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An Extended Airy Function of the First Kind

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1. Introduction

The linear differential equation

$$z^n \frac{d^n y}{dz^n} - z^q y = 0, \tag{1.1}$$

where z is a complex variable and q is an integer larger than n, has an extended form of the well-known Airy equation. For $n=2$ and $q=3$ (1.1) is exactly the Airy equation which has a long history of investigations. Two linearly independent entire solutions of the Airy equation $Ai(z)$ and $Bi(z)$ are called the Airy functions of the first and second kind, respectively. Their properties have been studied in great detail (see [5, 6]). For instance, we here give a brief exposition of the global behavior of the Airy function of the first kind

$$Ai(z) = \sum_{m=0}^{\infty} \frac{z^{3m}}{3^{2m+2/3} m! \Gamma\left(m + \frac{2}{3}\right)} - \sum_{m=0}^{\infty} \frac{z^{3m+1}}{3^{2m+4/3} m! \Gamma\left(m + \frac{4}{3}\right)}. \tag{1.2}$$

$Ai(z)$ is recessive on the positive real axis $\arg z = 0$ and admits the following asymptotic behavior as z tends to infinity:

$$(1.3)\quad \begin{cases} Ai(z) \sim \dfrac{-i}{2\sqrt{\pi}} \exp\left(\dfrac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}} \displaystyle\sum_{s=0}^{\infty} \left(\frac{3}{4}\right)^s \frac{\Gamma\left(s+\frac{1}{6}\right)\Gamma\left(s+\frac{5}{6}\right)}{\Gamma(s+1)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} z^{-\frac{3}{2}s} \\ \qquad\qquad \text{in} \quad -\dfrac{4}{3}\pi < \arg z < -\pi, \\ Ai(z) \sim \dfrac{1}{2\sqrt{\pi}} \exp\left(-\dfrac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}} \displaystyle\sum_{s=0}^{\infty} \left(-\frac{3}{4}\right)^s \frac{\Gamma\left(s+\frac{1}{6}\right)\Gamma\left(s+\frac{5}{6}\right)}{\Gamma(s+1)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} z^{-\frac{3}{2}s} \\ \qquad\qquad \text{in} \quad -\pi < \arg z < \pi, \\ Ai(z) \sim \dfrac{i}{2\sqrt{\pi}} \exp\left(\dfrac{2}{3} z^{\frac{3}{2}}\right) z^{-\frac{1}{4}} \displaystyle\sum_{s=0}^{\infty} \left(\frac{3}{4}\right)^s \frac{\Gamma\left(s+\frac{1}{6}\right)\Gamma\left(s+\frac{5}{6}\right)}{\Gamma(s+1)\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} z^{-\frac{3}{2}s} \\ \qquad\qquad \text{in} \quad \pi < \arg z < \dfrac{4}{3}\pi. \end{cases}$$

The negative real axis $\arg z = \pm\pi$ is the actual Stokes line of $Ai(z)$. By means of the asymptotic behavior (1.3) and an application of the principle of the argument, we see that zeros of $Ai(z)$ are located in a small sector including the Stokes line, more precisely, just on the Stokes line, and moreover can obtain an exact number of zeros contained in a disk with a sufficiently large radius, though we can also use Lommel's method to know the zero-free domains of $Ai(z)$.

C. A. Swanson and V. B. Headley [7] defined the Airy functions of the first and second kind satisfying a second order linear differential equation of the form (1.1), where q is an arbitrary integer larger than 2, in terms of the modified Bessel function of the first kind and investigated continuation formulas, linear dependence relations, zero-free domains, the distribution of zeros and other properties.

In this paper we shall define the Airy function of the first kind satisfying the higher order linear differential equation (1.1) which reduces to the original $Ai(z)$ when $n=2$ and $q=3$. As not explained explicitly anywhere, the Airy function of the first kind should be defined as a particular entire solution of linear differential equations of the form (1.1) which is principally recessive on the positive real axis $\arg z=0$. For that reasoning, we have to assume that n is even, i.e., $n=2N$, N being a positive integer.

In order to define the extended Airy function of the first kind, we first investigate the global behaviors of solutions of the linear differential equation (1.1). Such investigations have been done by H. L. Turrittin [8], J. Heading [2] and B. L. J. Braaksma [1]. But we here use our theory of solving a two point connection problem established in the papers [3, 4] to obtain the desired result. We shortly explain our method (see [4: Section 8]). By the change of variables $z=t^n$ we can rewrite (1.1) in the form

$$[\mathscr{D}\{\mathscr{D}-n\}\cdots\{\mathscr{D}-n(n-1)\}-n^n t^{qn}]y=0, \tag{1.4}$$

where $\mathscr{D}$ denotes the differential operator $t\dfrac{d}{dt}$. If we put

$$\begin{cases} y_1 = y, \\ y_p = \{\mathscr{D}-n(n-p+1)\}\{\mathscr{D}-n(n-p+2)\}\cdots\{\mathscr{D}-n(n-1)\}y \end{cases} \tag{1.5}$$
$$(p=2, 3,\dots, n)$$

and denote the column vector $(y_1, y_2,\dots, y_n)$ by Y, then we have

$$t\frac{dY}{dt} = \begin{pmatrix} n(n-1) & 1 & & & 0 \\ & n(n-2) & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & n & 1 \\ n^n t^{qn} & 0 & \cdots\cdots & 0 & 0 \end{pmatrix} Y. \tag{1.6}$$

A further application of the shearing transformation

$$X = S(t)Y, \qquad S(t) = \operatorname{diag}(t^{-q(n-1)}, t^{-q(n-2)}, \ldots, t^{-q}, 1) \tag{1.7}$$

reduces (1.6) to the system of linear differential equations

$$t\frac{dX}{dt} = (A_0 + A_q t^q)X, \tag{1.8}$$

where

$$A_0 = \begin{pmatrix} \rho_1 & & 0 \\ & \rho_2 & \\ & & \ddots \\ 0 & & & \rho_n \end{pmatrix}, \ \rho_j = (n-j)(n+q) \qquad (j = 1, 2, \ldots, n),$$

$$A_q = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ n^n & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

