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A Correction to "Forced Oscillations in General Ordinary Differential Equations with Deviating Arguments"

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1. Introduction

In [1] this author presented conditions to ensure that all oscillatory solutions of the equation

$$(1)\qquad (r(t)y'(t))^{(n-1)} + a(t)y_\tau(t) = f(t), \quad y_\tau(t) \equiv y(t - \tau(t))$$

approach zero as $t \to \infty$. The proof of the main result (Lemma (1)) was based on the "truth" of the inequality

$$(2)\qquad \left| \int_{t_1}^{t_2} \int_s^p a(t)dt\, ds \right| \le \int_{t_1}^{t_2} \int_t^{t_2} |a(s)| ds\, dt$$

where $t_1 < p < t_2$ and $a(t)$ continuous in $[t_1, t_2]$.
But this inequality (cf. Staikos and Philos [2]) is false as the following counter example (due to Prof. T. Kusano of Hiroshima University) shows:

$$\int_\pi^{5\pi} \int_s^{5\pi} |f(t)| dt\, ds = 3\pi \quad \text{and} \quad \left| \int_\pi^{5\pi} \int_s^{2\pi} f(t)dt\, ds \right| = 5\pi$$

where

$$f(t) = \begin{cases} 0 & (\pi \le t < 2\pi) \\ \sin t & (2\pi \le t \le 3\pi) \\ 0 & (3\pi \le t \le 5\pi). \end{cases}$$

However the conclusion of this crucial lemma remains true with a very minor change. We shall consider the following more general equation

$$(3)\qquad (r(t)y'(t))^{(n-1)} + a(t)h(y(g(t))) = f(t)$$

subject to similar assumptions. More precisely we assume

(i) $a(t)$, $r(t)$, $g(t)$, $h(t)$, $f(t)$ are real, continuous on the whole real line R:

(ii) $r(t) > 0$, $g(t) \le t$, $g(t) \to \infty$ as $t \to \infty$;

(iii) $0 \le \dfrac{h(t)}{t} \le m$, for some $m > 0$, $t > 0$.

2. Main results

LEMMA (2.1). *Suppose*

(4) $$\int^{\infty} |a(t)|dt < \infty,$$

(5) $$\int^{\infty} |f(t)|dt < \infty,$$

and

(6) $$\frac{1}{r(t)} = O(1/t^{n-k}), \qquad 0 \leq k < 1;$$

then all oscillatory solutions of equation (3) *are bounded.*

PROOF. Let $y(t)$ be an oscillatory solution of (3). Let $T > t_0$ be large enough so that for $t > T$, $g(t) > t_0$. Since $(r(t)y'(t))^{(n-2)}$ is oscillatory, there exist a $T_1 > t_1 > t_0$ such that $(r(t_1)y'(t_1))^{(n-2)} = 0$ and for $t \geq T_1$, $g(t) \geq t_1$. Designate $C_0 = r(t_1)y'(t_1)$, $C_1 = (r(t_1)y'(t_1))'$, $C_2 = (r(t_1)y'(t_1))''/2!, \ldots,$ $C_{n-3} = \dfrac{(r(t_1)y'(t_1))^{(n-3)}}{(n-3)!}$.

From (3) on integration

(7) $$(r(t)y'(t))^{(n-2)} + \int_{t_1}^{t} a(s)h(y(g(s)))ds = \int_{t_1}^{t} f(s)ds.$$

On repeated integration from (7) we have

(8) $$\begin{aligned} r(t)y'(t) &= C_0 + C_1(t - t_1) + C_2(t - t_1)^2 + \cdots + C_{n-3}(t - t_1)^{n-3} \\ &\quad - \int_{t_1}^{t} \frac{(t - s)^{n-2}}{(n - 2)!} a(s) \frac{h(y(g(s)))}{y(g(s))} y(g(s))ds \\ &\quad + \int_{t_1}^{t} \frac{(t - s)^{n-2}}{(n - 2)!} f(s)ds. \end{aligned}$$

