty}(\mathbf{R})$, and let K be a common bound for the absolute value of u and its derivatives up to order three. Then u, $F(u_x) \in B(L)$ where, $L=K+\max\{|(F(t)|: |t|\leq K\}$. Thus $u+F(u_x)\in B(2L)$ and (6) follows.

Now we want to show that

$$\mathscr{D}(A_\varepsilon^k) \subset \mathscr{R}(I + \lambda A_\varepsilon^k)$$

for $\lambda>0$. For this we consider the equation

$$u + \lambda F(u_x) = h \tag{7}$$

for $h\in \mathscr{D}(A_\varepsilon^k)$ and seek a solution u in $\mathscr{D}(A_\varepsilon^k)$. To establish the existence of a solution we proceed as in [7] and [2]. Consider the regularized elliptic equation

$$u + \lambda F(u_x) - \varepsilon u_{xx} = h. \tag{8}$$

We will need the following known result.

Proposition 3.2 [7]. *Let $k>0$, $h\in B(k)$, and $\lambda>0$. Then there is a $u \in \mathscr{D}(A_k)$ such that* (7) *holds.*

For $\varepsilon>0$ let u^ε be the unique solution in $C^2(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$ of

$$u^\varepsilon + \lambda F(u_x^\varepsilon) - \varepsilon u_{xx}^\varepsilon = h;$$

u^ε exists. Then for some sequence $\varepsilon\downarrow 0$, $u^\varepsilon\to u$ uniformly on compact subsets of $\mathbf{R}$, and $u_x^\varepsilon\to u$ a.e. in $\mathbf{R}$.

The existence proof for u^ε can be found in both Aizawa [2] and Burch [7]. The full statement is proved in [7].

We next study u^ε further in order to see how u^ε and u inherit properties from h. The main tool we require is the following version of the maximum principle for elliptic equations.

Lemma 3.3. *Let $a\in L^\infty(\mathbf{R})$ and $\varepsilon>0$. If $v\in C^2(\mathbf{R})$ is bounded from above and*

$$Lv \equiv v + av_x - \varepsilon v_{xx} \leq 0 \qquad \textit{for all} \quad x\in\mathbf{R},$$

then $v(x)\leq 0$ for all $x\in\mathbf{R}$.

For a proof see [2], [7], or [13].

PROPOSITION 3.4. *Let* $\varepsilon>0$, $\lambda>0$, $F\in C^2(\mathbf{R})$, $F(0)=0$, $F''(x)\geq 0$ *and* $F(-x)=F(x)$ *for all* $x\in\mathbf{R}$. *Let* $u\in C^2(\mathbf{R})\cap W^{1,\infty}(\mathbf{R})$ *satisfy*

$$u+\lambda F(u_x)-\varepsilon u_{xx}=h, \qquad h\in \mathrm{BUC}(\mathbf{R}).$$

Then the following conclusions hold:

(i) *If* h *is an even function, then so is* u.

(ii) *If* $h\geq 0$, *then* $u\geq 0$.

(iii) *If* $h\in B(k)$, h *is even and* $h(0)=0$, *then* $u(0)\leq\varepsilon k$.

(iv) *If* $h(x+p)=h(x)$ *for all* $x\in\mathbf{R}$, *then* $u(x+p)=u(x)$ *for all* $x\in\mathbf{R}$.

PROOF. (i): Define $w(x)\equiv u(x)-u(-x)$. Since F and h are even we have, for all $x\in\mathbf{R}$,

$$\begin{aligned}0&=w(x)+\lambda[F(u_x(x))-F(u_x(-x))]-\varepsilon w_{xx}(x)\\&=w(x)+\lambda[F(u_x(x))-F(-u_x(-x))]-\varepsilon w_{xx}(x)\\&=w(x)+\lambda\left[\int_0^1 F'(\tau u_x(x)+(1-\tau)(-u_x(-x))d\tau\right]w_x(x)-\varepsilon w_{xx}(x).\end{aligned}$$

Define

$$Lv(x)\equiv v+\lambda\left[\int_0^1 F'(\tau u_x(x)+(1-\tau)(-u_x(-x))d\tau\right]v_x(x)-\varepsilon v_{xx}(x).$$

The above calculation shows that $L(\pm w)=0$, and so Lemma 3.3 implies $\pm w\leq 0$. Thus $w\equiv 0$ and consequently u is even.

(ii): Define

$$L_0v(x)\equiv v(x)+\lambda\left[\int_0^1 F'(\tau u_x(x))d\tau\right]v_x(x)-\varepsilon v_{xx}(x).$$

Then if $h\geq 0$,

$$L_0(-u)=-h\leq 0,$$

whence $u\geq 0$ by Lemma 3.3.

