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Estimates for the Coincidence Sets of Solutions of Elliptic Variational Inequalities

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1. Introduction

In this paper, we shall be concerned with the following elliptic variational inequalities with obstacle Ψ

$$\text{(VI)} \qquad \begin{cases} -\Delta u + \alpha u \geqq f & \text{in } \Omega, \\ u \geqq \Psi & \text{in } \Omega, \\ (u - \Psi)(-\Delta u + \alpha u - f) = 0 & \text{in } \Omega \end{cases}$$

under the three types of boundary conditions

$$\text{(DC)} \qquad u = \psi \quad \text{on } \Gamma,$$

$$\text{(NC)} \qquad \frac{\partial u}{\partial n} = \phi \quad \text{on } \Gamma$$

and

$$\text{(SC)} \qquad u \geqq \psi, \quad \frac{\partial u}{\partial n} \geqq \phi, \quad (u - \psi)\left(\frac{\partial u}{\partial n} - \phi\right) = 0 \quad \text{on } \Gamma,$$

where Ω is a bounded domain in R^N with smooth boundary Γ, n is the unit outer normal to Γ, Δ denotes the Laplace operator and α is a positive constant. The boundary conditions (DC), (NC) and (SC) are the Dirichlet condition, the Neumann condition and the Signorini condition, respectively. The variational inequalities (VI) have been investigated by many authors. For instance, we refer to the papers [3], [4] and [6]. Applications of the variational inequalities (VI) to physical problems have been given in [1] and [5].

Given a solution u of (VI), the domain Ω is divided into two parts Ω_1 and Ω_2 such that

$$\Omega_1 = \{x \in \Omega;\ u(x) = \Psi(x)\},$$

$$\Omega_2 = \{x \in \Omega;\ u(x) > \Psi(x)\}.$$

Ω_1 is called the coincidence set of u. It is of interest to give an estimate of the size of Ω_1. Recently, A. Bensoussan, H. Brézis and A. Friedman [2] gave an estimate of the size of Ω_1 under the Dirichlet boundary condition and N. Yamada [7] obtained an estimate under the other boundary conditions. These estimates are independent of α.

The purpose of this paper is to give an estimate depending on α such that the coincidence set Ω_1 converges to the whole set Ω as $\alpha \to \infty$, and to study the behavior of solutions of (VI) near the boundary Γ. To prove our results we shall make use of a comparison theorem (Theorem 2.1) and comparison functions constructed by using the Bessel functions.

2. Notation and Preliminaries

Let Ω be a bounded domain with smooth boundary Γ and let $W^{k,p}(\Omega)$ and $W^{k,p}(\Gamma)$ be the usual Sobolev spaces. For a maximal monotone graph β in R^2 with $0 \in \beta(0)$ we put

$$\begin{aligned}
&\beta^+(r) = \max\{z;\ z \in \beta(r)\} && \text{if} \quad r \in D(\beta),\\
&\beta^-(r) = \min\{z;\ z \in \beta(r)\} && \text{if} \quad r \in D(\beta),\\
&\beta^+(r) = \beta^-(r) = +\infty && \text{if} \quad r \notin D(\beta) \text{ and } r \geqq \sup D(\beta),\\
&\beta^+(r) = \beta^-(r) = -\infty && \text{if} \quad r \notin D(\beta) \text{ and } r \leqq \inf D(\beta),
\end{aligned}$$

where $D(\beta)$ is the domain of β.

We assume

$$f \in L^\infty(\Omega), \quad \Psi \in W^{2,\infty}(\Omega) \text{ and } \phi,\ \psi \in W^{1,\infty}(\Gamma) \tag{2.1}$$

and put

$$K = \{v \in W^{1,2}(\Omega);\ v \geqq \Psi \text{ a.e. in } \Omega\},$$

which is a closed convex set in $W^{1,2}(\Omega)$.