Dividing (8) by $r(t)$ and then integrating between t_1 and $g(t)$ for $t \geq T_1$ we have

$$\begin{aligned} y(g(t)) &= y(t_1) + C_0 \int_{t_1}^{g(t)} \frac{1}{r(s)} ds + C_1 \int_{t_1}^{g(t)} \frac{(s - t_1)}{r(s)} ds + \cdots \\ &\qquad + C_{n-3} \int_{t_1}^{g(t)} \frac{(s - t_1)^{n-3}}{r(s)} ds \\ &\quad - \int_{t_1}^{g(t)} 1/r(s) \int_{t_1}^{s} \frac{(s - x)^{n-2} a(x)h(y(g(x)))y(g(x))dx}{(n-2)!\, y(g(x))} ds \\ &\quad + \int_{t_1}^{g(t)} 1/r(s) \int_{t_1}^{s} \frac{(s - x)^{n-2}}{(n - 2)!} f(x)dxds. \end{aligned}$$

$$|y(g(t))| \leq K_0 + mK_1 \int_{t_1}^{t} \int_{t_1}^{s} \frac{(s-x)^{n-2}}{s^{n-k}} |a(x)|\,|y(g(x))|\; dxds$$
$$+ K_1 \int_{t_1}^{t} \int_{t_1}^{s} \frac{(s-x)^{n-2}}{s^{n-k}} |f(x)|\, dxds$$
$$\leq K_0 + mK_1 \int_{t_1}^{t} \int_{t_1}^{s} \frac{|a(x)|\, y}{s^{2-k}}\, dxds + K_1 \int_{t_1}^{t} \int_{t_1}^{s} \frac{|f(x)|\, dxds}{s^{2-k}}$$

where

$$\frac{1}{r(t)} \leq \frac{K_1}{t^{n-k}}.$$

Changing the order of integration in the above we get

$$|y(g(t))| \leq K_0 + mK_1 \left[\int_{t_1}^{t} \int_{x}^{t} \frac{1}{s^{2-k}}\, ds \right] |y(g(x))|\,|a(x)| dx \tag{9}$$
$$+ K_1 \int_{t_1}^{t} \left[\int_{x}^{t} \frac{1}{s^{2-k}}\, ds \right] |f(x)| dx.$$

Since $0 \leq k < 1$, $\int_x^t \frac{1}{s^{2-k}}\, ds \leq K_2$ for some $K_2 > 0$.
Let $mK_1K_2 = K_3$. We have from (9)

$$|y(g(t))| \leq K_0 + K_3 \int_{t_1}^{t} |a(x)|\,|y(g(x))| dx + K_1K_2 \int_{t_1}^{t} |f(x)| dx \tag{10}$$

and since $\int_{t_1}^{\infty} |f(x)| dx < \infty$, there exists $K_4 > 0$ such that

$$|y(g(t))| \leq K_4 + K_3 \int_{t_1}^{t} |a(x)|\,|y(g(x))| dx. \tag{11}$$

The conclusion of the lemma follows from (11) by application of Gronwall's inequality. The proof is complete.

REMARK. The following inequality is needed in the proof of our main theorem:
If $t_1 < t_2 < t_3$ then

$$\left| \int_{t_t}^{t_2} \int_{s}^{t_3} a(x) dxds \right| \leq \int_{t_1}^{t_3} \int_{s}^{t_3} |a(x)| dxds \leq \int_{t_1}^{\infty} \int_{s}^{\infty} |a(x)| dxds$$

which gives by induction

$$\left| \int_{t_1}^{t_2} \int_{s_3}^{t_3} \int_{s_4}^{t_4} \cdots \int_{s_n}^{t_n} a(x) dx ds_n \cdots ds_3 \right| \tag{12}$$
$$\leq \int_{t_1}^{t_n} \int_{s_3}^{t_n} \int_{s_4}^{t_n} \cdots \int_{s_n}^{t_n} |a(x)| dx ds_n \cdots ds_3$$

$$\leq \int_{t_1}^{\infty}\int_{s_3}^{\infty}\cdots\int_{s_n}^{\infty}|a(x)|dxds_n\cdots ds_3$$

where $t_1<t_2<t_3<\cdots<t_{n-1}<t_n$.

THEOREM (2.1). *Suppose* $\int^{\infty} t^{n-2}|a(t)|dt<\infty$, $\int^{\infty}|f(t)|t^{n-2}dt<\infty$ *and* $1/r(t) =O(t^{n-k})$, $0\leq k<1$, *then oscillatory solutions of* (3) *approach zero as* $t\to\infty$.

PROOF. Suppose to the contrary that some oscillatory solution $y(t)$ of (3) is such that $\limsup_{t\to\infty}|y(t)|>2d>0$ for some number d. By Lemma (2.1), $y(t)$ is bounded.