We now apply our general theory to this system of linear differential equations. We can immediately obtain a fundamental set of solutions of (1.8) which are expressed in terms of convergent power series

$$X_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m \qquad (j = 1, 2, \ldots, n), \tag{1.9}$$

where the coefficients $G_j(m)$ $(j=1, 2, \ldots, n)$ satisfy the systems of linear difference equations

$$\begin{cases} (m + \rho_j - A_0)G_j(m) = A_q G_j(m-q), \\ G_j(0) \neq 0, \quad G_j(r) = 0 \quad (r < 0) \qquad (j = 1, 2, \ldots, n). \end{cases} \tag{1.10}$$

On the other hand, we derive formal solutions of (1.8) at $t = \infty$, an irregular singular point of rank q, with the following form

$$X^k(t) = \exp\left(\frac{\lambda_k}{q} t^q\right) t^{\mu_k} \sum_{s=0}^{\infty} H^k(s) t^{-s} \qquad (k = 1, 2, \ldots, n), \tag{1.11}$$

where the characteristic constants $\lambda_k = n\omega_n^{k-1}$ $(\omega_n = \exp(2\pi i/n))$, $\mu_k = (n+q)(n-1)/2$ $(k=1, 2, \ldots, n)$ and the coefficients $H^k(s)$ $(k=1, 2, \ldots, n)$ satisfy the systems of linear difference equations

$$\begin{cases} (\lambda_k - A_q)H^k(s) = (A_0 - \mu_k + s - q)H^k(s-q), \\ H^k(0) \neq 0, \quad H^k(r) = 0 \quad (r < 0) \qquad (k = 1, 2, \ldots, n). \end{cases} \tag{1.12}$$

Then we can prove the following expansion formulas, together with the explicit determination of the Stokes multipliers T_{jl}^k:

$$X_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m \tag{1.13}$$

$$= \sum_{m=0}^{\infty} \Big(\sum_{k=1}^{n} \sum_{l=1}^{q} T_{jl}^k \sum_{s=0}^{\infty} H^k(s) g_{jl}^k(m+s) \Big) t^{m+\rho_j}$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{q} T_{jl}^k \sum_{s=0}^{\infty} H^k(s) x_{jl}^k(t, s) \qquad (j = 1, 2, \dots, n),$$

where we put

$$\begin{cases} x_{jl}^k(t, s) = t^{\rho_j} \sum_{m=0}^{\infty} g_{jl}^k(m+s) t^m \\ g_{jl}^k(m) = \dfrac{1}{q} \dfrac{\left\{ \left(\dfrac{\lambda_k}{q} \right)^{\frac{1}{q}} \omega_q^{-(l-1)} \right\}^{m+\rho_j-\mu_k}}{\Gamma\left(\dfrac{m+\rho_j-\mu_k}{q} + 1 \right)} \end{cases} \tag{1.14}$$

$$(\omega_q = \exp(2\pi i/q);\ j, k = 1, 2, \dots, n;\ l = 1, 2, \dots, q).$$

The functions $x_{jl}^k(t, s)$ satisfy nonhomogeneous linear differential equations of the first order and hence have the global integral representations which yield their global behaviors in the whole complex plane. From (1.13) and the global behaviors of $x_{jl}^k(t, s)$ we can consequently solve the connection problem for (1.8), obtaining the following result:

$$X_j(t) \sim \sum_{k=1}^{n} T_{jl_k}^k X^k(t) \qquad (j = 1, 2, \dots, n) \tag{1.15}$$

as $t \to \infty$ in the sector $S(l_1, l_2, \dots, l_n) = S_{l_1}(\lambda_1) \cap S_{l_2}(\lambda_2) \cap \cdots \cap S_{l_n}(\lambda_n)$, where

$$S_l(\lambda_k):\ -\frac{3\pi}{q} + \frac{2\pi}{q} l \leqq \arg \lambda_k^{\frac{1}{q}} t < -\frac{\pi}{q} + \frac{2\pi}{q} l, \tag{1.16}$$

l running over all integers.

Returning to the original linear differential equation (1.1), we rewrite the above result in the form of

THEOREM 1. *Let $y_j(z)$ $(j=1, 2, \dots, n)$ be a fundamental set of entire solutions of* (1.1) *of the form*

$$y_j(z) = z^{n-j} \sum_{m=0}^{\infty} \Big(\prod_{i=1}^{n} \Gamma\Big(m + 1 + \frac{i-j}{q} \Big) \Big)^{-1} (z^q q^{-n})^m \tag{1.17}$$

$$(j = 1, 2, \dots, n).$$

Then we have

$$(1.18)\qquad y_j(z)\sim\sum_{k=1}^{n}T^k_{jl_k}y^k(z)\qquad (j=1, 2,\dots, n)$$

as $z\to\infty$ *in the sector* $S(l_1, l_2,\dots, l_n)=S^1_{l_1}\cap S^2_{l_2}\cap\cdots\cap S^n_{l_n}$, S^k_l $(k=1, 2,\dots, n)$ *denoting the sectors*

$$(1.19)\qquad S^k_l\colon (2l-3)\pi-\frac{2\pi}{n}(k-1)\leqq\arg z^{\frac{q}{n}}<(2l-1)\pi-\frac{2\pi}{n}(k-1)$$

$$(l=0, \pm 1 \pm 2,\dots),$$

where $y^k(z)$ $(k=1, 2,\dots, n)$ *are formal solutions of* (1.1) *of the form*

$$(1.20)\qquad y^k(z)=\exp\left(\frac{n}{q}\,\omega_n^{k-1}z^{\frac{q}{n}}\right)z^{\frac{(n-q)(n-1)}{2n}}\sum_{s=0}^{\infty}h^k(s)z^{-\frac{q}{n}s}$$

$$(h^k(0)=1;\ k=1, 2,\dots, n),$$

$h^k(s)$ *being the first component of the column vector* $H^k(s)$. *In the above the Stokes multipliers* T^k_{jl} *are given by*

$$(1.21)\qquad T^k_{jl}=\frac{1}{(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}}}\left(\frac{\omega_n}{q}\right)^{-j+1-\frac{\rho_j-\mu_k}{q}}\omega_q^{(l-1)(\rho_j-\mu_k)}$$

$$=\frac{q^{\frac{(n+q)(n-1)}{2q}}}{(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}}}q^{-\frac{n}{q}(j-1)}\omega_q^{\{(k-1)-n(l-1)\}\left\{(j-1)+\frac{(n+q)(1-n)}{2n}\right\}}$$

