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Isometry Groups of Negatively Pinched 3-Manifolds

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§1. Introduction

Let M be a Riemannian manifold. For a 2-plane π tangent to M, let $k(\pi)$ denote the sectional curvature at π. M is said to be *negatively pinched* if there exist negative numbers c_1 and c_2 such that $c_1 \leq k(\pi) \leq c_2 < 0$ for all π.

In [8], p. 152, Margulis gave some unsolved problems. Among them we find:

PROBLEM. Let M be a simply connected, symmetric space of noncompact type. Let Γ_1 and Γ_2 be discrete groups of isometries of M with the factor spaces of finite volume. Is the ratio of the volume $\mathrm{vol}(M/\Gamma_1)$: $\mathrm{vol}(M/\Gamma_2)$ rational?

The main purpose of this paper is to establish the following theorem by which in particular the analogous problem to the above for negatively pinched 3-manifolds can be reduced to the case of hyperbolic spaces.

THEOREM. *Let M be a complete, simply connected, negatively pinched Riemannian manifold of dimension three. Suppose that there exists a discrete group Γ of isometries of M such that the factor space M/Γ is of finite volume. Then either M is of constant curvature or the group $I(M)$ of all isometries of M is discrete.*

Under the assumptions of the theorem, let G denote the identity component of the Lie group $I(M)$. Then by Heintze, G is a semisimple Lie group without compact factor and with trivial center, see §2, Lemma 6. Now suppose that $G \neq \{1\}$. Then we can see easily that G must be isomorphic with the adjoint group of either $SL(2, \mathbf{C})$ or $SL(2, \mathbf{R})$, and if we can exclude the latter case, the theorem follows directly. For the purpose we start with a little more general setting and construct a warped product of a certain symmetric space and a straight line $\mathbf{R}$, which does not admit a discrete group Γ of isometries with the factor space of finite volume. The exact statement is given in PROPOSITION of §3.

§2. Lemmas

For the later use we shall state some known results.

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LEMMA 1. *Let R be a connected Riemannian manifold, and let H be a closed subgroup of $I(R)$. Then for any $p \in R$ the orbit $H(p)$ is closed.*

See e.g. Theorem 2.2, Helgason [6], p. 167.

LEMMA 2. *Let R be a connected Riemannian manifold, and let H be a closed subgroup of $I(R)$. Suppose that there exists $p \in R$ such that* $\dim H - \dim H_p \geq \dim R$, *where H_p denotes the stability group at p. Then*

$$\dim H - \dim H_p = \dim R,$$

and H is transitive on R.

This follows directly from Lemma 1.

LEMMA 3. *Let R be a Riemannian manifold with the metric tensor g. A transformation φ of R onto itself is said to be homothetic if there is a positive number c such that $\varphi^* g = cg$. If R is complete and is not locally flat, then any homothetic transformation of R is an isometry.*

See e.g. Kobayashi-Nomizu [7], p. 242.

LEMMA 4. *Let R be a complete, simply connected Riemannian manifold of non-positive sectional curvature. Let σ be an isometry of R. Let $F(\sigma)$ denote the set of all fixed points under σ. Then $F(\sigma)$ is totally geodesic.*

PROOF. Suppose that $\sigma p = p$ and $\sigma q = q \neq p$. Then σ fixes every point of the unique geodesic passing through p and q. Q. E. D.

LEMMA 5. *Let G be a connected, non-compact, simple Lie group with trivial center. Let K be a maximal compact subgroup of G. Then*

(i) *K coincides with its normalizer in G.*

(ii) *Let $\mathfrak{G}$ and $\mathfrak{K}$ denote the Lie algebras of G and K, respectively. Let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ be a Cartan decomposition of $\mathfrak{G}$. Then the representation ψ of K given by*

$$\psi(\eta) = \mathrm{Ad}(\eta)|_{\mathfrak{P}} \qquad (\eta \in K)$$

is faithful and irreducible.