We consider the following elliptic variational inequalities

$$\int_\Omega (-\Delta u + \alpha u)(v - u)dx \geqq \int_\Omega f(v - u)dx \qquad \text{for any} \quad v \in K \tag{2.2}$$

with the boundary condition

$$-\frac{\partial u}{\partial n} + \phi \in \beta(u - \psi) \qquad \text{a.e. on} \quad \Gamma, \tag{2.3}$$

where α is a positive constant and n is the unit outer normal to Γ.

In the case

$$(2.4) \qquad \beta(r) = \begin{cases} (-\infty, +\infty) & \text{if} \quad r = 0, \\ \phi & \text{if} \quad r \neq 0, \end{cases}$$

the boundary condition (2.3) is the Dirichlet condition (DC).

In the case

$$(2.5) \qquad \beta(r) = 0 \qquad \text{for any} \quad r \in R^1,$$

the boundary condition (2.3) is the Neumann condition (NC).

In the case

$$(2.6) \qquad \beta(r) = \begin{cases} 0 & \text{if} \quad r > 0, \\ (-\infty, 0] & \text{if} \quad r = 0, \\ \phi & \text{if} \quad r < 0, \end{cases}$$

the boundary condition is the Signorini condition (SC).

By using the same method as in [3], we see that the problem (2.2) and (2.3) has a unique solution u in $W^{2,2}(\Omega)$ which is continuous on $\bar{\Omega}$, provided that the condition (2.1) and

$$(2.7) \qquad -\frac{\partial \Psi}{\partial n} + \beta^-(\Psi - \psi) \leqq \phi \qquad \text{a.e. on} \quad \Gamma$$

are satisfied. Hence, the problems (VI) with the three types of the boundary conditions (DC), (NC) and (SC) have a unique solution u which should be understood in the sense of the solution of the problem (2.2) and (2.3).

The following comparison theorem will be used in the proof of our main theorems. For the proof we refer to [3] and [7].

THEOREM 2.1. *For* $\tilde{f} \in L^\infty(\Omega)$, $\tilde{\Psi} \in W^{2,\infty}(\Omega)$ *and* $\tilde{\phi}, \tilde{\psi} \in W^{1,\infty}(\Gamma)$ *let* $\tilde{u} \in W^{2,2}(\Omega) \cap C(\bar{\Omega})$ *be a solution of the inequalities*

$$\begin{cases} -\Delta \tilde{u} + \alpha \tilde{u} \geqq \tilde{f}, \ \tilde{u} \geqq \tilde{\Psi} & \text{in} \quad \Omega, \\ -\dfrac{\partial \tilde{u}}{\partial n} + \tilde{\phi} \in \tilde{\beta}(\tilde{u} - \tilde{\psi}) & \text{on} \quad \Gamma, \end{cases}$$

where $\tilde{\beta}$ *is a maximal monotone graph in* R^2.

If $f \leqq \tilde{f}$, $\Psi \leqq \tilde{\Psi}$ *a.e. in* Ω, $\phi \leqq \tilde{\phi}$, $\psi \leqq \tilde{\psi}$ *a.e. on* Γ *and* $\tilde{\beta}^- \leqq \beta^+$, *then for the solution* u *of* (2.2) *and* (2.3) *we obtain* $u \leqq \tilde{u}$ *a.e. in* Ω.

3. Comparison functions

In this section we construct a comparison function which plays an important role in the proof of our main theorems.

Consider the initial value problem for the ordinary differential equation

$$\text{(ODE)} \qquad \begin{cases} \mu''(t) + \dfrac{N-1}{t}\mu'(t) - \alpha\mu(t) = \gamma & \text{in} \quad (0, \infty), \\ \mu(0) = \mu'(0) = 0, \end{cases}$$

where α and γ are positive constants. This problem (ODE) has a solution μ of the following form