Let T be large enough so that for $T_1>T$, $\int_{T_1}^{\infty}1/r(t)dt<1$,

$$m\int_{T_1}^{\infty}x^{n-2}|a(x)|dx < d/M_1^2, \tag{13}$$

and

$$\int_{T_1}^{\infty}x^{n-2}|f(x)|dx < d/M_1, \tag{14}$$

where $M_1=\sup\{|y(t)|,\ t\geq T\}$. Let t_1, t_2 be zeros of $y(t)$ such that $\mathrm{Max}\,|y(t)|>d$ for $t\in[t_1, t_2]$. Let $p_1<p_2<p_3<\cdots<p_{n-2}$, $(p_1>t_2)$ be zeros of $(r(t)y'(t))'$, $(r(t)y'(t))''$,..., $(r(t)y'(t))^{(n-2)}$. On repeated integration from (3) for $t<p_1$

$$\pm\,(r(t)y'(t))' = -\int_{t}^{p_1}\int_{s_2}^{p_2}\cdots\int_{s_{n-2}}^{p_{n-2}}a(x)h(y(g(x)))dxds_{n-2}\cdots ds_2$$

$$+\int_{t}^{p_1}\int_{s_2}^{p_2}\cdots\int_{s_{n-2}}^{p_{n-2}}f(x)dxds_{n-2}\cdots ds_2$$

which gives by (12)

$$|(r(t)y'(t))'| \leq m\int_{t}^{p_{n-2}}\int_{s_2}^{p_{n-2}}\cdots\int_{s_{n-2}}^{p_{n-2}}|a(x)|\,|y(g(x))|dx\cdots ds_2$$

$$+\int_{t}^{p_{n-2}}\int_{s_2}^{p_{n-2}}\cdots\int_{s_{n-2}}^{p_{n-2}}|f(x)|dx\cdots ds_2, \qquad \text{for all}\quad t\in[t_1, t_2].$$

Therefore

$$\int_{t_1}^{t_2}|(r(t)y'(t))'|dt \leq m\int_{t_1}^{p_{n-2}}\int_{s_1}^{p_{n-2}}\int_{s_2}^{p_{n-2}}\cdots\int_{s_{n-2}}^{p_{n-2}}|a(x)|\,|y|dx\cdots ds_2ds_1 \tag{15}$$

$$+\int_{t_1}^{p_{n-2}}\int_{s_1}^{p_{n-2}}\int_{s_2}^{p_{n-2}}\cdots\int_{s_{n-2}}^{p_{n-2}}|f(x)|dx\cdots ds_2ds_1$$

$$\leq \frac{m}{(n-2)!}\int_{t_1}^{\infty}(x-t_1)^{n-2}|a(x)|\,|y(g(x))|dx$$

$$+\frac{1}{(n-2)!}\int_{t_1}^{\infty}(x-t_1)^{n-2}|f(x)|dx.$$

Let

$$T_0 \in [t_1, t_2] \text{ such that } M = |y(T_0)| = \max|y(t)| \text{ in } [t_1, t_2].$$

Now

$$M = \int_{t_1}^{T_0} y'(t)dt = -\int_{T_0}^{t_2} y'(t)dt \tag{16}$$

which gives

$$2M \leq \int_{t_1}^{t_2} |y'(t)|dt = \int_{t_1}^{t_2} (r(t))^{1/2}|y'(t)|^{1/2}(r(t))^{-1/2}|y'(t)|^{1/2}dt. \tag{17}$$

By Schwarz's inequality

$$4M^2 \leq \int_{t_1}^{t_2} 1/r(t)dt \int_{t_1}^{t_2} (r(t)y'(t))y'(t)dt, \tag{18}$$

Integrating second integral by parts gives

$$4M \leq \left[\int_{t_1}^{t_2} 1/r(t)dt\right]\left[\int_{t_1}^{t_2} |(r(t)y'(t))'|dt\right] \tag{19}$$

since $|y(t)| \leq M$. Without any loss of generality we can assume that $d \leq MM_1$. From (15) and (19) we have

$$4M \leq \left(\int_{t_1}^{\infty} 1/r(t)dt\right)\left[\frac{m}{(n-2)!}\int_{t_1}^{\infty}(x-t_1)^{n-2}|a(x)|\,|y(g(x))|dx + \frac{1}{(n-2)!}\int_{t_1}^{\infty}(x-t_1)^{n-2}|f(x)|dx\right] \tag{20}$$

From (20), (13) and (14)

$$4\frac{d}{M_1} \leq \left(\int_{t_1}^{\infty} 1/r(t)\,dt\right)\frac{2d}{M_1}. \tag{21}$$

Conclusion follows from contradiction apparent in (21), since

$$\int_{t_1}^{\infty} 1/r(t)dt < 1.$$

The proof is complete.

References

[1] Bhagat Singh, Forced oscillations in general ordinary differential equations with deviating arguments, Hiroshima Math. J., **6** (1976), 7–14.

[2] V. A. Staikos and C. G. Philos, Some oscillation and asymptotic properties of linear differential equations, Bull. Fac. Sci., Ibaraki Univ. Math., **8** (1976), 25–30.

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