(iii): Let h be as in the statement of (iii). u satisfies

$$u(0)+\lambda F(u_x(0))-\varepsilon u_{xx}(0)=h(0)=0. \tag{9}$$

u is even since h is (by (i)), thus u_x is odd and so $u_x(0)=0$. Also, $F(0)=0$, whence (9) reduces to $u(0)=\varepsilon u_{xx}(0)$. Furthermore, $h\in B(k)$ implies $u\in B(k)$. (For a proof see [7].) Thus $u_{xx}(x)\leq k$ for all $x\in\mathbf{R}$ as $u\in C^2(\mathbf{R})\cap B(k)$. Consequently

$$u(0) = \varepsilon u_{xx}(0) \leq \varepsilon k.$$

(iv): (Note that the assumption $F(x)=F(-x)$ is not needed here.) Let $w(x)\equiv u(x+p)-u(x)$ and proceed as in (i) above.

We can now interpret the above proposition in terms of the operators A_e^k, A_π^k, and A_p^k.

PROPOSITION 3.5. *$\mathscr{D}(A_p^k)\subset\mathscr{R}(I+\lambda A_p^k)$, $\mathscr{D}(A_e^k)\subset\mathscr{R}(I+\lambda A_e^k)$, and $\mathscr{D}(A_\pi^k)\subset\mathscr{R}(I+\lambda A_\pi^k)$ for all $\lambda>0$.*

PROOF. Let $h\in\mathscr{D}(A_e^k)$ and $\lambda>0$. Proposition 3.2 provides a $u\in\mathscr{D}(A_k)$ satisfying $u+\lambda F(u_x)=h$ and $u=\lim_{\varepsilon\to 0} u^\varepsilon$, uniformly on compacta, etc. Since h is even each u_ε is even by Proposition 3.4 (i), whence so is the limit function u. Thus $u\in\mathscr{D}(A_e^k)$.

Now suppose $h\in\mathscr{D}(A_\pi^k)$. We just proved that $u\in\mathscr{D}(A_e^k)$, since $h\in\mathscr{D}(A_\pi^k)\subset\mathscr{D}(A_e^k)$. Furthermore, h even, $h\geq 0$, and $h(0)=0$ implies $u^\varepsilon\geq 0$ and $u^\varepsilon(0)\leq\varepsilon k$ by Proposition 3.4 (ii), (iii). Thus $u\geq 0$ and $u(0)=0$. In other words, $u\in\mathscr{D}(A_\pi^k)$, and so $\mathscr{D}(A_\pi^k)\subset\mathscr{R}(I+A_\pi^k)$. Similarly, $\mathscr{D}(A_p^k)\subset\mathscr{R}(I+\lambda A_p^k)$.

We could now apply the Crandall-Liggett theorem to the closures of A_p^k, A_e^k, and A_π^k to get contraction semigroups. Instead we proceed as in [7] and define extensions of A_p^k, A_e^k, A_π^k, and then apply Crandall-Liggett to their closures.

Define A_e in BUC(**R**) by:

$$\mathscr{D}(A_e) = \cup\ \{\mathscr{D}(A_e^k)\colon k>0\},\ A_e(u) = F(u_x) \qquad \text{for} \quad u\in\mathscr{D}(A_e).$$

Define A_p in BUC(**R**) by:

$$\mathscr{D}(A_p) = \cup\ \{\mathscr{D}(A_p^k)\colon k>0\},\ A_p(u) = F(u_x) \qquad \text{for} \quad u\in\mathscr{D}(A_p).$$

Define A_π in BUC(**R**) by:

$$\mathscr{D}(A_\pi) = \cup\ \{\mathscr{D}(A_\pi^k)\colon k>0\},\ A_\pi(u) = F(u_x) \qquad \text{for} \quad u\in\mathscr{D}(A_\pi).$$

First note that $\mathscr{D}(A_a)\subset\mathscr{R}(I+\lambda A_a)$ for $a\in\{p, e, \pi\}$ and for all $\lambda>0$ by Proposition 3.5. Secondly, A_p, A_e, and A_π are accretive, being restrictions of A_∞. Thus their closures in BUC(**R**) are accretive. These remarks allow us to conclude

PROPOSITION I. *$\bar{A}_p$ [resp. $\bar{A}_e$, $\bar{A}_\pi$] determines a contraction semigroup on $\overline{\mathscr{D}(\bar{A}_p)}=\mathrm{BUC}_p(\mathbf{R})$ [resp. $\overline{\mathscr{D}(\bar{A}_e)}=\mathrm{BUC}_e(\mathbf{R})$, $\overline{\mathscr{D}(\bar{A}_\pi)}=\mathrm{BUC}_\pi(\mathbf{R})$] via the Crandall-Liggett exponential formula.*

The assertions concerning the closures of the domains of $\bar{A}_p$, $\bar{A}_e$, $\bar{A}_\pi$ are routine to check once Lemma 3.1 is noted. See [7].