$$\text{(3.1)} \qquad \begin{aligned} \mu(t) &= \frac{\gamma}{\alpha}\left\{\Gamma\left(\frac{N}{2}\right)\left(\frac{\sqrt{\alpha}t}{2}\right)^{1-\frac{N}{2}} I_{\frac{N}{2}-1}(\sqrt{\alpha}t) - 1\right\} \\ &= \frac{\gamma}{\alpha}\left\{\Gamma\left(\frac{N}{2}\right)\Sigma_{m=0}^{\infty}\frac{1}{m!\Gamma\left(\frac{N}{2}+m\right)}\left(\frac{\sqrt{\alpha}t}{2}\right)^{2m} - 1\right\}, \end{aligned}$$

where $I_\nu(t)$ is the modified Bessel function of the first kind of order ν and $\Gamma(t)$ is the gamma function (for the definitions, see [8]). By using the relation $\{t^{-\nu}I_\nu(t)\}' = t^{-\nu}I_{\nu+1}(t)$, we get

$$\text{(3.2)} \qquad \begin{aligned} \mu'(t) &= \frac{\gamma}{\sqrt{\alpha}}\Gamma\left(\frac{N}{2}\right)\left(\frac{\sqrt{\alpha}t}{2}\right)^{1-\frac{N}{2}} I_{\frac{N}{2}}(\sqrt{\alpha}t) \\ &= \frac{\gamma}{2}\Gamma\left(\frac{N}{2}\right)\Sigma_{m=0}^{\infty}\frac{1}{m!\Gamma\left(\frac{N}{2}+m+1\right)}\left(\frac{\alpha}{4}\right)^{m} t^{2m+1}. \end{aligned}$$

It follows from (3.1) and (3.2) that $\mu(t)>0$ and $\mu'(t)>0$ in $(0, \infty)$, and that $\theta\mu'(t)\geqq\mu'(\theta t)$ for $0\leqq\theta\leqq1$ and $t\geqq0$.

LEMMA 3.1. *For any positive constant C there exists a positive constant R_α such that*

(i) $\mu(R_\alpha) = C$ *and* $\mu(t) > C$ *for* $t > R_\alpha$,

(ii) $R_\alpha < \left(\dfrac{2NC}{\gamma}\right)^{\frac{1}{2}}$,

(iii) $R_\alpha \nearrow \left(\dfrac{2NC}{\gamma}\right)^{\frac{1}{2}}$ *as* $\alpha \searrow 0$,

(iv) $R_\alpha \searrow 0$ *as* $\alpha \nearrow \infty$.

PROOF. Since $\mu(t)\nearrow\infty$ as $t\nearrow\infty$ and $\mu(0)=0$, there exists a positive constant R_α such that $\mu(R_\alpha)=C$ and $\mu(t)>C$ for $t>R_\alpha$. Noting the relation

$$\mu\left(\left(\frac{2NC}{\gamma}\right)^{\frac{1}{2}}\right)=C+\frac{\gamma}{\alpha}\Gamma\left(\frac{N}{2}\right)\sum_{m=2}^{\infty}\frac{1}{m!\,\Gamma\left(\frac{N}{2}+m\right)}\left(\frac{\alpha}{4}\right)^m\left(\frac{2NC}{\gamma}\right)^m>C,$$

we obtain $R_\alpha<\left(\frac{2NC}{\gamma}\right)^{\frac{1}{2}}$.

Next it follows from $\mu(R_\alpha)=C$ that

$$\Gamma\left(\frac{N}{2}\right)\sum_{m=1}^{\infty}\frac{1}{m!\,\Gamma\left(\frac{N}{2}+m\right)}\alpha^{m-1}\left(\frac{R_\alpha}{2}\right)^{2m}=\frac{C}{\gamma},$$

which implies that R_α is strictly decreasing in α. Since $\mu(R_\alpha)=C$, we get

$$\begin{aligned}\left|R_\alpha^2-\frac{2NC}{\gamma}\right| &\leqq 2N\Gamma\left(\frac{N}{2}\right)\sum_{m=2}^{\infty}\frac{1}{m!\,\Gamma\left(\frac{N}{2}+m\right)}\alpha^m\left(\frac{R_\alpha}{2}\right)^{2m}\\ &\leqq 2N\Gamma\left(\frac{N}{2}\right)\left(\frac{NC}{\gamma}\right)^2\alpha^2 e^{\frac{\alpha NC}{\gamma}}.\end{aligned}$$