$$(j, k=1, 2,\dots, n;\ l=0, \pm 1, \pm 2,\dots).$$

As an example illustrating the above theorem, we consider the most simple differential equation, where $n=q$. In this case the linear differential equation (1.1) has only constant coefficients and $\exp(\omega_n^{k-1}z)$ $(k=1, 2,\dots, n)$ are its global solutions. The Stokes multipliers corresponding to (1.21) are rewritten in the form

$$(1.22)\qquad T^k_{jl}=(2\pi)^{-\frac{n-1}{2}}n^{-\frac{1}{2}}\left(\frac{\omega_n^{k-1}}{n}\right)^{j-n}\qquad (j, k=1, 2,\dots, n),$$

which are independent of l, i.e., the sectors (1.19), and then the relations (1.18) imply the identical formulas

$$(1.23)\qquad y_j(z)=z^{n-j}\sum_{m=0}^{\infty}\left(\prod_{i=1}^{n}\Gamma\left(m+1+\frac{i-j}{n}\right)\right)^{-1}(z^n n^{-n})^m$$

$$=\frac{n^{n-j}}{(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}}}nz^{n-j}\sum_{m=0}^{\infty}\frac{z^{nm}}{\Gamma(nm+n-j+1)}$$

$$\sim \sum_{k=1}^{n} T_{jl_k}^k y^k(z)$$

$$= \frac{n^{n-j}}{(2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}}} \sum_{k=1}^{n} (\omega_n^{k-1})^{j-n} \exp(\omega_n^{k-1} z) \qquad (j = 1, 2,\dots, n).$$

This fact is easily checked by expanding the exponential functions in the last line of (1.23) in terms of power series and taking account of the relation

$$\sum_{k=1}^{n} \omega_n^{(k-1)(nm+h)} = \sum_{k=1}^{n} \omega_n^{(k-1)h} = \begin{cases} n & (h = 0), \\ 0 & (1 < h < n-1). \end{cases}$$

As to the Stokes multipliers in other cases, refer to and compare with the results in [8, 2, 1].

2. Definition of the extended Airy function of the first kind $Ai(z)$

We now define, as already explained, the extended Airy function of the first kind $Ai(z)$ by an entire solution of an even order, i.e., $n=2N$ ($N \geqq 1$)-th order linear differential equation (1.1) which is principally recessive on the positive real axis $\arg z=0$. As a matter of course, such a function is uniquely determined. In fact, we put and then obtain

$$(2.1) \qquad Ai(z) = \sum_{j=1}^{n} c_j y_j(z) \sim \sum_{k=1}^{n} \left(\sum_{j=1}^{n} c_j T_{jl_k}^k \right) y^k(z)$$

as z tends to infinity in the sector $S(l_1, l_2,\dots, l_n)$. Since the positive real axis $\arg z=0$ is included in the sector

$$(2.2) \quad S(1, 1,\dots, 1, 2, 2,\dots, 2) = S_1^1 \cap \cdots \cap S_1^N \cap S_2^{N+1} \cap \cdots \cap S_2^{2N}\colon 0 \leqq \arg z^{\frac{q}{n}} < \frac{\pi}{N}$$

and among the exponential factors of formal solutions (1.20) the exponential factor for $k=N+1$ is principally recessive on the line $\arg z=0$, we only have to determine the constants c_j ($j=1, 2,\dots, n$) by the equation

$$(2.3) \qquad \sum_{j=1}^{n} c_j T_{jl_k}^k = \delta_{k\,N+1} \qquad (k = 1, 2,\dots, n),$$

where δ_{ij} denotes the Kronecker delta and l_k is set equal to 1 for $1 \leqq k \leqq N$ and 2 for $N+1 \leqq k \leqq 2N$. We now calculate the determinant composed of the Stokes multipliers appearing in (2.3) and its cofactors.

$$
(2.4)\qquad
\begin{vmatrix}
T_{12}^{N+1} & T_{22}^{N+1}\cdots T_{2N2}^{N+1} \\
T_{12}^{N+2} & T_{22}^{N+2}\cdots T_{2N2}^{N+2} \\
\vdots & \vdots \qquad \vdots \\
T_{12}^{2N} & T_{22}^{2N}\cdots T_{2N2}^{2N} \\
T_{11}^{1} & T_{21}^{1}\cdots T_{2N1}^{1} \\
T_{11}^{2} & T_{21}^{2}\cdots T_{2N1}^{2} \\
\vdots & \vdots \qquad \vdots \\
T_{11}^{N} & T_{21}^{N}\cdots T_{2N1}^{N}
\end{vmatrix}
$$

$$
= \alpha^n q^{-\frac{n}{p}\frac{n(n-1)}{2}} \omega_q^{-NM}
\begin{vmatrix}
1 & \omega_q^{-N} & \cdots \omega_q^{-N(j-1)} & \cdots \omega_q^{-N(n-1)} \\
1 & \omega_q^{-N+1} & \cdots \omega_q^{(-N+1)(j-1)} & \cdots \omega_q^{(-N+1)(n-1)} \\
\vdots & \vdots & \vdots & \vdots \\
1 & \omega_q^{-1} & \cdots \omega_q^{-1(j-1)} & \cdots \omega_q^{-1(n-1)} \\
1 & 1 & \cdots 1 & \cdots 1 \\
1 & \omega_q & \cdots \omega_q^{j-1} & \cdots \omega_q^{n-1} \\
\vdots & \vdots & \vdots & \vdots \\
1 & \omega_q^{N-1} & \cdots \omega_q^{(N-1)(j-1)} & \cdots \omega_q^{(N-1)(n-1)}
\end{vmatrix}
$$

$$
= \alpha^n q^{-\frac{n}{q}\frac{n(n-1)}{2}} \omega_q^{-NM} V_{2N}(\omega_q^{-N}, \omega_q^{-N+1},\dots, \omega_q^{-1}, 1, \omega_q,\dots, \omega_q^{N-1}),
$$

where we have put

$$
(2.5)\qquad \alpha = \frac{q^{\frac{(n+q)(n-1)}{2q}}}{(2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}}}, \qquad M = \frac{(n+q)(1-n)}{2n},
$$

