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On some semilinear evolution equations with time-lag

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§1. Introduction

In the theory of combustion, the Cauchy problem is considered for the equation

$$(1.1) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \quad t>0, \quad x \in \boldsymbol{R},$$

with the initial condition

$$(1.2) \qquad u(0, x) = a(x).$$

Here $f(\mu)$ is a Lipschitz continuous function on $[0, 1]$ with $f(0)=f(1)=0$ and $f(\mu)\geq 0$ for $0\leq\mu\leq 1$. The solution $u(t, x)$ of (1.1) and (1.2) corresponds to the temperature and the function $f(\mu)$ is the speed of chemical reaction. Kanel' considered the asymptotic behavior of the solution of (1.1) and (1.2) in the following cases I and II (cf. [9]).

Case I. $f(0)=f(1)=0, f(\mu)>0$ for $0<\mu<1, f'(0)>0$ and $a(x)$ is nonnegative in $\boldsymbol{R}$, positive on some interval and dominated by 1 in $\boldsymbol{R}$.

Case II. $f(\mu)=0$ for $0\leq\mu<\mu_0<1, f(1)=0, f(\mu)>0$ for $\mu_0<\mu<1$ (μ_0 is a positive constant), and $a(x)=1$ for $|x|\leq\ell$, $a(x)=0$ for $|x|>\ell>0$.

In [9] the following results were obtained: In the case I, the "burning up" occurs, that is, the solution $u(t, x)$ of (1.1) with the initial condition (1.2) converges to 1 uniformly in x on each finite interval in $\boldsymbol{R}$ as $t\to\infty$. In the case II there exist positive numbers ℓ_1 and ℓ_2 such that for any initial value $a(x)$ with $\ell\leq\ell_1$ the "extinction of flame" occurs (that is, the solution $u(t, x)$ of (1.1) with the initial condition (1.2) converges to 0 uniformly in x as $t\to\infty$), and for any initial value $a(x)$ with $\ell\geq\ell_2$ the "burning up" occurs. In this paper we shall consider a similar problem of "burning up" when $\frac{\partial^2}{\partial x^2}$ in (1.1) is replaced by a fractional power of Laplacian as in (I_1); namely we shall find a sufficient condition on $f(\mu)$ under which the "burning up" occurs for any nonnegative initial value $a(x)\not\equiv 0$. The equations of type (1.1) occur also in population genetics, population growth models, etc. (see A. Kolmogoroff-I. Petrovsky-N. Piscounoff [12] and D. G. Aronson-H. F. Weinberger [1]). In these fields equations with time-lag are considered. (For example, in a herbivore population grazing on vegetation, the effect of overgrazing affects a later generation rather than that existing at that time. For such models it is natural to hypothesize a time-lag in $f(\mu)$ corresponding to the effect of population size on growth rate.) Also, in connection with some control problems the parabolic equations with time-lag have been studied by several authors. A. Inoue-T. Miyakawa-K. Yoshida [7] considered the initial boundary value problem of the equation of type (1.1) with time-lag in a domain of $\boldsymbol{R}^3$.

In this paper we are concerned with the asymptotic behavior of the positive solution of the semilinear equation

$$(\mathrm{I}_1) \qquad \frac{\partial}{\partial t}u(t, x) = -(-\Delta)^\alpha u(t, x) + F(u(t-r(t, x), x), u(t, x)), \ t>0, \ x \in \boldsymbol{R}^d,$$

with the initial condition

$$(\mathrm{I}_2) \qquad u(t, x) = a(t, x), \qquad -r_0 \leq t \leq 0, \quad x \in \boldsymbol{R}^d,$$

where $r(t, x)$ is a bounded continuous given function with $0 \leq r(t, x) \leq r_0$ and α a given constant with $0<\alpha\leq 1$. Define $p(t, x)$ by the Fourier transform

$$(1.3) \qquad p(t, x) = (2\pi)^{-d}\int_{\boldsymbol{R}^d} \exp(-iz\cdot x - t|z|^{2\alpha})dz, \qquad 0 < \alpha \leq 1.$$

Then $p(t, x)$ satisfies $p(t+s, \cdot) = p(t, \cdot) * p(s, \cdot)$ and is the fundamental solution of

$$(1.4) \qquad \frac{\partial u}{\partial t} = -(-\Delta)^\alpha u.$$

By a solution of (I_1) with the initial condition (I_2) (abbreviated: a solution of (I)) we always mean a solution of the integral equation

$$(\mathrm{I}') \qquad \begin{cases} u(t, x) = P_t a(0, x) + \int_0^t ds P_{t-s} F(u(s-r(s, \cdot), \cdot), u(s, \cdot))(x), & t>0, \\ u(t, x) = a(t, x), \qquad -r_0 \leq t \leq 0, \quad x \in \boldsymbol{R}^d, \end{cases}$$

where

$$(1.5) \qquad P_t a(0, x) = \int_{\boldsymbol{R}^d} p(t, x-y) a(0, y) dy.$$

We treat the following two cases; these are called the case $F(\lambda, 1)=0$ and the case $F(\lambda, \mu)>0$ for $\lambda>0$, $\mu>0$, for simplicity.

Case $F(\lambda, 1)=0$: The functions $a(t, x)$ and $F(\lambda, \mu)$ satisfy the conditions (a.1°) and (F.1°).

(a.1°) $a(t, x)$ is a nonnegative, bounded and uniformly continuous function on $[-r_0, 0]\times \boldsymbol{R}^d$ with $0\leq a(t, x)\leq 1$ and $a(0, x)\not\equiv 0$.