Hence, $R_\alpha\nearrow\left(\frac{2NC}{\gamma}\right)^{\frac{1}{2}}$ as $\alpha\searrow 0$. Finally, $R_\alpha<2\alpha^{-\frac{1}{4}}$ for sufficiently large α, since

$$\mu(2\alpha^{-\frac{1}{4}})=\frac{\gamma}{\alpha}\Gamma\left(\frac{N}{2}\right)\sum_{m=1}^{\infty}\frac{1}{m!\,\Gamma\left(\frac{N}{2}+m\right)}\alpha^{\frac{m}{2}}>C$$

for sufficiently large α. Therefore, $R_\alpha\searrow 0$ as $\alpha\nearrow\infty$.

LEMMA 3.2. *For any positive constant C there exists a positive constant $\hat{R}_\alpha$ such that*

(i) $\mu'(\hat{R}_\alpha)=C$ *and* $\mu'(t)\geqq C$ *for* $t\geqq\hat{R}_\alpha$,

(ii) $\hat{R}_\alpha<\frac{NC}{\gamma}$,

(iii) $\hat{R}_\alpha\searrow 0$ *as* $\alpha\nearrow\infty$.

PROOF. Note that $\mu'(0)=0$ and $\mu'(t)\nearrow\infty$ as $t\nearrow\infty$ by the relation (3.2). Hence, there exists a positive constant $\hat{R}_\alpha$ such that $\mu'(\hat{R}_\alpha)=C$ and $\mu'(t)\geqq C$ for $t\geqq\hat{R}_\alpha$.

By the relation

$$\mu'\left(\frac{NC}{\gamma}\right)=C+\frac{\gamma}{2}\Gamma\left(\frac{N}{2}\right)\sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma\left(\frac{N}{2}+m+1\right)}\left(\frac{\alpha}{4}\right)^m\left(\frac{NC}{\gamma}\right)^{2m+1}>C,$$

we get $\hat{R}_\alpha < \dfrac{NC}{\gamma}$.

It follows from $\mu'(\hat{R}_\alpha) = C$ that

$$\frac{\gamma}{2}\Gamma\left(\frac{N}{2}\right)\sum_{m=0}^{\infty}\frac{1}{m!\Gamma\left(\frac{N}{2}+m+1\right)}\left(\frac{\alpha}{4}\right)^m \hat{R}_\alpha^{2m+1} = C.$$

Hence, $\hat{R}_\alpha$ is strictly decreasing in α.

Finally we obtain

$$\mu'(2\alpha^{-\frac{1}{4}}) = \frac{2\gamma}{N}\alpha^{-\frac{1}{4}} + \gamma\Gamma\left(\frac{N}{2}\right)\sum_{m=1}^{\infty}\frac{1}{m!\Gamma\left(\frac{N}{2}+m+1\right)}\alpha^{\frac{m}{2}-\frac{1}{4}} > C$$

for sufficiently large α, which implies that $\hat{R}_\alpha < 2\alpha^{-\frac{1}{4}}$ for sufficiently large α. Therefore, $\hat{R}_\alpha \searrow 0$ as $\alpha \nearrow \infty$.

4. Estimates of the coincidence sets in the interior

In this section, we will always assume that there exists a positive constant γ such that

$$f + \Delta\Psi - \alpha\Psi \leqq -\gamma \quad \text{a.e. in } \Omega \tag{4.1}$$

and we consider the solutions u of (2.2) and (2.3) under the hypotheses (2.7) for the three types β of (2.4), (2.5) and (2.6), that is, u is the solution of (VI) under the three types of boundary conditions (DC), (NC) and (SC).