(iii) *The Killing form of $\mathfrak{G}$ gives rise to a Riemannian metric g of the factor space G/K such that $S = G/K$ is an irreducible symmetric space of non-compact type. In this case, G is the identity component of the isometry group $I(S)$ and the factor group $I(S)/G$ is finite.*

(iv) *Let N be a Riemannian manifold. Suppose that G acts transitively on N, and that for some $p \in N$ the stability group G_p coincides with K. Then after identifying S and N by the map $\sigma K \mapsto \sigma p$ $(\sigma \in G)$, the metric tensor of N is* *given by fg, where f is a positive constant.*

See Helgason [6], Wolf [9] and Goto-Grosshans [4].

Next, let M be a complete, simply connected, Riemannian manifold of non-positive sectional curvature. The set $M^{(\infty)}$ of all asymptote classes of geodesic rays in M forms the *boundary* of M. The set $\overline{M}=M \cup M^{(\infty)}$ has a natural topology as a closed disk in $\mathbf{R}^{n+1}$, where $\dim M=n+1$, and $M^{(\infty)}$ is homeomorphic to the bounding sphere S^n. If for any pair of points x and y in $M^{(\infty)}$, there exists a geodesic connecting x and y, then the manifold M is said to satisfy the *visibility axiom*. If the sectional curvature of M is bounded above by a negative constant, then the visibility axiom holds for M. Let H be a subgroup of $I(M)$, and let p be in M. The intersection of the closure of the orbit $H(p)$ in $\overline{M}$ with $M^{(\infty)}$ does not depend on the choice of p, is called the *limit set* of H, and is denoted by $L(H)$. If H is discrete and the factor space M/H is of finite volume, then $L(H)=M^{(\infty)}$. (The details of these results can be found in Eberlein-O'Neill [2].)

LEMMA 6 (Heintze [5]). *Let M be a complete, simply connected Riemannian manifold of non-positive sectional curvature satisfying the visibility axiom. Suppose that there exists a discrete subgroup Γ of $I(M)$ such that $L(\Gamma)=M^{(\infty)}$. Let G denote the identity component of $I(M)$. Then G is a semisimple Lie group with no compact factor and with trivial center.*

§3. A warped product

PROPOSITION. *Let M be a complete, simply connected, negatively pinched, Riemannian manifold of dimension $n+1$. Let G denote the identity component of $I(M)$. Let K be a maximal compact subgroup of G. Suppose that G is a non-compact, simple Lie group with trivial center, that G is not transitive on M, and that*

$$\dim G - \dim K = n.$$

Let $S=G/K$ denote the irreducible symmetric space of non-compact type with the metric tensor ds^2 given by the Killing form of the Lie algebra of G. Then

(i) *S is a symmetric space of rank one, and M is a warped product of S and $\mathbf{R}$, i.e. M can be identified with the direct product space $S\times\mathbf{R}$ with the metric given by*

$$f(t)^2ds^2 + dt^2,$$

where $f(t)$ is a positive, strictly convex smooth function on $\mathbf{R}$ with a unique minimum.

(ii) *G is of finite index in $I(M)$, and for any discrete subgroup Γ of $I(M)$, the volume of the factor space M/Γ is infinite.*

PROOF. We fix a maximal compact subgroup K of G, once and for all, and put

$$B = \{q \in M;\ \eta q = q \text{ for all } \eta \in K\}.$$

Let p be in M. Then the stability group G_p, being compact, is contained in some maximal compact subgroup, say $\sigma K \sigma^{-1}$ for $\sigma \in G$. If $G_p \neq \sigma K \sigma^{-1}$, i.e. $\dim G_p < \dim K$, then $\dim G - \dim G_p \geq \dim M$, and G acts transitively on M, by Lemma 2, which is a contradiction. Hence $G_p = \sigma K \sigma^{-1}$. Then for any $\eta \in K$ we have $\sigma \eta \sigma^{-1} p = p$, i.e. $\eta \sigma^{-1} p = \sigma^{-1} p$. Hence $\sigma^{-1} p \in B$ and $p \in \sigma B$. Thus every point of M can be written in a form σb ($\sigma \in G$, $b \in B$).

Let φ denote the map

$$(G/K) \times B \ni (\sigma K, b) \longmapsto \sigma b \in M.$$

We have seen that φ is onto. In order to prove that φ is one-one, let us suppose that

$$\sigma b = \sigma' b' \qquad \text{for} \quad \sigma, \sigma' \in G \quad \text{and} \quad b, b' \in B.$$

Then $b = (\sigma^{-1}\sigma')b'$ and $G_b = (\sigma^{-1}\sigma')G_{b'}(\sigma^{-1}\sigma')^{-1}$. Since $G_b = G_{b'} = K$, we have that $\sigma^{-1}\sigma'$ is contained in the normalizer of K. By Lemma 5 (i), we get $\sigma^{-1}\sigma' \in K$ and $b = b'$.