(F.1°) $F(\lambda, \mu)$ is a nonnegative Lipschitz continuous function on $[0, 1]\times[0, 1]$ with $F(\lambda, 1)=0$ for $\lambda\in[0, 1]$, $F(\lambda, \mu)>0$ for $(\lambda, \mu)\in(0, 1]\times(0, 1)$ and nondecreasing in λ for each fixed μ.

Case $F(\lambda, \mu)>0$ *for* $\lambda>0$, $\mu>0$: The functions $a(t, x)$ and $F(\lambda, \mu)$ satisfy the conditions (a.1) and (F.1).

(a.1) $a(t, x)$ is a nonnegative, bounded and uniformly continuous function on $[-r_0, 0]\times \boldsymbol{R}^d$ with $a(0, x)\not\equiv 0$.

(F.1) $F(\lambda, \mu)$ is a nonnegative locally Lipschitz continuous function on $\boldsymbol{R}_+\times \boldsymbol{R}_+=[0, \infty)\times[0, \infty)$ with $F(\lambda, \mu)>0$ for $\lambda>0$, $\mu>0$, and nondecreasing in λ for each fixed μ.

In the case $F(\lambda, 1)=0$, the equation (I) (namely, (I′)) has a unique solution $u(t, x)$ with $0<u(t, x)\leq 1$ for $t>0$ by virtue of Lemma 2.1 and Theorem 2.2. We call such a solution the *positive solution dominated by* 1 of the equation (I) and denote it by $u(t, x)$ or $u(t, x; a, F; r)$ when we want to stress the initial value $a(t, x)$, the nonlinear term $F(\lambda, \mu)$ and the time-lag $r(t, x)$. In the case $F(\lambda, \mu)>0$ for $\lambda>0$, $\mu>0$, by Lemma 2.1 and Theorem 2.2 the equation (I) has a unique positive local solution. That is, there exist positive T and $u(t, x)$ such that

(i) $u(t, x)$ is defined in $[0, T)\times \boldsymbol{R}^d$, strictly positive in $(0, T)\times \boldsymbol{R}^d$, and satisfies the integral equation (I′), and

(ii) for any $T'<T$, $u(t, x)$ is bounded and continuous in $[0, T']\times \boldsymbol{R}^d$.

Let $T_\infty=T_\infty(a, F; r)$ be the supremum of all T satisfying the above conditions (i) and (ii). In case $T_\infty=\infty$, $u(t, x)$ is a *global solution* of the equation (I), and in the contrary case ($T_\infty<\infty$), $u(t, x)$ is said to *blow up in a finite time* and T_∞ is called the *blowing-up time* of the solution $u(t, x)$. In general we have $T_\infty\leq\infty$, and the existence and uniqueness theorems hold for $t<T_\infty$, of course. Similarly to the case $F(\lambda, 1)=0$, such a solution is called simply the *positive solution* of (I) and denoted by $u(t, x)$ or $u(t, x; a, F; r)$. In the case $F(\lambda, 1)=0$, we say that the positive solution $u(t, x)$ dominated by 1 *grows up to* 1 as $t\to\infty$ if $u(t, x)$ converges to 1 uniformly on each compact set K in $\boldsymbol{R}^d$ as $t\to\infty$. In the case $F(\lambda, \mu)>0$ for $\lambda>0$, $\mu>0$ we say that the positive global solution $u(t, x)$ *grows up to infinity* as $t\to\infty$ if for each positive constant M and each compact set K in $\boldsymbol{R}^d$ there exists a positive time $T<\infty$ such that $t>T$ and $x\in K$ imply $u(t, x)>M$.

Now our problems can be stated.

Find a (sufficient) condition on F for each of the following:

Case $F(\lambda, 1)=0$:

(A.1) Any positive solution of (I) dominated by 1 grows up to 1 as $t\to\infty$.

Case $F(\lambda, \mu)>0$ *for* $\lambda>0$, $\mu>0$:

(A.2) Any positive global solution of (I) grows up to infinity as $t\to\infty$.
(A.3) Any positive solution of (I) blows up in a finite time.

When $r(t, x)\equiv 0$ (the case without time-lag), these problems were considered by many authors. In this case, putting $F(\lambda, \mu)=f(\mu)$ and $a(t, x)=a(x)$, the equation (I) can be written as follows.

$$\text{(III)} \qquad \begin{cases} \dfrac{\partial}{\partial t}u(t, x) = -(-\Delta)^{\alpha}u(t, x) + f(u(t, x)), & t > 0, \\ u(0, x) = a(x), & x \in \boldsymbol{R}^d. \end{cases}$$

Case $f(1)=0$ (or $F(\lambda, 1)=0$): When $\alpha=1$ (the Laplacian case), the problems were considered by Ya. I. Kanel' [9], N. Ikeda-Y. Kametaka (unpublished), K. Masuda [13], K. Hayakawa [6] and K. Kobayashi-T. Sirao-H. Tanaka [11]. The time-lag case with $\alpha=1$ (the Laplacian case) was also treated by K. Kobayashi [10]. In [11] and [10] considerably sharp sufficient conditions for (A.1) (as well as (A.2) and (A.3)) were obtained. Our present results are generalizations of the results of [11] and [10] to the time-lag case (I) with $0<\alpha\leq 1$. The main results in the case $F(\lambda, 1)=0$ are the following (see § 5 and § 7).

THEOREM 5.1. *Let $F(\lambda, \mu)$ satisfy* (F.1°) *and the following conditions.*

(F.3) $\displaystyle\int_0^{\delta} F_{\delta}(\lambda)/\lambda^{2+\frac{2\alpha}{d}}\, d\lambda = \infty$ *for some* $\delta > 0$,

where $F_{\delta}(\lambda)=\inf\{F(\xi, \eta): \lambda\leq\xi\leq\delta, \lambda\leq\eta\leq\delta\}$ *for* $0\leq\lambda\leq\delta$.