Put $\bar{u} = u - \Psi$. We see that $\bar{u}$ satisfies the following variational inequalities

$$\begin{cases} -\Delta\bar{u} + \alpha\bar{u} \geqq \bar{f} & \text{in } \Omega, \\ \bar{u} \geqq 0 & \text{in } \Omega, \\ \bar{u}(-\Delta\bar{u} + \alpha\bar{u} - \bar{f}) = 0 & \text{in } \Omega, \\ -\dfrac{\partial\bar{u}}{\partial n} + \bar{\phi} \in \beta(\bar{u} - \bar{\psi}) & \text{on } \Gamma, \end{cases} \tag{4.2}$$

where $\bar{f} = f + \Delta\Psi - \alpha\Psi$, $\bar{\phi} = \phi - \dfrac{\partial\Psi}{\partial n}$ and $\bar{\psi} = \psi - \Psi$.

Let $\mu(t)$ be the solution of (ODE) and for a point $x_0 \in \Omega$ define the function $w(x)$ on $\bar{\Omega}$ by

$$w(x) = \mu(|x - x_0|). \tag{4.3}$$

It is clear that

$$-\Delta w(x)+\alpha w(x)=-\gamma \quad \text{in} \quad \Omega.$$

4.1. The Dirichlet problem

Taking β as in (2.4), the boundary condition yields the Dirichlet condition (DC). In this case we have the following estimate.

THEOREM 4.1. *Assume that* (4.1) *holds and* $\delta_1=\operatorname{ess\cdot sup}_{\Gamma}(\psi-\Psi)>0$. *Then there exists a positive constant* R_α *such that*

(i) $u(x)=\Psi(x)$ *for* $x\in\Omega$ *and* $\operatorname{dist}(x,\Gamma)\geqq R_\alpha$,

(ii) $R_\alpha<\left(\frac{2N}{\gamma}\delta_1\right)^{\frac{1}{2}}$,

(iii) $R_\alpha\nearrow\left(\frac{2N}{\gamma}\delta_1\right)^{\frac{1}{2}}$ *as* $\alpha\searrow 0$,

(iv) $R_\alpha\searrow 0$ *as* $\alpha\nearrow\infty$.

PROOF. Taking $C=\delta_1$ in Lemma 3.1, we obtain a positive constant R_α satisfying $\mu(R_\alpha)=\delta_1$, $\mu(t)>\delta_1$ for $t>R_\alpha$ and (ii)~(iv) in Theorem 4.1.

Let $x_0\in\Omega$ such that $\operatorname{dist}(x_0,\Gamma)\geqq R_\alpha$. The function $w(x)$ defined on $\bar{\Omega}$ by (4.3) satisfies

$$w\geqq 0,\quad -\Delta w+\alpha w=-\gamma\geqq\bar{f}\quad \text{in} \quad \Omega.$$

Since $\mu(R_\alpha)=\delta_1$, $\mu(t)>\delta_1$ for $t>R_\alpha$ and $\operatorname{dist}(x_0,\Gamma)\geqq R_\alpha$, for $x\in\Gamma$ we get

$$w(x)=\mu(|x-x_0|)\geqq\delta_1=\bar{\psi}(x)=\psi(x)-\Psi(x).$$

Hence, by using Theorem 2.1 we obtain $\bar{u}\leqq w$ in Ω, which implies $\bar{u}(x_0)=u(x_0)-\Psi(x_0)\leqq w(x_0)=0$. Thus, $u(x_0)=\Psi(x_0)$.

4.2. The Neumann problem

Taking β as in (2.5), the boundary condition yields the Neumann condition (NC).

For a given $x_0\in\Omega$ we put

$$\theta_0(x_0)=\inf\{\cos(n(x),\,x-x_0);\ x\in\Gamma\}, \tag{4.4}$$

where $n(x)=(n_1(x),n_2(x),\dots,n_N(x))$ is the unit outer normal to Γ at $x\in\Gamma$ and $(n(x),x-x_0)$ denotes the angle between $n(x)$ and $x-x_0$. If Ω is convex, then $\theta_0(x_0)>0$ for $x_0\in\Omega$.