and $V_n(x_1, x_2,\dots, x_n)$ denotes the Vandermonde determinant. Let us denote the cofactor corresponding to $(1, j)$ element $\omega_q^{-N(j-1)}$ of the Vandermonde determinant in (2.4) by $(-1)^{j+1}\Delta_j$. Putting $x_j=\omega_q^{-N-1+j}$ $(j=2, 3,\dots, n)$, each Δ_j $(j=1, 2,\dots, n)$ has the following form:

$$
(2.6)\qquad
\begin{vmatrix}
1 & x_2\cdots x_2^{j-2} & x_2^j\cdots x_2^{n-1} \\
1 & x_3\cdots x_3^{j-2} & x_3^j\cdots x_3^{n-1} \\
\vdots & \vdots \quad \vdots & \vdots \quad \vdots \\
1 & x_n\cdots x_n^{j-2} & x_n^j\cdots x_n^{n-1}
\end{vmatrix}
= \prod_{2\le i<k\le n} (x_k - x_i) P_{n-j}(x_2, x_3,\dots, x_n)
$$

$$(j = 1, 2,\dots, n),$$

where $P_{n-j}(x_2, x_3,\dots, x_n)$ is a fundamental symmetric function of degree $n-j$, i.e.,

$$
(2.7)\qquad
\begin{cases}
P_{n-j}(x_2, x_3,\dots, x_n) = \displaystyle\sum_{2\le i_1<i_2<\cdots<i_{n-j}\le n} x_{i_1}x_{i_2}\cdots x_{i_{n-j}}, \\
P_0(x_2, x_3,\dots, x_n) = 1
\end{cases}
\qquad (j = 1, 2,\dots, n).
$$

Taking account of (2.4–7), we find from (2.3) that the constants c_j $(j=1, 2,\dots, n)$ are given by

$$(2.8)\qquad c_j = \alpha^{-1} q^{\frac{n}{q}(j-1)} \omega_q^{NM} \frac{(-1)^{j+1} \Delta_j}{V_{2N}(\omega_q^{-N}, \omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1})}$$

$$= \alpha^{-1} q^{\frac{n}{q}(j-1)} \frac{(-1)^{j+1} P_{2N-j}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1})}{\omega_q^{-NM}(\omega_q^{-N+1}-\omega_q^{-N})(\omega_q^{-N+2}-\omega_q^{-N})\cdots(\omega_q^{N-1}-\omega_q^{-N})}$$

$$= \frac{(2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}} q^{\frac{(n+q)(1-n)}{2q}}}{2^{n-1}\left[\prod_{i=1}^{n-1} \sin\left(\frac{\pi}{q} i\right)\right]} q^{\frac{n}{q}(j-1)} (-1)^j P_{n-j}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1})$$

$$(j = 1, 2,\dots, n).$$

In the above $P_{n-j}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1})$ can be calculated as follows:

$$(2.9)\qquad \not p(x) = (x+\omega_q^{-N+1})(x+\omega_q^{-N+2})\cdots(x+1)\cdots(x+\omega_q^{N-2})(x+\omega_q^{N-1})$$

$$= (x+1)\prod_{i=1}^{N-1}\left\{x^2 + 2x\cos\left(\frac{2i}{q}\pi\right) + 1\right\}$$

$$= P_0 x^{2N-1} + P_1 x^{2N-2} + \cdots + P_{n-2}x + P_{n-1},$$

whence

$$(2.10)\qquad P_{n-j}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1}) = \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{dx^{j-1}} \not p(x)\right]_{x=0}$$

$$(j = 1, 2,\dots, n).$$

From this, we easily see that all $P_{n-j}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1})$ are real number, that is, all c_j are real number, and moreover we have the symmetric relations

$$(2.11)\qquad P_{n-j}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1}) = P_{j-1}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1})$$

$$(j = 1, 2,\dots, N).$$

Thus we have defined the extended Airy function of the first kind $Ai(z)$ as follows:

$$(2.12)\qquad \begin{cases} Ai(z) = \gamma \sum_{j=1}^{n} (-1)^j q^{\frac{n}{q}(j-1)} P_{n-j}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1}) y_j(z), \\ \gamma = \left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} \dfrac{n^{\frac{1}{2}} q^{\frac{(n+q)(1-n)}{2q}}}{\prod_{i=1}^{n-1} \sin\left(\frac{\pi}{q} i\right)}. \end{cases}$$

We here make a remark. In the definition of the original Airy function of the first kind and the like by C. A. Swanson and V. B. Headley the constant factor

γ is dropped and is included in the Stokes multipliers, but, for simplicity of later considerations, we take the above definition (2.12) in this paper.

We shall now show linear dependence relations. It is easy to see that $Ai(\omega_q^k z)$ $(k=0, \pm 1, \pm 2,\dots)$ are solutions of the linear differential equation (1.1) and

$$Ai(\omega_q^k z) = \sum_{j=1}^{n} c_j y_j(\omega_q^k z) = \sum_{j=1}^{n} c_j \omega_q^{k(n-j)} y_j(z) \tag{2.13}$$

holds. Let k_i $(i=1, 2,\dots, n)$ be mutually distinct modulo q. Then we have

$$Ai(z) = \sum_{i=1}^{n} \alpha_i Ai(\omega_q^{k_i} z), \tag{2.14}$$

where

$$\alpha_i = \frac{(\omega_q^{k_1}-1)(\omega_q^{k_2}-1)\cdots(\omega_q^{k_{i-1}}-1)(\omega_q^{k_{i+1}}-1)\cdots(\omega_q^{k_n}-1)}{(\omega_q^{k_1}-\omega_q^{k_i})(\omega_q^{k_2}-\omega_q^{k_i})\cdots(\omega_q^{k_{i-1}}-\omega_q^{k_i})(\omega_q^{k_{i+1}}-\omega_q^{k_i})\cdots(\omega_q^{k_n}-\omega_q^{k_i})} \quad (i=1, 2,\dots, n), \tag{2.15}$$

which obviously reduces to the well-known relation and the like in [7] when $n=2$. We can moreover calculate the Wronskian

$$\begin{aligned} &W[Ai(\omega_q^{k_1} z), Ai(\omega_q^{k_2} z),\dots, Ai(\omega_q^{k_n} z); z] \\ &= c_1 c_2 \cdots c_n \prod_{1\leqq i<j\leqq n} (\omega_q^{k_j} - \omega_q^{k_i}) \\ &= c_1 c_2 \cdots c_n \prod_{1\leqq i<j\leqq n} \sin\left(\frac{k_j-k_i}{q}\pi\right) \exp\left\{\left(\frac{k_j+k_i}{q}+\frac{1}{2}\right)\pi i\right\}. \end{aligned} \tag{2.16}$$

From this, it follows that $Ai(\omega_q^{k_i} z)$ $(i=1, 2,\dots, n)$ make a fundamental set of solutions of the linear differential equation (1.1).