(F.4) *There exist positive constants δ and c_0 (≤ 1) such that*

$$F_{\delta}(\lambda_1\lambda_2) \geq c_0\lambda_2^{1+\frac{2\alpha}{d}}F_{\delta}(\lambda_1) \qquad \textit{for} \quad 0 < \lambda_1 \leq \lambda_2,\ \lambda_1 < c_0,\ \lambda_1\lambda_2 < c_0.$$

Then, for any initial value $a(t, x)$ satisfying the condition (a.1°) *and for any nonnegative bounded continuous time-lag $r(t, x)$, the positive solution $u(t, x; a, F; r)$ of the equation* (I) *dominated by* 1 *grows up to* 1 *as $t\to\infty$.*

THEOREM[1]. *Let $F(\lambda, \mu)$ be nondecreasing in $\mu\in[0, c_2']$ (for some positive constant c_2') with $F(0, 0)=0$ satisfying* (F.1°) *and the following conditions.*

(F.3#) $\displaystyle\int_0^{\delta} F_{\Delta}(\lambda)/\lambda^{2+\frac{2\alpha}{d}}\, d\lambda < \infty$ *for some* $\delta > 0$, *where* $F_{\Delta}(\lambda) = F(\lambda, \lambda)$.

(F.4') *There exists a positive constant c_2' (≤ 1) such that*

$$F_{\Delta}(\lambda_1\lambda_2) \geq c_2'\lambda_2 F_{\Delta}(\lambda_1) \qquad \textit{for} \quad 0 < \lambda_1 < c_2',\ \lambda_2 \geq 1,\ \lambda_1\lambda_2 < c_2'.$$

1) In § 7 this result is stated as Theorem 7.1 removing the condition $F(\lambda, 1)=0$.

Then, for some small initial value $a(t, x)$ *satisfying* (a.1°), *the positive solution* $u(t, x; a, F; r)$ *of the equation* (I) *converges to* 0 *uniformly in* x *as* $t\to\infty$.

Case $f(\mu)>0$ *for* $\mu>0$ (or $F(\lambda, \mu)>0$ *for* $\lambda>0, \mu>0$): When $\alpha=1$ (the Laplacian case) and $f(\mu)=\mu^{1+\beta}$ with $\beta>0$, the problems were considered by H. Fujita [3], [4], K. Hayakawa [5] and K. Kobayashi-T. Sirao-H. Tanaka [11] (with general f). When $0<\alpha\leq 1$ (the case of the fractional power of Laplacian) and $f(\mu)=\mu^{1+\beta}$, there are works of M. Nagasawa-T. Sirao [14] and S. Sugitani [15]. Our present results (*case* $F(\lambda, \mu)>0$ *for* $\lambda, \mu>0$) will generalize these earlier results (especially, [11: Theorems 3.5 and 4.1] and [10: Theorem 3]) to the time-lag case with $0<\alpha\leq 1$. Namely, we shall obtain the following results in §6 and §8.

THEOREM 6.1. *Let* $F(\lambda, \mu)$ *satisfy the conditions* (F.1), (F.3) *and* (F.4). *Then, for any initial value* $a(t, x)$ *satisfying* (a.1) *and for any nonnegative bounded continuous time-lag* $r(t, x)$, *the positive global solution* $u(t, x; a, F; r)$ *of the equation* (I), *if it exists, grows up to infinity as* $t\to\infty$.

THEOREM 8.1. *Let* $F(\lambda, \mu)$ *satisfy the conditions of Theorem* 6.1 *and the following condition.*

(F.5) *There exist positive constants* λ_0, μ_0 *and* c_3 *such that*

(a) $F(\lambda_0, \mu_2)\geq c_3F(\lambda_0, \mu_1)$ *for* $\mu_0\leq\mu_1\leq\mu_2$,

(b) $\int^{\infty}\frac{d\mu}{F(\lambda_0, \mu)}<\infty.$

Then, for any initial value $a(t, x)$ *satisfying* (a.1) *and for any nonnegative bounded continuous time-lag* $r(t, x)$, *the positive solution* $u(t, x; a, F; r)$ *of the equation* (I) *blows up in a finite time.*

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§2. Comparison theorems

In this section we prepare some theorems of comparison type (Theorems 2.2, 2.2′ and 2.5) for later use.

2.1. First we deal with the existence and uniqueness of solutions for the initial value problem:

$$\text{(I)}\qquad \begin{cases} \dfrac{\partial u}{\partial t}=-(-\Delta)^{\alpha}u+F(u(t-r(t, x), x), u(t, x)), & t>0,\\ u(t, x)=a(t, x), \quad -r_0\leq t\leq 0, & x\in\boldsymbol{R}^d. \end{cases}$$

Here α and r_0 are positive constants with $0<\alpha\leq 1$ and $r(t, x)$ is a continuous function on $[0, \infty)\times \boldsymbol{R}^d$ with $0\leq r(t, x)\leq r_0$. As stated in § 1, by a solution of the equation (I) we mean a continuous solution of the corresponding integral equation (I′). The following lemma can be proved by a routine iteration method and so is omitted.