4. The boundary problem

The four generators ($\bar{A}_p$, $\bar{A}_e$, $\bar{A}_\pi$, A) that we have discussed satisfy

$$\bar{A}_\pi \subset \bar{A}_e \subset A \quad \text{and} \quad \bar{A}_p \subset A.$$

Thus the corresponding semigroups satisfy the same restriction (or extension) inequalities. In particular, we can consider the semigroup $T=\{T(t): t\in \mathbf{R}^+\}$ determined by A and treat the other semigroups are restrictions of this semigroup T to appropriate domains. Recall that for $v\in \mathrm{BUC}(\mathbf{R})$, $t\in \mathbf{R}^+$,

$$T(t)v = \lim_{n\to\infty}\left(I+\frac{t}{n}A\right)^{-n} v \in \mathrm{BUC}(\mathbf{R}).$$

PROPOSITION 4.1. *Let* $u(t)=T(t)v$ *for* $v\in \mathrm{BUC}(\mathbf{R})$, $t\in\mathbf{R}^+$. *Let* $u(t, x) =u(t)(x)$ *for* $x\in R$.

(i) *If* $v\in B(k)$, *then* $u(t)\in B(k)$ *for each* $t>0$.

(ii) *If* $v\in \cup\{B(k): k>0\}$, *then* $u(\cdot, \cdot)$ *is Lipschitz continuous on* $[0, T]\times\mathbf{R}$ *for each* $T>0$.

(iii) *If* $v\in \cup\{B(k): k>0\}$, *then* $u_t+F(u_x)=0$ *a.e. in* $\mathbf{R}^+\times\mathbf{R}$.

(iv) *If* $v\in B_e(k)$, *then* $u(t)\in B_e(k)$ *for all* $t>0$.

(v) *If* $v\in \cup\{B(k): k>0\}$ *and* $v\geq 0$, *then* $u(t)\geq 0$ *for each* $t>0$.

(vi) *If* $v\in B_\pi(k)$, *then* $u(t)\in B_\pi(k)$ *for each* $t>0$.

(vii) *If* $v\in B_p(k)$, *then* $u(t)\in B_p(k)$ *for each* $t>0$.

PROOF. (i), (ii), and (iii) are proved in Burch [7].

To prove (iv), let $v\in B_e(k)$ and let $t>0$. Define u^1 by $u^1+tF(u^1_x)=v$, i.e. $u^1=(I+tA)^{-1}v$. Then $u^1\in B_e(k)$ by Proposition 3.4 (i) and a limiting argument. (Cf. the proof of Proposition 3.5.) Defining u^n to be $\left(I+\frac{t}{n}A\right)^{-n} v$, the above argument plus induction gives $u^n\in B_e(k)$. Since $u(t)=\lim_{n\to\infty} u^n$ and since $B_e(k)$ is closed in BUC($\mathbf{R}$), $u(t)\in B_e(k)$ follows.

The proofs of (v), (vi), and (vii) are similar and are omitted.

COROLLARY 4.2. *Let* u, v *be as in Proposition* 4.1. *If* $v\in B_\pi(k)$, *then* $u(t, 0)=u_x(t, 0)=0$ *for each* $t>0$.

PROOF. Proposition 4.1 (vi) and Lemma 3.1 (i).

For $u\in C(\mathbf{R}^+)$ we define $\tilde{u}\in C(\mathbf{R})$ by

$$\tilde{u}(x) = \begin{cases} u(x), & x\geq 0,\\ u(-x), & x<0.\end{cases}$$

Thus $\tilde{u}$ is the even extension of u.

Define

$$X(k) = \{u \in C(\mathbf{R}^+) \colon \tilde{u} \in B(k)\}$$

for $k>0$.

If $u \in C^1(\mathbf{R}^+)$ and $u_x(0^+)>0$, then $\tilde{u}_x$ has an upward jump at the origin, and so $\tilde{u} \notin \cup \{B(k) \colon k>0\}$. Hence if $u \in X(k)$, then $u_x(0^+) \le 0$. We also note that an argument using mollifiers shows that $\cup \{X(k) \colon k>0\}$ is dense in $\mathrm{BUC}_e(\mathbf{R})$. (Here we identify $u \in X(k)$ with $\tilde{u} \in B_e(k)$.)