Then φ is a continuous one-one map from $S \times B$ onto M. Because G is not transitive on M, the set B contains at least two points. On the other hand, $B = \bigcap_{\eta \in K} F(\eta)$ is totally geodesic, by Lemma 4. Hence B contains a non-constant complete geodesic γ. Suppose that B contains a point which does not lie on γ. Then B contains an open 2-cell D, and the restriction of φ on $S \times D$ gives rise to a one-one smooth map from $\mathbf{R}^{n+2}$ into $\mathbf{R}^{n+1}$, which is impossible. Hence B coincides with γ. Thus

$$\varphi_0 \colon S \times \mathbf{R} \ni (\sigma K, t) \longmapsto \sigma\gamma(t) \in M$$

is a homeomorphism by Brouwer's invariance theorem of domains, where we suppose that the parameter t is given by arclength.

For each $t \in \mathbf{R}$, we set $S(t) = G(\gamma(t))$. Then the map $\varphi(t) \colon S \ni \sigma K \to \sigma\gamma(t) \in S(t) \subset M$ is smooth and is a homeomorphism. By the implicit function theorem, for some $\sigma_0 \in G$, the infinitesimal linear map $d\varphi(t)_{\sigma_0 K}$ is one-one.

Let τ be in G. Since $\tau\sigma_0^{-1}$ is a diffeomorphism of M and $(\tau\sigma_0^{-1})(\sigma_0\gamma(t)) = \tau\gamma(t)$, we see immediately that $d\varphi(t)_{\tau K}$ is one-one. Hence $S(t)$ is a submanifold, which is obviously regularly embedded. By Lemma 5 (iv), that $G_{\gamma(t)} = K$ implies that $S(t)$ is an irreducible symmetric space of non-compact type with the metric $f(t)^2ds^2$, where $f(t)$ is a positive number, by identifying S and $S(t)$.

Let σ be in G. Then $(\sigma\gamma)(t)=\sigma(\gamma(t))$ is a geodesic, and for a fixed t we have that $S(t)\cap(\sigma\gamma)=\{\sigma\gamma(t)\}$. We shall see that $S(t)\perp\sigma\gamma$. For that purpose it obviously suffices to show that $S(t)\perp\gamma$ at $\gamma(t)$.

Let us put $T_{\gamma(t)}M=\mathfrak{T}$ and $T_{\gamma(t)}S(t)=\mathfrak{S}\subset\mathfrak{T}$. Also let $\dot{\gamma}$ denote the vector tangent to γ at $\gamma(t)$. Since $K(\gamma(t))=\gamma(t)$, K acts on $\mathfrak{T}$ as a group of isometries. The subspace $\mathfrak{S}$ is invariant under K, and can be identified with $\mathfrak{P}$ in Lemma 5 (ii). Also the action of K on $\mathfrak{S}$ is identified with ψ there. Therefore, the action of K on $\mathfrak{S}$ is irreducible and faithful. On the other hand, K acts trivially on γ, and so on $\dot{\gamma}$. Hence $\mathfrak{S}$ and $\mathbf{R}\dot{\gamma}$ are irreducible, mutually inequivalent invariant subspaces under K. Since $\dim\mathfrak{S}+\dim\mathbf{R}\dot{\gamma}=\dim\mathfrak{T}$, we have that

$$\mathfrak{T}=\mathfrak{S}\oplus\mathbf{R}\dot{\gamma},\qquad \mathfrak{S}\perp\dot{\gamma}.$$

This fact implies, in particular, that φ_0 is a diffeomorphism, and that the metric of M is given by

$$f(t)^2ds^2+dt^2.$$

Therefore, M is a warped product of S and $\mathbf{R}$, see Bishop-O'Neill [1] and Eberlein [2].