LEMMA 2.1. *Suppose that $F(\lambda, \mu)$ and $a(t, x)$ satisfy the following conditions:*
(2.1) *$F(\lambda, \mu)$ is a locally Lipschitz continuous function on $\boldsymbol{R}\times\boldsymbol{R}$.*
(2.2) *$a(t, x)$ is a bounded continuous function on $[-r_0, 0]\times \boldsymbol{R}^d$.*
Then, there exist $u(t, x)=u(t, x; a, F; r)$ and positive $T_\infty=T_\infty(a, F; r)$ $(\leq\infty)$ satisfying the following conditions (i), (ii) *and* (iii).
(i) *$u(t, x)$ is defined in $[0, T_\infty)\times\boldsymbol{R}^d$, bounded and continuous on $[0, T]\times\boldsymbol{R}^d$ for any $T<T_\infty$.*
(ii) *$u(t, x)$ satisfies the integral equation* (I′) *for $0\leq t<T_\infty$.*
(iii) *When $T_\infty<\infty$, $u(t, x)$ can not be prolonged to a solution of* (I′) *beyond T_∞.*
Moreover such $u(t, x)$ and T_∞ are unique.

THEOREM 2.2. *Suppose that $F_i(\lambda, \mu)$, $i=1, 2$, satisfy the condition* (2.1) *and at least one of the functions $F_i(\lambda, \mu)$, $i=1, 2$, is nondecreasing in λ for each fixed μ. Let $a_i(t, x)$, $i=1, 2$, satisfy the following condition.*
(2.3) *$a(t, x)$ is bounded and uniformly continuous function on $[-r_0, 0]\times\boldsymbol{R}^d$.*
If $F_1\geq F_2$ and $a_1\geq a_2$, then

$$u(t, x; a_1, F_1; r)\geq u(t, x; a_2, F_2; r)$$

for $0\leq t<T_\infty(a_1, F_1; r)$ and $x\in\boldsymbol{R}^d$.

For proving this theorem we prepare two lemmas. First, for any ε with $0<\varepsilon<r_0$, we consider the following auxiliary equation.

$$(\mathrm{I}_\varepsilon)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + F(u(t-r_\varepsilon(t, x), x), u(t, x)), & t>0, \\ u(t, x) = a(t, x), \quad -r_0\leq t\leq 0, & x\in\boldsymbol{R}^d, \end{cases}$$

where $r_\varepsilon(t, x)=r(t, x)\vee\varepsilon$ $(\equiv\max(r(t, x), \varepsilon))$. This equation (I_ε) has a unique (local) solution $u(t, x; a, F; r_\varepsilon)$ by Lemma 2.1.

LEMMA 2.3. *Let $F(\lambda, \mu)$ and $a(t, x)$ satisfy the conditions* (2.1) *and* (2.3), *respectively. Then we have*

(i) $\varliminf_{\varepsilon\downarrow 0} T_\infty(a, F; r_\varepsilon)\geq T_\infty(a, F; r)$;

(ii) *for any $0<T<T_\infty(a, F; r)$, $u(t, x; a, F; r_\varepsilon)$ converges to $u(t, x; a, F; r)$*

uniformly on $[0, T]\times \boldsymbol{R}^d$ *as* $\varepsilon \downarrow 0$.

PROOF. Put $u(t, x)=u(t, x; a, F; r)$ and $u_\varepsilon(t, x)=u(t, x; a, F; r_\varepsilon)$.

Step 1. We prove the lemma in the case when $F(\lambda, \mu)$ is bounded and Lipschitz continuous on $\boldsymbol{R}\times\boldsymbol{R}$. In this case the equations (I) and (I_ε) have global solutions, that is, $T_\infty(a, F; r)=T_\infty(a, F; r_\varepsilon)=\infty$, and hence (i) is obvious. Since $u_\varepsilon(t, x)$ satisfies the integral equation (I′) with $r(s, \cdot)$ replaced by $r_\varepsilon(s, \cdot)$, the Lipschitz continuity of F implies that

$$(2.4)\quad |u_\varepsilon(t, x) - u(t, x)|$$
$$\leq L\int_0^t dsP_{t-s}\{|u_\varepsilon(s-r_\varepsilon(s, \cdot), \cdot) - u(s-r(s,\cdot), \cdot)| + |u_\varepsilon(s, \cdot) - u(s, \cdot)|\}(x),$$

where L is the Lipschitz constant of F. Putting

$$v_\varepsilon(t) = \sup_{-r_0\leq s\leq t} \|u_\varepsilon(s, \cdot) - u(s, \cdot)\|_\infty,$$

$$w_\varepsilon(t) = \sup_{\substack{-r_0\leq t_1\leq t_2\leq t\\ t_2-t_1\leq\varepsilon}} \|u(t_1, \cdot) - u(t_2, \cdot)\|_\infty,$$

we have

$$|u_\varepsilon(s-r_\varepsilon(s, x), x) - u(s-r(s, x), x)|$$
$$\leq |u_\varepsilon(s-r_\varepsilon(s, x), x) - u(s-r_\varepsilon(s, x), x)| + |u(s-r_\varepsilon(s, x), x) - u(s-r(s, x), x)|$$
$$\leq v_\varepsilon(s) + w_\varepsilon(s),$$

and hence by (2.4)

$$v_\varepsilon(t) \leq 2L\int_0^t v_\varepsilon(s)ds + Lw_\varepsilon(t)t.$$

Therefore we have $v_\varepsilon(t)\leq Lw_\varepsilon(t)te^{2Lt}$, $t\geq 0$. Since what we have to prove was that $v_\varepsilon(t)\to 0$ as $\varepsilon\downarrow 0$, it is enough to prove that $w_\varepsilon(t)\to 0$ as $\varepsilon\downarrow 0$ for each fixed $t\geq 0$. First, assuming $0<t_1\leq t_2\leq t$ and $t_2-t_1\leq\varepsilon$ we estimate $|u(t_2, x)-u(t_1, x)|$; we have

$$u(t_2, x) - u(t_1, x) = P_{t_1}\{P_{t_2-t_1}a(0, \cdot) - a(0, \cdot)\}(x)$$
$$+ \int_0^{t_1} dsP_{t_1-s}\{P_{t_2-t_1}F(u(s-r(s, \cdot), \cdot), u(s, \cdot)) - F(u(s-r(s, \cdot), \cdot), u(s, \cdot))\}(x)$$
$$+ \int_{t_1}^{t_2} dsP_{t_2-s}F(u(s-r(s, \cdot), \cdot), u(s, \cdot))(x)$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