3. Stokes phenomenon of $Ai(z)$

We shall now investigate the Stokes phenomenon of the extended Airy function of the first kind $Ai(z)$.

We have defined $Ai(z)$ such that

$$Ai(z) \sim y^{N+1}(z) \quad \text{as } z\to\infty \text{ in } S(1,\dots, \underset{\hat{N}}{1}, \underset{\widehat{N+1}}{2},\dots, 2)\colon\ 0 \leqq \arg z^{\frac{q}{n}} < \frac{\pi}{N}.$$

The asymptotic behavior of $Ai(z)$ in the sector $S(1,\dots, \underset{\hat{N}}{1}, \underset{\widehat{N+1}}{1}, 2,\dots, 2)$ lying below is immediately derived from (2.1), replacing T_{j2}^{N+1} by T_{j1}^{N+1} $(j=1, 2,\dots, n)$ as follows:

$$Ai(z) \sim y^{N+1}(z) \quad \text{as } z\to\infty \text{ in } S(1,\dots, \underset{\hat{N}}{1}, \underset{\widehat{N+1}}{1}, 2,\dots, 2)\colon\ -\frac{\pi}{N} \leqq \arg z^{\frac{q}{n}} < 0,$$

whence we have

$$(3.1)\qquad Ai(z)\sim y^{N+1}(z)\quad \text{as } z\to\infty \text{ in } -\frac{\pi}{N}\leqq \arg z^{\frac{q}{n}}<\frac{\pi}{N}.$$

Like this, in order to investigate the Stokes phenomenon of $Ai(z)$, we have only to evaluate the coefficients in the right hand side of (2.1), which become the Stokes multipliers of $Ai(z)$. We first calculate them. From (1.21) and (2.8) it follows that

$$(3.2)\qquad \sum_{j=1}^{n} c_j T_{jl}^{k}=\sum_{j=1}^{n}\frac{\omega_q^{\{(k-1)-n(l-1)\}M}\omega_q^{\{(k-1)-n(l-1)\}(j-1)}(-1)^{j+1}\Delta_j}{\omega_q^{-NM}V_n(\omega_q^{-N},\omega_q^{-N+1},\dots,1,\dots,\omega_q^{N-1})}$$

$$=\frac{\omega_q^{\{(k-1)-n(l-1)\}M}V_n(\omega_q^{\{(k-1)-n(l-1)\}},\omega_q^{-N+1},\dots,1,\dots,\omega_q^{N-1})}{\omega_q^{-NM}V_n(\omega_q^{-N},\omega_q^{-N+1},\dots,1,\dots,\omega_q^{N-1})}$$

$$=\frac{\exp\left[\frac{(1-n)\{(k-1)-n(l-1)\}}{n}\pi i\right]\prod_{j=-N+1}^{N-1}\sin\left[\frac{j-\{(k-1)-n(l-1)\}}{q}\pi\right]}{\exp\left[\frac{(1-n)(-N)}{n}\pi i\right]\prod_{j=-N+1}^{N-1}\sin\left(\frac{j+N}{q}\pi\right)}$$

$$=D_l^k\qquad\qquad (k=1,2,\dots,n;\ l=0,\pm1,\pm2,\dots).$$

We can therefore rewrite (2.1) in the following form:

$$(3.3)\qquad Ai(z)\sim\sum_{k=1}^{n}D_{l_k}^{k}y^k(z)$$

as z tends to infinity in the sector $S(l_1, l_2,\dots, l_n)$.

We are now in a position to analyze the global behavior of $Ai(z)$ in the whole complex plane $0\leqq\arg z<2\pi$. Considering (1.19), we see that if z lies in the sector $\mathscr{S}_k$: $\frac{\pi}{N}k\leqq\arg z^{\frac{q}{n}}<\frac{\pi}{N}(k+1)$ when k successively runs over $1, 2,\dots, q-1$, then the subscript l_{N+1-k} of $S_{l_{N+1-k}}^{N+1-k}$ only increases its value by 1, the others being unchanged, successively, where $N+1-k$ is considered as a number $\hat{k}=N+1-k\,(\mathrm{mod}\,2N)$ $(1\leqq\hat{k}\leqq 2N)$. From this fact, taking account of (3.2) and (3.3), we can obtain the asymptotic behavior of $Ai(z)$ in the whole complex plane $0\leqq\arg z<2\pi$ and summarize results derived in the following

THEOREM 2. *Let* $q=\mu n+\nu\geqq n$ $(\mu\geqq1,\ 0\leqq\nu<n)$. *We put*

$$(3.4)\qquad d_k=\exp\left[\frac{(n-1)k}{n}\pi i\right]\frac{\prod_{j=1}^{2N-1}\sin\left(\frac{j+k}{q}\pi\right)}{\prod_{j=1}^{2N-1}\sin\left(\frac{j}{q}\pi\right)}\qquad (k=0,1,2,\dots).$$

Then we have

$$Ai(z) \sim d_{pn}y^{N+1}(z) + d_{pn+1}y^{N}(z) + \cdots + d_{pn+r}y^{N+1-r}(z) + d_{(p-1)n+r+1}y^{N-r}(z) + d_{(p-1)n+r+2}y^{N-r-1}(z) + \cdots + d_{(p-1)n+n-1}y^{N+2}(z) \tag{3.5}$$

as z tends to infinity in the sector

$$\mathscr{S}_{pn+r}: \left(2p + \frac{r}{N}\right)\pi \leqq \arg z^{\frac{q}{n}} < \left(2p + \frac{r+1}{N}\right)\pi \quad (p = 0, 1,\dots, \mu;\ r = 0, 1,\dots, 2N-1;\ pn + r < q), \tag{3.6}$$

where the superscript $N-r$ of $y^{N-r}(z)$ is considered as a number $\hat{r}=N-r \pmod{2N}$ $(1 \leqq \hat{r} \leqq 2N)$, d_{-k} $(k>0)$ is set equal to zero and moreover the last $(2N-1)$ Stokes multipliers are vanishing, i.e.,

$$d_{(\mu-1)n+\nu+1} = d_{(\mu-1)n+\nu+2} = \cdots = d_{\mu n+\nu-1} = 0. \tag{3.7}$$