THEOREM 4.2. *Assume that* (4.1) *holds,* Ω *is convex and* $\delta_2=\operatorname{ess\cdot sup}_{\Gamma}$

$\left(\phi - \frac{\partial \Psi}{\partial n}\right) > 0$. *Then there exists a positive constant* $\hat{R}_\alpha$ *such that*

(i) $u(x) = \Psi(x)$ *for* $x \in \Omega$ *and* $\theta_0(x) \operatorname{dist}(x, \Gamma) \geqq \hat{R}_\alpha$,

(ii) $\hat{R}_\alpha < \frac{N}{\gamma} \delta_2$,

(iii) $\hat{R}_\alpha \searrow 0$ *as* $\alpha \nearrow \infty$.

PROOF. Taking $C = \delta_2$ in Lemma 3.2, we obtain a positive constant $\hat{R}_\alpha$ such that $\mu'(\hat{R}_\alpha) = \delta_2$, $\mu'(t) \geqq \delta_2$ for $t \geqq \hat{R}_\alpha$, and (ii) and (iii) are satisfied.

Let $x_0 \in \Omega$ be such that $\theta_0(x_0) \operatorname{dist}(x_0, \Gamma) \geqq \hat{R}_\alpha$. Noting that $0 < \theta_0(x_0) \leqq 1$ and $\theta \mu'(t) \geqq \mu'(\theta t)$ for $0 \leqq \theta \leqq 1$, for $x \in \Gamma$ we get

$$\begin{aligned}(4.5) \qquad \frac{\partial w}{\partial n}(x) &= \textstyle\sum_{i=1}^{N} \frac{\partial w}{\partial x_i}(x) n_i(x) = \mu'(|x - x_0|) \cos(n(x), x - x_0) \\ &\geqq \mu'(|x - x_0|) \theta_0(x_0).\end{aligned}$$

It follows from $\theta_0(x_0)|x - x_0| \geqq \hat{R}_\alpha$ and (4.5) that

$$\frac{\partial w}{\partial n}(x) \geqq \delta_2 \geqq \bar{\phi}(x) = \phi(x) - \frac{\partial \Psi}{\partial n}(x) \qquad \text{on } \Gamma.$$

Hence, by Theorem 2.1 we obtain $u(x_0) = \Psi(x_0)$.

4.3. The Signorini problem

Taking β as in (2.6), we obtain the Signorini boundary condition (SC). In this case we have the following estimate for the coincidence set.

THEOREM 4.3. *Assume that* (4.1) *holds,* Ω *is convex,* $\delta_1 = \operatorname{ess\cdot sup}_\Gamma (\psi - \Psi) > 0$ *and* $\delta_2 = \operatorname{ess\cdot sup}_\Gamma \left(\phi - \frac{\partial \Psi}{\partial n}\right) > 0$. *Then there are positive constants* R_α *and* $\hat{R}_\alpha$ *such that*

(i) $u(x) = \Psi(x)$ *for* $x \in \Omega$ *and* $\operatorname{dist}(x, \Gamma) \geqq \max\left\{R_\alpha, \frac{\hat{R}_\alpha}{\theta_0(x)}\right\}$,
where $\theta_0(x)$ *is the same one as in* (4.4),

(ii) $R_\alpha < \left(\frac{2N}{\gamma}\delta_1\right)^{\frac{1}{2}}$ *and* $\hat{R}_\alpha < \frac{N}{\gamma}\delta_2$,

(iii) $R_\alpha \searrow 0$ *and* $\hat{R}_\alpha \searrow 0$ *as* $\alpha \nearrow \infty$.

PROOF. Taking $C = \delta_1$ in Lemma 3.1 and $C = \delta_2$ in Lemma 3.2, we obtain positive constants R_α and $\hat{R}_\alpha$ such that $\mu(R_\alpha) = \delta_1$, $\mu'(\hat{R}_\alpha) = \delta_2$, and the condition (ii) and (iii) are satisfied. By using the same methods as in the proof of Theorems 4.1 and 4.2, we get

$$w \geqq \bar{\psi},\ \frac{\partial w}{\partial n} \geqq \bar{\phi} \quad \text{on} \quad \Gamma.$$

Hence, we obtain the assertion of Theorem 4.3.