It is easy to see that $|\mathrm{III}|\leq M\varepsilon$, where $M=\sup_{\lambda, \mu\in\boldsymbol{R}}|F(\lambda, \mu)|$. Putting

$$w_\varepsilon^0 = \sup_{0<\delta\le\varepsilon} \|P_\delta a(0,\cdot) - a(0,\cdot)\|_\infty,$$

we have easily $|\mathrm{I}| \le w_\varepsilon^0 \to 0$ as $\varepsilon \downarrow 0$. The second term II can be estimated as follows. Take an arbitrary positive constant h. Then, in the case $t_1 \le h$ we have $|\mathrm{II}| \le 2Mt_1 \le 2Mh$, while in the other case ($t_1 > h$)

$$\begin{aligned} \mathrm{II} &= \int_0^{t_1-h} (\cdots) ds + \int_{t_1-h}^{t_1} (\cdots) ds \\ &= \mathrm{II}_1 + \mathrm{II}_2. \end{aligned}$$

Putting $w_\varepsilon^h = \sup\{\|P_\delta P_h \varphi - P_h \varphi\|_\infty : 0<\delta\le\varepsilon \text{ and } \|\varphi\|_\infty \le M\}$, we see easily that w_ε^h converges to 0 as $\varepsilon \downarrow 0$ for each fixed $h>0$, because $\{P_h\varphi : \|\varphi\|_\infty \le M\}$ is an equicontinuous family. Then, we have for $t_1 > h$

$$\begin{aligned} |\mathrm{II}_1| \le \int_0^{t_1-h} ds |P_{t_2-t_1} P_h \{P_{t_1-h-s} F(u(s-r(s,\cdot),\cdot), u(s,\cdot))\}(x) \\ - P_h\{P_{t_1-h-s} F(u(s-r(s,\cdot),\cdot), u(s,\cdot))\}(x)| \\ \le \int_0^{t_1-h} w_\varepsilon^h ds < w_\varepsilon^h t, \end{aligned}$$

and $|\mathrm{II}_2| \le 2Mh$. Therefore, if $0<t_1\le t_2\le t$ and $t_2 - t_1 \le \varepsilon$, we have

$$|u(t_2, x) - u(t_1, x)| \le w_\varepsilon^0 + w_\varepsilon^h t + 2Mh + M\varepsilon.$$

In the case when $-r_0 \le t_1 \le 0$, $0<t_2\le t$ and $t_2 - t_1 \le \varepsilon$, we have

$$|u(t_2, x) - u(t_1, x)| \le w_\varepsilon^0 + \sup_{-\varepsilon\le t_1\le 0} \|a(0,\cdot) - a(t_1,\cdot)\|_\infty + M\varepsilon,$$

and in the remaining case ($-r_0 \le t_1 \le t_2 \le 0$, $t_2 - t_1 \le \varepsilon$) we have

$$|u(t_2, x) - u(t_1, x)| \le \sup_{0\le t_2-t_1\le\varepsilon} \|a(t_2,\cdot) - a(t_1,\cdot)\|_\infty.$$

Consequently we have, for any positive h,

$$\begin{aligned} w_\varepsilon(t) \le w_\varepsilon^0 + w_\varepsilon^h t + 2Mh + M\varepsilon \\ + \sup_{-\varepsilon\le t_1\le 0} \|a(0,\cdot) - a(t_1,\cdot)\|_\infty + \sup_{0\le t_2-t_1\le\varepsilon} \|a(t_2,\cdot) - a(t_1,\cdot)\|_\infty, \end{aligned}$$

and hence we obtain $\lim_{\varepsilon\downarrow 0} w_\varepsilon(t) = 0$.

Step 2. Let $F(\lambda, \mu)$ be locally Lipschitz continuous on $\boldsymbol{R}\times\boldsymbol{R}$. If $0<T<T_\infty(a, F; r)$, then there exists a positive constant M such that

$$|u(t, x; a, F; r)| \le M \quad \text{for} \quad 0 \le t \le T, \quad x \in \boldsymbol{R}^d.$$

Let $F_M(\lambda, \mu)$ be a bounded and Lipschitz continuous function on $\boldsymbol{R}\times\boldsymbol{R}$ which is equal to $F(\lambda, \mu)$ for $|\lambda|, |\mu| \leq 2M$. Then by Step 1, for any T' such that $0 < T' < T_\infty(a, F_M; r)$ $(=\infty)$, $u(t, x; a, F_M; r_\varepsilon)$ converges to $u(t, x; a, F_M; r)$ uniformly on $[0, T'] \times \boldsymbol{R}^d$ as $\varepsilon \downarrow 0$. Since $u(t, x; a, F_M; r) = u(t, x; a, F; r) \in [-M, M]$ on $[0, T] \times \boldsymbol{R}^d$, there exists a positive constant ε_0 such that $|u(t, x; a, F_M; r_\varepsilon)| \leq 2M$ on $[0, T] \times \boldsymbol{R}^d$, provided $0 < \varepsilon \leq \varepsilon_0$. Therefore, $u(t, x; a, F_M; r_\varepsilon) = u(t, x; a, F; r_\varepsilon)$ on $[0, T] \times \boldsymbol{R}^d$, provided $0 < \varepsilon \leq \varepsilon_0$, and hence $u(t, x; a, F; r_\varepsilon)$ converges to $u(t, x; a, F; r)$ uniformly on $[0, T] \times \boldsymbol{R}^d$ as $\varepsilon \downarrow 0$. Thus we have proved (i) and (ii) of Lemma 2.3 for $F(\lambda, \mu)$ satisfying (2.1).