Similarly, in the whole complex plane $-2\pi \leqq \arg z < 0$, we have

$$Ai(z) \sim \bar{d}_{pn}y^{N+1}(z) + \bar{d}_{pn+1}y^{N+2} + \cdots + \bar{d}_{pn+r}y^{N+1+r}(z) + \bar{d}_{(p-1)n+r+1}y^{N+r+2}(z) + \bar{d}_{(p-1)n+r+2}y^{N+r+3}(z) + \cdots + \bar{d}_{(p-1)n+n-1}y^{N}(z) \tag{3.8}$$

as z tends to infinity in the sector

$$\mathscr{S}_{-(pn+r)}: -\left(2p + \frac{r+1}{N}\right)\pi \leqq \arg z^{\frac{q}{n}} < -\left(2p + \frac{r}{N}\right)\pi \quad (p = 0, 1,\dots, \mu;\ r = 0, 1,\dots, 2N-1;\ pn + r < q), \tag{3.9}$$

where the superscript again keeps the meaning stated above and $\bar{d}_k$ denotes the complex conjugate of d_k.

In order to obtain the asymptotic behavior of $Ai(z)$ on the Riemann surface, we may only make p and r run over positive integers and $0, 1,\dots, 2N-1$, respectively, with the Stokes multipliers d_k and $\bar{d}_k$.

From the above relations (3.5) we can see where the Stokes phenomenon of $Ai(z)$ occurs. Consider the Stokes phenomenon of $Ai(z)$ in the whole complex plane $0 \leqq \arg z < 2\pi$. We put

$$\theta_k = \frac{\pi}{2N} + \frac{\pi}{N}k \qquad (k=0, 1,\dots, q-1). \tag{3.10}$$

In the first n sectors, according as z moves from $\mathscr{S}_k$ to $\mathscr{S}_{k+1}$ $(k=0, 1,\dots, n-1)$, $y^{N+1-k}(z)$ appears for the first time, and then all $y^k(z)$ are appearing in the right hand side of (3.5) as far as z reaches the sector $\mathscr{S}_{(\mu-1)n+\nu+1}$. From this fact we have, putting $\theta=\arg z^{\frac{q}{n}}$,

$$(3.11) \qquad Ai(z) \sim y^{N+1}(z) \qquad (0 \leqq \theta < \theta_N),$$

$$(3.12) \qquad Ai(z) \sim d_{k+1-N} y^{n-k}(z) \qquad (\theta_k \leqq \theta < \theta_{k+1})$$

$$(k = N, N+1,\dots, (\mu-1)n+\nu-1).$$

We have to pay attention to the last $(n-1)$ sectors because some of the Stokes multipliers vanish there. We see that $y^{N+1-(\nu+1)}(z)$, $y^{N+1-(\nu+2)}(z),\dots$, $y^{N+1-(n+\nu-1)}(z)$ one after another disappear in the sector $\mathscr{S}_k$ according as k takes $(\mu-1)n+\nu+1$, $(\mu-1)n+\nu+2,\dots, \mu n+\nu-1$ successively. On the other hand, $y^{n-\nu}(z)$, $y^{n-\nu-1}(z),\dots, y^{N-\nu+1}(z)$ one after another become dominant in the sector $\theta_k \leqq \theta < \theta_{k+1}$ when k takes $(\mu-1)n+\nu$, $(\mu-1)n+\nu+1,\dots, (\mu-1)n+\nu+N-1$ in succession, and after that, because of the consecutive disappearance of others, $y^{N-\nu+1}(z)$ becomes dominant. Consequently, we have

$$(3.13) \qquad Ai(z) \sim d_{k+1-N} y^{n-k}(z) \qquad (\theta_k \leqq \theta < \theta_{k+1})$$

$$(k = (\mu-1)n+\nu, (\mu-1)n+\nu+1,\dots, (\mu-1)n+\nu+N-2),$$

$$(3.14) \qquad Ai(z) \sim d_{(\mu-1)n+\nu} y^{N-\nu+1}(z) \qquad \left(\theta_{q-N-1} \leqq \theta < \frac{q}{N}\pi\right).$$

The half-lines $\arg z^{\frac{q}{n}}=\theta_k$ $(k=N, N+1,\dots, q-N-1)$ therefore are the actual Stokes lines of $Ai(z)$ in the whole complex plane $0 \leqq \arg z < 2\pi$.

4. The distribution of zeros

From the global behavior derived in the preceeding section we can now investigate the distribution of zeros of $Ai(z)$ on the Stokes lines.

Let Q, P_1 and P_2 be points of intersection of the circle $|z|=\rho$ with one of the Stokes lines θ_k $(k=N, N+1,\dots, q-N-1)$, the rays $\theta=\theta_k-\varepsilon$ and $\theta=\theta_k+\varepsilon$, ε being an arbitrarily small positive number, respectively. From (2.12) and (3.12–14) we then have for a sufficiently large ρ

$$(4.1) \quad \Delta_{QP_1} \arg Ai(z) = \arg d_{k-N}$$

$$+ \arg\left\{\exp\frac{n}{q}\rho^{\frac{q}{n}}\left(\cos\left(\theta_k-\varepsilon+\frac{n-k}{N}\pi\right)+i\sin\left(\theta_k-\varepsilon+\frac{n-k}{N}\pi\right)\right)\right.$$

$$+\frac{(n-q)(n-1)}{2n}\left(\log\rho+i\frac{n}{q}(\theta_k-\varepsilon)\right)+\log(1+(\rho^{-\frac{q}{n}}))\Big\}$$

$$-\arg Ai(z)|_{OP_1\ni z=0}$$

$$=\frac{n}{q}\rho^{\frac{q}{n}}\sin\left(\theta_k-\varepsilon+\frac{n-k}{N}\pi\right)+\frac{(n-q)(n-1)}{2q}(\theta_k-\varepsilon)$$

$$+\frac{(n-1)(k-N)}{n}\pi+o(1)-\arg Ai(z)|_{OP_1\ni z=0},$$

(4.2) $\Delta_{P_1Q}\arg Ai(z)$

$$=\left[\frac{n}{q}\rho^{\frac{q}{n}}\sin\left(\theta+\frac{n-k}{N}\pi\right)+\frac{(n-q)(n-1)}{2q}\theta+\frac{(n-1)(k-N)}{n}\pi+o(1)\right]_{\theta_k-\varepsilon}^{\theta_k},$$