Let π be a two-dimensional plane in $\mathfrak{T}=T_{\gamma(t)}M$. Then we can find an orthonormal basis $\{c\dot{\gamma}+V, W\}$ of π such that $c\in\mathbf{R}$ and $V, W\in\mathfrak{S}$. When $\{V, W\}$ is linearly independent, we denote by $l(V, W)$ the sectional curvature of the plane spanned by V and W in S, and we put $l(V, W)=0$ otherwise. Then the sectional curvature $k(\pi)$ is given by

$$k(\pi)=\frac{-f''(t)}{f(t)}c^2+\frac{l(v, w)-f'(t)^2}{f(t)^2}\|v\|^2,$$

see Bishop-O'Neill [1]. Since $k(\pi)<0$ for all π, we have $f''(t)>0$ for all t and f is a strictly convex function. Furthermore, by Eberlein [3], we have that the sectional curvature of S is strictly negative, i.e. S is a symmetric space of rank one, and that f has a unique minimum. After this, we suppose that γ is parametrized so that $f(0)$ is the minimum.

Next, let ρ be in $I(M)$. Since $\rho G=G\rho$, we have that $\rho S(0)=\rho G(\gamma(0))=G(\rho\gamma(0))$ is an orbit. Hence there exists $t_0\in\mathbf{R}$ with $\rho S(0)=S(t_0)$. Therefore ρ induces an isometry between $S(0)$ and $S(t_0)$. We denote

$$\varphi(t)\colon S\ni\sigma K\longmapsto\sigma\gamma(t)\qquad\text{for}\quad t\in\mathbf{R}.$$

Let X be a vector tangent to S. Then we have

$$\|d\varphi(t)X\|=f(t)\|X\|.$$

Let us put $\mu=\varphi(t_0)^{-1}\circ\rho\circ\varphi(0)$. Then μ is a diffeomorphism of S onto itself and

$$\|d\mu X\| = \frac{f(0)}{f(t_0)}\|X\|,$$

i.e. μ is homothetic. Since S is complete and is not locally flat, by Lemma 3, we have $f(0)=f(t_0)$ and $t_0=0$. Thus we have seen that $I(M)S(0)=S(0)$.

Let ρ be in $I(M)$, and suppose that ρ acts trivially on $S(0)$. Since γ is orthogonal to $S(0)$, ρ leaves the geodesic γ invariant. Since $\rho\gamma(0)=\gamma(0)$, we have either $\rho\gamma(t)=\gamma(t)$ or $\rho\gamma(t)=\gamma(-t)$ for $t\in\mathbf{R}$. Hence there are at most two choices for ρ, i.e. the kernel of the homomorphism

$$I(M)\ni\rho\longmapsto\rho|_{S(0)}\in I(S(0))$$

is of order at most two. On the other hand, G is of finite index in $I(S)$, by Lemma 5 (iii). Hence $I(M)/G$ is finite.

Let Γ be a discrete subgroup of $I(M)$. We put $G\cap\Gamma=\Gamma_0$. Then Γ_0 is of finite index in Γ. Let v denote the volume of S/Γ_0. Then the volume of M/Γ_0 is

$$v\int_{-\infty}^{\infty}f(t)^n dt \geq v\int_{-\infty}^{\infty}f(0)^n dt = \infty.$$

Hence the volume of M/Γ cannot be finite. Q.E.D.

§4. Proof of Theorem

Let M be a complete, simply connected, negatively pinched 3-manifold. Suppose that there exists a discrete subgroup Γ of $I(M)$ such that M/Γ is of finite volume. Then by Lemma 6, the identity component G of $I(M)$ is a semisimple Lie group without compact factor and with trivial center. Let K be a maximal compact subgroup of G. Since $\dim M=3$, we have

$$\dim G\leq 6 \quad\text{and}\quad \dim G-\dim K\leq 3.$$

Then the only choices for $G\neq\{1\}$ are the adjoint groups of either $SL(2,\mathbf{C})$ or $SL(2,\mathbf{R})$. In the first case, $\dim G-\dim K=3$, and G is transitive on M, and we know that M is of constant curvature in this case. On the other hand, if G is the adjoint group of $SL(2,\mathbf{R})$, then all the assumptions in the Proposition are satisfied, and we have a contradiction. Q.E.D.

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Added in Proof:

In this paper, by the factor space M/Γ is "of finite volume" we mean that there is a fundamental domain of finite volume. Cf. Heintze [5].