Next we consider the equation without time-lag:

$$\text{(II)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + f(t, x, u), & t > 0, \\ u(0, x) = a(x), & x \in \boldsymbol{R}^d, \end{cases}$$

where it is assumed that $f(t, x, \mu)$ and $a(x)$ satisfy the following conditions (2.5) and (2.6), respectively.

(2.5) $f(t, x, \mu)$ is defined and continuous in $[0, \infty) \times \boldsymbol{R}^d \times \boldsymbol{R}$, also for any constants $T > 0$ and $M > 0$ (a) $f(t, x, \mu)$ is bounded on $[0, T] \times \boldsymbol{R}^d \times [-M, M]$ and (b) there exists $L = L_{T,M} > 0$ such that

$$|f(t, x, \mu_1) - f(t, x, \mu_2)| \leq L|\mu_1 - \mu_2|$$

for $0 \leq t \leq T$, $x \in \boldsymbol{R}^d$ and $|\mu_1|, |\mu_2| \leq M$.

(2.6) $a(x)$ is a bounded continuous function in $\boldsymbol{R}^d$.

As in the case of (I), by a solution of (II) we mean a continuous solution of the corresponding integral equation. The existence of a unique (local) solution $u(t, x) = u(t, x; a, f)$ of (II) is well-known. The following comparison lemma is also well-known in case $\alpha = 1$ (cf. [12]); the proof for the case $0 < \alpha \leq 1$ is similar.

LEMMA 2.4. *Let $f_i(t, x, \mu)$ and $a_i(x)$, $i = 1, 2$, satisfy the conditions* (2.5) *and* (2.6), *respectively. If $f_1 \geq f_2$ and $a_1 \geq a_2$, then $u(t, x; a_1, f_1) \geq u(t, x, a_2, f_2)$ for $0 \leq t < T_\infty(a_1, f_1)$ and $x \in \boldsymbol{R}^d$, where $T_\infty(a_1, f_1)$ is the blowing-up time of $u(t, x; a_1, f_1)$.*

PROOF OF THEOREM 2.2. For each $i = 1, 2$, let $u_\varepsilon^i(t, x) = u(t, x; a_i, F_i; r_\varepsilon)$ be the solution of the equation (I_ε) with a and F replaced by a_i and F_i, respectively. By virtue of Lemma 2.3 it is sufficient to show that for any sufficiently small $\varepsilon > 0$

$$u_\varepsilon^1(t, x) \geq u_\varepsilon^2(t, x) \qquad \text{for} \quad 0 \leq t < T_\infty(a_1, F_1; r_\varepsilon), \qquad x \in \boldsymbol{R}^d.$$

We assume here that $F_1(\lambda, \mu)$ is nondecreasing in λ for each fixed μ and we shall prove, by induction in n, that

(2.7) $u_\varepsilon^1(t, x) \geq u_\varepsilon^2(t, x)$ for $-r_0 \leq t \leq n\varepsilon, \quad x \in \boldsymbol{R}^d, \quad n = 0, 1, 2, \ldots$.

The case when $F_2(\lambda, \mu)$ is nondecreasing in λ can be treated similarly. When $n=0$, (2.7) is valid since $u_\varepsilon^i(t, x)=a_i(t, x)$ for $-r_0 \leq t \leq 0$. Assume that (2.7) is true for n. Let $n\varepsilon < t \leq (n+1)\varepsilon$. Since $t-r_\varepsilon(t, x) \leq n\varepsilon$, the induction hypothesis implies that

$$u_\varepsilon^1(t-r_\varepsilon(t, x), x) \geq u_\varepsilon^2(t-r_\varepsilon(t, x), x) \qquad \text{for} \quad n\varepsilon < t \leq (n+1)\varepsilon, \quad x \in \boldsymbol{R}^d.$$

Put $f_i(t, x, \mu)=F_i(u_\varepsilon^i(t-r_\varepsilon(t, x), x), \mu)$ for $n\varepsilon < t \leq (n+1)\varepsilon$ and $x \in \boldsymbol{R}^d$. Then we have

$$f_1(t, x, \mu) \geq F_1(u_\varepsilon^2(t-r_\varepsilon(t, x), x), \mu) \geq f_2(t, x, \mu).$$

Since $u_\varepsilon^i(t, x)$, $i=1, 2$, satisfy the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + f_i(t, x, u), & n\varepsilon < t \leq (n+1)\varepsilon, \\ u(n\varepsilon, x) = u_\varepsilon^i(n\varepsilon, x), & x \in \boldsymbol{R}^d, \end{cases}$$

an application of Lemma 2.4 yields

$$u_\varepsilon^1(t, x) \geq u_\varepsilon^2(t, x) \qquad \text{for} \quad n\varepsilon < t \leq (n+1)\varepsilon, \qquad x \in \boldsymbol{R}^d,$$

and hence

$$u_\varepsilon^1(t, x) \geq u_\varepsilon^2(t, x) \qquad \text{for} \quad -r_0 \leq t \leq (n+1)\varepsilon, \qquad x \in \boldsymbol{R}^d.$$

This completes the proof.