(4.3) $\Delta_{OP_2}\arg Ai(z)$

$$=\frac{n}{q}\rho^{\frac{q}{n}}\sin\left(\theta_k+\varepsilon+\frac{n-k-1}{N}\pi\right)+\frac{(n-q)(n-1)}{2q}(\theta_k+\varepsilon)$$

$$+\frac{(n-1)(k+1-N)}{n}\pi+o(1)-\arg Ai(z)|_{OP_2\ni z=0},$$

(4.4) $\Delta_{P_2Q}\arg Ai(z)$

$$=\left[\frac{n}{q}\rho^{\frac{q}{n}}\sin\left(\theta+\frac{n-k-1}{N}\pi\right)\right.$$

$$\left.+\frac{(n-q)(n-1)}{2q}\theta+\frac{(n-1)(k+1-N)}{n}\pi+o(1)\right]_{\theta_k+\varepsilon}^{\theta_k}.$$

Hence we have for a sufficiently large ρ and a sufficiently small ε

(4.5) $\Delta_{OP_1QP_2O}\arg Ai(z)$

$$=\frac{n}{q}\rho^{\frac{q}{n}}\left\{\sin\left(\theta_k+\frac{n-k}{N}\pi\right)-\sin\left(\theta_k+\frac{n-k-1}{N}\pi\right)\right\}-\left(\frac{n-1}{n}\right)\pi+o(1)$$

$$=\frac{n}{q}\rho^{\frac{q}{n}}2\sin\left(\frac{\pi}{n}\right)-\left(\frac{n-1}{n}\right)\pi+o(1),$$

from which it follows that putting

$$\rho=\left\{\frac{m+\left(\frac{n-1}{2n}\right)\pi}{\frac{n}{q}\sin\left(\frac{\pi}{n}\right)}\right\}^{\frac{n}{q}}, \tag{4.6}$$

$$(4.7) \qquad \Delta_{OP_1QP_2O} \arg Ai(z) = 2m\pi + o(1).$$

Consequently, we obtain the following result on the distribution of zeros of $Ai(z)$ on the Stokes lines.

THEOREM 3. *There exists an integer m_0 such that for any integer $m \geqq m_0$ exactly m zeros of $Ai(z)$ are located on each Stokes line* $\arg z = \frac{\pi}{q}(2k+1)$ $(k=N, N+1,..., q-N-1)$ *in the disk $|z|<\rho$, ρ denoting the number* (4.6).

5. Lommel's method

The well-known Lommel's method is very effective for the investigation of the location of zeros of $Ai(z)$ satisfying a second order linear differential equation of the form (1.1) (see [6, 7]). As for the extended Airy function of the first kind, we can only use Lommel's method to investigate the location of N-zeros of $Ai(z)$.

Let a be an N-zero of $Ai(z)$, i.e.,

$$(5.1) \qquad Ai(a) = Ai'(a) = \cdots = Ai^{(N-1)}(a) = 0.$$

From the definition (2.12) we easily see that $\overline{Ai^{(k)}(z)} = A^{(k)}(\bar{z})$ $(k=0, 1,...)$, which implies that $\bar{a}$ is also an N-zero of $Ai(z)$.

By the change of variables $z=ax$, where x is a real variable, the linear differential equation (1.1) becomes

$$(5.2) \qquad x^n \frac{d^n}{dx^n}(Ai(ax)) - a^q x^q Ai(ax) = 0.$$

This and the integration by parts lead to

$$\begin{aligned}(5.3) \qquad a^q \int_{x_0}^{1} x^{q-n} Ai(ax)Ai(bx)dx &= \int_{x_0}^{1} \frac{d^n}{dx^n}(Ai(ax))Ai(bx)dx \\ &= [a^{n-1}Ai^{(n-1)}(ax)Ai(bx) + (-1)a^{n-2}bAi^{(n-2)}(ax)Ai'(bx) \\ &\quad + \cdots + (-1)^{N-1}a^N b^{N-1} Ai^{(N)}(ax)Ai^{(N-1)}(bx)]_{x_0}^{1} \\ &\quad + (-1)^N \int_{x_0}^{1} \frac{d^N}{dx^N}(Ai(ax)) \frac{d^N}{dx^N}(Ai(bx))dx,\end{aligned}$$

where b is an arbitrary complex number and x_0 is an arbitrary real number. Interchanging a with b in the above, an another relation can be obtained. From two relations just derived, we then obtain the so-called Green's symmetric identities

$$(5.4)\qquad (a^q - b^q)\int_{x_0}^1 x^{q-n} Ai(ax)Ai(bx)\,dx$$
$$= [\{a^{n-1}Ai^{(n-1)}(ax)Ai(bx) - b^{n-1}Ai(ax)Ai^{(n-1)}(bx)\}$$
$$+ (-1)\{a^{n-2}bAi^{(n-2)}(ax)Ai'(bx) - ab^{n-2}Ai'(ax)Ai^{(n-2)}(bx)\}$$
$$+$$
$$\vdots$$
$$+ (-1)^{N-1}\{a^N b^{N-1}Ai^{(N)}(ax)Ai^{(N-1)}(bx)$$
$$- a^{N-1}b^N Ai^{(N-1)}(ax)Ai^{(N)}(bx)\}]_{x_0}^1$$

$$(5.5)\qquad (a^q - b^q)\int_{x_0}^1 \frac{d^N}{dx^N}(Ai(ax))\frac{d^N}{dx^N}(Ai(bx))\,dx$$
$$= [\{a^{q+N-1}b^N Ai^{(N-1)}(ax)Ai^{(N)}(bx) - a^N b^{q+N-1}Ai^{(N)}(ax)Ai^{(N-1)}(bx)\}$$
$$+ (-1)\{a^{q+N-2}b^{N+1}Ai^{(N-2)}(ax)Ai^{(N+1)}(bx)$$
$$- a^{N+1}b^{q+N-2}Ai^{(N+1)}(ax)Ai^{(N-2)}(bx)\}$$
$$+$$
$$\vdots$$
$$+ (-1)^{N-1}\{a^q b^{n-1}Ai(ax)Ai^{(n-1)}(bx)$$
$$- a^{n-1}b^q Ai^{(n-1)}(ax)Ai(bx)\}]_{x_0}^1.$$

