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The Minimal Condition for Subideals of Lie Algebras Implies that Every Ascendant Subalgebra is a Subideal

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Tôgô [5] has shown that various minimal conditions on ascendant subalgebras of Lie algebras are equivalent to each other. These results generalize earlier ones on minimal conditions for subideals (Amayo and Stewart [2], Stewart [4]). The purpose of this note is to point out a stronger result:

THEOREM. *If L is a Lie algebra satisfying the minimal condition for subideals, then every ascendant subalgebra of L is a subideal.*

PROOF. We use the notation of Amayo and Stewart [1]. Suppose that A asc L. Let B be a subideal of L, minimal subject to $A \leq B$. Then the ideal closure A^B of A in B must be B itself. Let K be the core of A in B (the largest ideal of B contained in A). Passing to the quotient B/K we may assume that A is corefree, and the condition $A^B = B$ remains valid. Let F be the unique ideal of B minimal with respect to B/F having finite dimension (see Amayo and Stewart [1] p. 165). Then $F+A$ si B so, by definition of B, we have $F+A=B$.

Let $Z=\zeta_1(F)$. Then $Z \cap A$ is idealized both by F and by A, so is an ideal of B. Since A is corefree in B we have $Z \cap A = 0$.

If $F \neq Z$, choose M minimal subject to $M \lhd B$, $F \geq M > Z$. By [1] theorem 8.2.3 p. 165, M/Z is infinite-dimensional simple. If $A \cap M \neq 0$ then $(A \cap M) + Z/Z$ asc M/Z. By Levič [3] a simple Lie algebra can have no nontrivial ascendant subalgebras, so we have $A \cap M + Z = Z$ or $A \cap M + Z = M$. But in the first case $A \cap M = A \cap Z = 0$. In the second, $A \cap M \cong M/Z$ which is simple, and $A \cap M$ asc B. But [1] proposition 1.3.5 p. 11 implies that $A \cap M \lhd B$, contrary to A being corefree.

Hence $A \cap M = 0$. Now $A+Z$ asc $A+M$. Consider an ascending series from $A+Z$ to $A+M$, which must be of the form $(A+X_\alpha)_{\alpha\leq\sigma}$ where $(X_\alpha)_{\alpha\leq\sigma}$ is a series from Z to M. Since M/Z is simple, Levič [3] implies that $A+Z \lhd A+M$. It follows that $A \leq C_B(M/Z)$, since $[M, A] \leq M \cap (A+Z) = (M \cap A) + Z = Z$. But $C_B(M/Z) \lhd B = A^B$, so $B = C_B(M/Z)$, which is absurd since $M \leq B$ and M/Z is simple and infinite-dimensional. This is a contradiction.

Thus the case $F \neq Z$ does not occur, so $F = Z$ and $F = F^2 = Z^2 = 0$. Hence $A = B$ and A is a subideal of L as claimed.

COROLLARY. *If L satisfies* Min-si *over a field of characteristic zero, then the Gruenberg radical $\gamma(L)$ is a nilpotent finite-dimensional characteristic ideal, equal to each of $\beta(L)$, $\rho(L)$, $\nu(L)$.*

In [5] Tôgô defines classes Min-$\lhd^{\sigma}$ and Min-(asc of step$<\sigma$), where σ is an ordinal: these consist of Lie algebras satisfying the minimal condition for the relevant types of subalgebra. Now Min-$\lhd^3$ is equivalent to Min-si by Amayo and Stewart [2] (Min-$\lhd^2$ for characteristic zero), hence for $\sigma\geq 3$ we have Min-$\lhd^{\sigma}\leq$Min-si, and for $\sigma\geq 4$ we have Min-(asc of step$<\sigma$)$\leq$Min-si. By the theorem, Min-si$=$Min-asc, hence all of the above classes are equal to Min-si. This improves the main theorem of Tôgô [5] p. 685.

References

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