2.2. We consider the following equations:

$$(\underline{\mathrm{I}}) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + F(u_*(t, x), u(t, x)), & t > 0, \\ u(t, x) = a(t, x), \quad -r_0 \leq t \leq 0, & x \in \boldsymbol{R}^d, \end{cases}$$

$$(\bar{\mathrm{I}}) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + F(u^*(t, x), u(t, x)), & t > 0, \\ u(t, x) = a(t, x), \quad -r_0 \leq t \leq 0, & x \in \boldsymbol{R}^d, \end{cases}$$

where $u_*(t, x) = \min_{t-r_0 \leq s \leq t} u(s, x)$ and $u^*(t, x) = \max_{t-r_0 \leq s \leq t} u(s, x)$.
Writing down the integral equations corresponding to ($\underline{\mathrm{I}}$) and ($\bar{\mathrm{I}}$) and employing the iteration method, we can prove the following lemma.

LEMMA 2.1′. *Let $F(\lambda, \mu)$ and $a(t, x)$ satisfy the conditions* (2.1) *and* (2.2) *in Lemma* 2.1, *respectively. Then there exists a unique* (*local*) *solution, in the* *same sense as in Lemma* 2.1, *for each of the equations* ($\underline{\mathrm{I}}$) *and* ($\bar{\mathrm{I}}$).

The solutions of the equations ($\underline{\mathrm{I}}$) and ($\bar{\mathrm{I}}$) are denoted by $\underline{u}(t, x; a, F; r_0)$ and $\bar{u}(t, x; a, F; r_0)$, respectively; they are called the minimum solution and the maximum solution; the corresponding blowing-up times ($\leq \infty$) are denoted by $\underline{T}_\infty(a, F; r_0)$ and $\bar{T}_\infty(a, F; r_0)$, respectively.

For any ε with $0<\varepsilon<r_0$, we consider the following auxiliary equations:

$$(\underline{\mathrm{I}}_\varepsilon) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + F(u_{*\varepsilon}(t, x), u(t, x)), & t > 0, \\ u(t, x) = a(t, x), \quad -r_0 \leq t \leq 0, & x \in \boldsymbol{R}^d, \end{cases}$$

$$(\bar{\mathrm{I}}_\varepsilon) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + F(u_\varepsilon^*(t, x), u(t, x)), & t > 0, \\ u(t, x) = a(t, x), \quad -r_0 \leq t \leq 0, & x \in \boldsymbol{R}^d, \end{cases}$$

where

$$u_{*\varepsilon}(t, x) = \min_{t-r_0 \leq s \leq t-\varepsilon} u(s, x),$$

$$u_\varepsilon^*(t, x) = \max_{t-r_0 \leq s \leq t-\varepsilon} u(s, x).$$

As in Lemma 2.1, under the conditions (2.1) and (2.2) there exists a unique (local) solution for each of the equations ($\underline{\mathrm{I}}_\varepsilon$) and ($\bar{\mathrm{I}}_\varepsilon$). We denote the solutions of the equations ($\underline{\mathrm{I}}_\varepsilon$) and ($\bar{\mathrm{I}}_\varepsilon$) by $\underline{u}_\varepsilon(t, x; a, F; r_0)$ and $\bar{u}_\varepsilon(t, x; a, F; r_0)$, respectively.

The following lemma can be proved by a method similar to that of Lemma 2.3.

LEMMA 2.3′. *Let* $F(\lambda, \mu)$ *and* $a(t, x)$ *satisfy the conditions* (2.1) *and* (2.3) *in Theorem* 2.2. *Then we have*

(i) $\varliminf_{\varepsilon \downarrow 0} \underline{T}^\varepsilon_\infty(a, F; r_0) \geq \underline{T}_\infty(a, F; r_0)$;

(ii) $\varliminf_{\varepsilon \downarrow 0} \bar{T}^\varepsilon_\infty(a, F; r_0) \geq \bar{T}_\infty(a, F; r_0)$;

(iii) *for any* $0<T<\underline{T}_\infty(a, F; r_0)$, $\underline{u}_\varepsilon(t, x; a, F; r_0)$ *converges to* $\underline{u}(t, x; a, F; r_0)$ *uniformly on* $[0, T] \times \boldsymbol{R}^d$ *as* $\varepsilon \downarrow 0$;

(iv) *for any* $0<T<\bar{T}_\infty(a, F; r_0)$, $\bar{u}_\varepsilon(t, x; a, F; r_0)$ *converges to* $\bar{u}(t, x; a, F; r_0)$ *uniformly on* $[0, T] \times \boldsymbol{R}^d$ *as* $\varepsilon \downarrow 0$.

Making use of Lemma 2.3′ and Lemma 2.4, we can prove the following theorem; the proof is quite similar to that of Theorem 2.2 and so is omitted.

THEOREM 2.2′. *Suppose that* $F_i(\lambda, \mu)$ *and* $a_i(t, x)$, $i=1, 2$, *satisfy the conditions* (2.1) *and* (2.3), *respectively, and that at least one of the functions* $F_i(\lambda, \mu)$, $i=1, 2$, *is nondecreasing in* λ *for each fixed* μ. *If* $F_1 \geq F_2$ *and* $a_1 \geq a_2$, *then*

(i) $\underline{u}(t, x; a_1, F_1; r_0) \geq \underline{u}(t, x; a_2, F_2; r_0)$ *for* $0 \leq t < \underline{T}_\infty$, $x \in \boldsymbol{R}^d$,

(ii) $\bar{u}(t, x; a_1, F_1; r_0) \geq \bar{u}(t, x; a_2, F_2; r_0)$ *for* $0 \leq t < \overline{T}_\infty$, $x \in \boldsymbol{R}^d$,

where $\underline{T}_\infty = \underline{T}_\infty(a_1, F_1; r_0)$ *and* $\overline{T}_\infty = \overline{T}_\infty(a_1, F_1; r_0)$.