5. Estimates near the boundary

We continue to impose the condition (4.1) on f.

Let $\mu(t)$ be the solution of (ODE) and for arbitrary fixed constant $a>0$ put

$$v(t) = \mu(t-a) \qquad \text{for} \quad t > a.$$

It is easy to see that

$$v''(t) + \frac{N-1}{t} v'(t) - \alpha v(t) < \gamma \qquad \text{in} \quad (a, \infty).$$

For a given $x_0 \in \Gamma$ we define the function $w(x)$ on $\bar{\Omega}$ by

$$w(x) = \begin{cases} 0 & \text{if} \quad |x - x_0| \leqq a, \\ v(|x - x_0|) & \text{if} \quad |x - x_0| > a. \end{cases}$$

Then the function $w(x)$ satisfies the following

$$\text{(5.1)} \qquad \begin{cases} w \in W^{2,2}(\Omega) \cap C(\bar{\Omega}), & \\ w \geqq 0 & \text{in} \quad \Omega, \\ -\Delta w + \alpha w \geqq -\gamma \geqq \bar{f} & \text{in} \quad \Omega, \end{cases}$$

where $\bar{f} = f + \Delta\Psi - \alpha\Psi$. This function $w(x)$ plays an important role in deriving the estimates near the boundary.

In the case of the Dirichlet problem we obtain

Theorem 5.1. *Assume that* (4.1) *holds and* $\delta_1 = \operatorname{ess\cdot sup}_{\Gamma} (\psi - \Psi) > 0$. *Suppose that there exist a point* $x_0 \in \Gamma$ *and a positive number* $r > R_\alpha$, *where* R_α *is the same one as in Theorem* 4.1, *such that* $\psi(x) = \Psi(x)$ *for* $x \in \Gamma$ *and* $|x - x_0| \leqq r$. *Then, we have* $u(x) = \Psi(x)$ *for* $x \in \Omega$ *and* $|x - x_0| \leqq a_\alpha = r - R_\alpha$. *Hence,* $a_\alpha \to r$ *as* $\alpha \to \infty$ *by Theorem* 4.1.

Proof. Define the comparison function $w(x)$ on $\bar{\Omega}$ by

$$w(x) = \begin{cases} 0 & \text{if} \quad |x - x_0| \leqq a_\alpha, \\ \mu(|x - x_0| - a_\alpha) & \text{if} \quad |x - x_0| > a_\alpha. \end{cases}$$

We shall show that $w \geqq \bar{\psi} = \psi - \Psi$ on Γ. In case $x \in \Gamma$ and $|x - x_0| > r$, we get

$$\begin{aligned} w(x) &= \mu(|x-x_0|-a_\alpha) \geqq \mu(r-a_\alpha) \\ &= \mu(R_\alpha) = \delta_1 \geqq \bar{\psi}(x). \end{aligned}$$

In case $x \in \Gamma$ and $|x-x_0| \leqq r$, it is clear that $w(x) \geqq \bar{\psi}(x)$. Hence, by Theorem 2.1 we get $u(x_0) = \Psi(x_0)$.

Next we shall consider the Neumann problem. For a given $x_0 \in \Gamma$ and $r>0$ define $\theta_0(x_0; r)$ by

$$\theta_0(x_0; r) = \inf\{\cos(n(x), x-x_0);\ x \in \Gamma \quad \text{and} \quad |x-x_0| \geqq r\}.$$

If Ω is strictly convex, that is, Γ does not contain any line segment, then $\theta_0(x_0; r) > 0$.