Assuming that $a=re^{i\theta}$ is a nonreal N-zero of $Ai(z)$ and putting $b=\bar{a}$ in these identities (5.4) and (5.5), we can investigate N-zero-free domains of $Ai(z)$. For instance, if $0<\theta<\frac{n}{2q}\pi$, then, taking account of (3.11) and letting x_0 tend to infinity in (5.4), we have

$$(5.6)\qquad (a^q - \bar{a}^q)\int_1^\infty x^{q-n}|Ai(ax)|^2 dx = 0,$$

which is a contradiction. Thus there are no N-zeros of $Ai(z)$ in the sectorial domain $0<\arg z<\frac{n}{2q}\pi$, and like this, we can conclude from (5.6) that there are no N-zeros of $Ai(z)$ in sectorial domains where $Ai(z)\to 0$ as $z\to\infty$.

We now put $x_0=0$ and then obtain

$$(5.7)\qquad \sin q\theta\int_0^1 x^{q-n}|Ai(ax)|^2 dx$$
$$= (-1)\sin(n-1)\theta\, r^{n-1-q}Ai^{(n-1)}(0)Ai(0)$$
$$+ (-1)^2\sin(n-3)\theta\, r^{n-1-q}Ai^{(n-2)}(0)Ai(0)$$
$$+$$
$$\vdots$$
$$+ (-1)^N\sin\theta\, r^{n-1-q}Ai^{(N)}(0)Ai^{(N-1)}(0),$$

(5.8)
$$\begin{aligned}\sin q\theta \int_0^1 |Ai^{(N)}(ax)|^2 dx &= (-1)\sin(q-1)\theta\, r^{-1} Ai^{(N)}(0) Ai^{(N-1)}(0) \\ &\quad + (-1)^2 \sin(q-3)\theta\, r^{-1} Ai^{(N-1)}(0) Ai^{(N-2)}(0) \\ &\quad + \\ &\quad \vdots \\ &\quad + (-1)^N \sin(q-n+1)\theta\, r^{-1} Ai^{(n-1)}(0) Ai(0).\end{aligned}$$

We here calculate $Ai^{(j-1)}(0)Ai^{(n-j)}(0)$ $(j=1, 2,\dots, N)$. Since

(5.9)
$$\begin{aligned}Ai^{(j-1)}(0) &= \gamma(-1)^{n-j+1} q^{\frac{n}{q}(n-j)} P_{j-1}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1}) \\ &\quad \times \left\{\prod_{i=1}^{n} \Gamma\left(1 + \frac{i-n+j-1}{q}\right)\right\}^{-1} (j-1)! \\ &\qquad (j = 1, 2,\dots, n),\end{aligned}$$

(5.10)
$$\begin{aligned}&Ai^{(j-1)}(0) Ai^{(n-j)}(0) \\ &= \gamma^2(-1)^{n+1} q^{\frac{n}{q}(n-1)} \{P_{j-1}(\omega_q^{-N+1},\dots, 1,\dots, \omega_q^{N-1})\}^2 \\ &\quad \times \left\{\prod_{i=1}^{n} \Gamma\left(1 + \frac{i-j}{q}\right) \Gamma\left(1 + \frac{i-n+j-1}{q}\right)\right\}^{-1} (j-1)!(n-j)! \\ &< 0 \qquad (j = 1, 2,\dots, N),\end{aligned}$$

where we have used the symmetric relations (2.11). Taking account of (5.10), we can see that sectorial domains where one of the relations (5.7) and (5.8) does not hold are N-zero-free domains of $Ai(z)$. For instance, it is easily seen that the sectorial domains

(5.11)
$$\left\{\theta \mid (-1)\frac{\sin(n-1)\theta}{\sin q\theta} > 0,\ (-1)^2 \frac{\sin(n-3)\theta}{\sin q\theta} > 0,\dots, (-1)^N \frac{\sin\theta}{\sin q\theta} > 0\right\}$$

and

(5.12)
$$\left\{\theta \mid (-1)\frac{\sin(q-1)\theta}{\sin q\theta} > 0,\ (-1)^2 \frac{\sin(q-3)\theta}{\sin q\theta} > 0,\dots, (-1)^N \frac{\sin(q-n+1)\theta}{\sin q\theta} > 0\right\}$$

are N-zero-free domains. We have already shown that zeros of $Ai(z)$ are located on the Stokes lines $\arg z = \frac{\pi}{q}(2k+1)$ ($k=N, N+1,\dots, q-N-1$), where $\sin q\theta$ is vanishing. We can therefore mention that as long as either of the right hand sides of (5.7) and (5.8) is not vanishing on a Stokes line, there exist no N-zeros of $Ai(z)$ on that Stokes line. As an example illustrating the above fact, we consider a case when $n=4$ and $q=8$. In this case the Stokes lines are given by

$$\arg z = \frac{5}{8}\pi,\ \frac{7}{8}\pi,\ \frac{9}{8}\pi,\ \frac{11}{8}\pi.$$

Only on the Stokes line $\arg z = \frac{5}{8}\pi$ in the upper half-plane we have

$$(-1)\sin 3\theta > 0, \quad (-1)^2 \sin\theta > 0$$

and also

$$(-1)\sin 7\theta < 0, \quad (-1)^2 \sin 5\theta < 0.$$

From this, we can insist that there exist no 2-zeros of $Ai(z)$ on the Stokes lines $\arg z = \frac{5}{8}\pi$ and $\frac{11}{8}\pi$.

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