Proposition 4.3. *Let* $u_0 \in \cup \{X(k) \colon k>0\}$ *and let* $u(t, \cdot)=T(t)\tilde{u}_0$. *Then* (1), (2), *and* (3) *hold. If, in addition,* $u_0 \ge 0$ *and* $u_0(0)=0$, *then* (4) *and* (5) *hold as well.*

Proof. For $u_0 \in \cup \{X(k) \colon k>0\}$, (1) and (2) hold by Proposition 4.1. (3) holds in a generalized sense. More precisely, $u(t, \cdot)$ is an even function and is the uniform limit of C^∞ functions whose first derivatives vanish at the origin. If also $u_0(0)=0$ and $u_0 \ge 0$, then $u(t, \cdot) \in B_\pi(k)$ for all $t>0$ by Proposition 4.1 and (3), (4) and (5) hold by Corollary 4.2. Note that (3) holds in the usual sense in this case.

Theorem 1 is an immediate consequence of Proposition 4.3. For the problem (1)–(3), the initial data u_0 should be in the set $\cup \{X(k) \colon k>0\}$, which is dense in $\mathrm{BUC}(\mathbf{R}^+)$. (Cf. the sentence preceding the statement of Proposition 4.3). For the problem (1)–(5), u_0 should be in $\cup \{X(k) \colon k>0\} \cap \{v \colon v(0)=0, v \ge 0\}$. This set is dense in $\mathrm{BUC}_\pi(\mathbf{R}^+)$.

5. Remarks

1. It is obvious that if a semigroup leaves a set invariant, then the solution of the associated differential equation takes values in that set for all times $t>0$ if it does initially. This gives us a method for finding solutions satisfying boundary conditions, positivity conditions, etc., and this method actually works in some cases, as the present paper illustrates. This method underscores the importance of the Crandall-Liggett condition

$$\mathscr{D}(A) \subset \mathscr{R}(I + \lambda A). \tag{10}$$

If A is a proper restriction of an accretive operator, then A is not maximal accretive, hence A is not m-accretive. Nevertheless, (10) can be verified in some cases, and (10) is precisely what is used to construct the semigroup governing A.

2. The method of this paper can be used to treat the Neumann problem

(1)–(3) for the Hamilton-Jacobi equation assuming only that $F: \mathbf{R} \to \mathbf{R}$ is continuous, even, and $F(0)=0$. Such a proof is based on Aizawa's paper [2]. Our method for (1)–(5), however, seems to be based in a crucial way on the convexity of F and the sets $B(k)$.

3. The method of this paper can be used to treat the Dirichlet problem for the Hamilton-Jacobi equation (i.e. (1), (2), and (3′) $u(t, 0)=0$ for $t \in \mathbf{R}^+$) assuming F is continuous, *odd*, and $F(0)=0$. As was the case in the previous remark, the proof is based on [2].

4. As is well-known, the one-dimensional Hamilton-Jacobi equation (1) and the one-dimensional conservation law

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(F(v)) = 0 \tag{11}$$

are connected via the formula $v=u_x$. The Neumann condition (3) for u becomes the Dirichlet condition

$$v(t, 0) = 0 \qquad (t \geq 0) \tag{12}$$

for v. This suggests that using the results of Crandall [8], the mixed problem for (11) [i.e. (11), (12) plus $v(0, x)=v_0(x)$] is governed by a semigroup in $L^1(\mathbf{R}^+)$ if F is continuous and even. We are confident that this is so, although we haven't worked out the details. For a treatment of (11), (12) with F strictly monotone increasing see [9].

5. If $u_t+F(u_x)=0$, then $u_{tt}=-(F(u_x))_t=-F'(u_x)u_{tx}=(F'(u_x))^2u_{xx}$; thus any solution u of (1) is a (generalized) solution of the hyperbolic equation

$$u_{tt} - (F'(u_x))^2 u_{xx} = 0. \tag{13}$$

This nonlinear wave equation is of substantial interest and has been studied by MacCamy and Mizel [14] among others. Our results give insight into the study of (13). For instance the formation of shocks for (11) leads to discontinuities in the first derivative of u, and thus we cannot expect to get classical solutions, in general, for (13), when F is strictly convex. However, Theorem I provides us with many nonnegative solutions of (13). More precisely, equation (13) together with the boundary conditions

$$u(t, 0) = 0 = u_x(t, 0) \qquad (t \geq 0)$$

and the initial conditions

$$u(0, x) = u_0(x) \geq 0,\ u_t(0, x) = -F\Big(\frac{d}{dx}u_0(x)\Big) \qquad (x \geq 0)$$

has a (generalized) nonnegative solution according to Theorem I.

Our results further suggest that in a semigroup approach to the Cauchy problem for (13) one should work in a space with the supremum norm.

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