THEOREM 5.2. *Assume that* (4.1) *holds, Ω is strictly convex and* $\delta_2 = \operatorname{ess\cdot sup}_{\Gamma}\left(\phi - \frac{\partial \Psi}{\partial n}\right) > 0$. *Suppose that there are a point $x_0 \in \Gamma$ and a number $r > \hat{R}_\alpha/\theta_0(x_0; r)$, where $\hat{R}_\alpha$ is the same as in Theorem* 4.2, *such that* $\phi(x) = \frac{\partial \Psi}{\partial n}(x)$ *for $x \in \Gamma$ and $|x-x_0| \leqq r$. Then $u(x) = \Psi(x)$ for $x \in \Omega$ and $|x-x_0| \leqq \hat{a}_\alpha = r - \hat{R}_\alpha/\theta_0(x_0; r)$. Hence, $\hat{a}_\alpha \to r$ as $\alpha \to \infty$ by Theorem* 4.2.

PROOF. Define the comparison function $w(x)$ on $\bar{\Omega}$ by

$$w(x) = \begin{cases} 0 & \text{if} \quad |x-x_0| \leqq \hat{a}_\alpha, \\ \mu(|x-x_0|-\hat{a}_\alpha) & \text{if} \quad |x-x_0| > \hat{a}_\alpha. \end{cases}$$

In case $x \in \Gamma$ and $|x-x_0| \geqq r$, we get

$$\begin{aligned} \frac{\partial w}{\partial n}(x) &= \mu'(|x-x_0|-\hat{a}_\alpha)\cos(n(x), x-x_0) \\ &\geqq \mu'(|x-x_0|-\hat{a}_\alpha)\theta_0(x_0; r) \geqq \mu'(r-\hat{a}_\alpha)\theta_0(x_0; r) \\ &\geqq \mu'((r-\hat{a}_\alpha)\theta_0(x_0; r)) = \mu'(\hat{R}_\alpha) = \delta_2 \\ &\geqq \bar{\Phi}(x) = \phi(x) - \frac{\partial \Psi}{\partial n}(x). \end{aligned}$$

In case $x \in \Gamma$ and $|x-x_0| < r$, it is clear that $\frac{\partial w}{\partial n}(x) \geqq \bar{\Phi}(x)$, since $\bar{\Phi}(x) = 0$ for $x \in \Gamma$ and $|x-x_0| < r$. Thus, the proof is complete.

Finally, for the Signorini problem we have

THEOREM 5.3. *Assume that* (4.1) *holds, Ω is strictly convex,* $\delta_1 = \operatorname{ess\cdot sup}_{\Gamma}(\psi - \Psi) > 0$ *and* $\delta_2 = \operatorname{ess\cdot sup}_{\Gamma}\left(\phi - \frac{\partial \Psi}{\partial n}\right) > 0$. *Suppose that there are a point $x_0 \in \Gamma$ and a number $r > \max\{R_\alpha, \hat{R}_\alpha/\theta_0(x_0; r)\}$, where R_α and $\hat{R}_\alpha$ are the same as in*

Theorem 4.3, *such that* $\psi(x)=\Psi(x)$ *and* $\phi(x)=\frac{\partial\Psi}{\partial n}(x)$ *for* $x\in\Gamma$ *and* $|x-x_0|\leqq r$. *Then* $u(x)=\Psi(x)$ *for* $x\in\Omega$ *and* $|x-x_0|\leqq\bar{a}_\alpha=r-\max\{R_\alpha, \hat{R}_\alpha/\theta_0(x_0; r)\}$. *Hence,* $\bar{a}_\alpha\to r$ *as* $\alpha\to\infty$ *by Theorem* 4.3.

PROOF. Define the comparison function $w(x)$ on $\bar{\Omega}$ by

$$w(x)=\begin{cases} 0 & \text{if } |x-x_0|\leqq \bar{a}_\alpha, \\ \mu(|x-x_0|-\bar{a}_\alpha) & \text{if } |x-x_0|\geqq \bar{a}_\alpha. \end{cases}$$

The same calculations as in the proof of Theorems 5.1 and 5.2 show that

$$w(x)\geqq\bar{\psi}(x) \quad\text{and}\quad \frac{\partial w}{\partial n}(x)\geqq\bar{\phi}(x) \qquad \text{on } \Gamma.$$

This completes the proof of Theorem 5.3.

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