Now we can state the final main theorem of this section; the proof can be accomplished by making use of Lemmas 2.3′ and 2.4 as in Theorem 2.2.

THEOREM 2.5. *Let* $F(\lambda, \mu)$ *and* $a(t, x)$ *satisfy the conditions* (2.1) *and* (2.3), *respectively. Then we have*

(i) $\underline{u}(t, x; a, F; r_0) \leq u(t, x; a, F; r)$ *for* $0 \leq t < T_\infty(a, F; r)$, $x \in \boldsymbol{R}^d$,

(ii) $u(t, x; a, F; r) \leq \bar{u}(t, x; a, F; r_0)$ *for* $0 \leq t < \overline{T}_\infty(a, F; r_0)$, $x \in \boldsymbol{R}^d$,

where $0 \leq r = r(t, x) \leq r_0$.

§3. A sufficient condition for the growing-up of minimum solutions

We begin with some simple properties of the fundamental solution $p(t, x)$ of (1.4).

LEMMA 3.1. *Let* $t>0$ *and* $x, y \in \boldsymbol{R}^d$. *Then we have the following properties*:

(3.1) $p(ts, x) = t^{-d/(2\alpha)} p(s, t^{-1/(2\alpha)} x)$.

(3.2) $p(t, x) < p(t, y)$ *for* $|x| > |y|$.

(3.3) $p(t, x-y) \geq \dfrac{1}{p(t, 0)} p(t, 2x) p(t, 2y)$.

(3.4) *If* $a(x)$ *is a nonnegative continuous function on* $\boldsymbol{R}^d$ *not being identically zero, then for each positive t we can find positive numbers* β *and* t_0 *such that* $P_t a(x) \geq \beta p(t_0, x)$ *for any* $x \in \boldsymbol{R}^d$, *where the operator* P_t *is defined by* (1.5).

PROOF. (3.1) follows immediately from the definition (1.3) of $p(t, x)$ by making a change of variable. Let θ_t be a one-sided stable process with index α and define $q(t, s)ds = P(\theta_t \in ds) \geq 0$, namely

$$\int_0^\infty e^{-\lambda s} q(t, s) ds = \exp(-t\lambda^\alpha).$$

Then $p(t, x)$ can be written in the following form ([2]):

$$p(t, x) = \begin{cases} \int_0^\infty q(t, s)(4\pi s)^{-d/2} \exp\left(-\frac{|x|^2}{4s}\right) ds & \text{for} \quad 0<\alpha<1, \\ (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right) & \text{for} \quad \alpha=1, \end{cases}$$

and hence we see (3.2). Since $|x-y| \leq |2x| \vee |2y|$, we have by (3.2)

$$\frac{p(t, x-y)}{p(t, 0)} \geq \frac{p(t, 2x)}{p(t, 0)} \wedge \frac{p(t, 2y)}{p(t, 0)} \geq \frac{p(t, 2x)}{p(t, 0)} \cdot \frac{p(t, 2y)}{p(t, 0)},$$

where $a \wedge b$ ($a \vee b$) denotes $\min(a, b)$ (respectively $\max(a, b)$). Finally we show (3.4). By the assumption on $a(x)$, there exist $\varepsilon > 0$ and a measurable subset A of $\boldsymbol{R}^d$ with positive Lebesgue measure such that $a(x) \geq \varepsilon$ on A. Using (3.3) and (3.1) we have

$$\begin{aligned} p(t, x-y) &\geq \frac{1}{p(t, 0)} p(t, 2x) p(t, 2y) \\ &= \frac{1}{p(t, 0)} 2^{-d} p(2^{-2\alpha} t, x) p(t, 2y). \end{aligned}$$

Therefore (3.4) follows from

$$\begin{aligned} P_t a(x) &= \int_{\boldsymbol{R}^d} p(t, x-y) a(y) dy \geq \varepsilon \int_A p(t, x-y) dy \\ &\geq \left\{\varepsilon p(t, 0)^{-1} 2^{-d} \int_A p(t, 2y) dy\right\} p(2^{-2\alpha} t, x). \end{aligned}$$

The proof of the lemma is finished.

In the sequel, we assume that $F(\lambda, \mu)$ is a nonnegative locally Lipschitz continuous function on $\boldsymbol{R}_+ \times \boldsymbol{R}_+ = [0, \infty) \times [0, \infty)$ and $a(t, x)$ is a nonnegative, bounded and uniformly continuous function on $[-r_0, 0] \times \boldsymbol{R}^d$ such that $a(0, x)$ does not vanish identically, unless explicitly mentioned otherwise. We consider the equation

$$(\underline{\mathrm{I}}) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + F(u_*(t, x), u(t, x)), & t > 0, \\ u(t, x) = a(t, x), \qquad -r_0 \leq t \leq 0, & x \in \boldsymbol{R}^d. \end{cases}$$

By virtue of Theorem 2.5 in the preceding section, if the solution $\underline{u}(t, x)$ of the equation ($\underline{\mathrm{I}}$) grows up to infinity then the solution $u(t, x)$ of the equation (I) with the same initial value as $\underline{u}(t, x)$ also grows up to infinity as $t \to \infty$. Therefore, in this section, we seek a sufficient condition in order that the solution $\underline{u}(t, x)$ of the equation ($\underline{\mathrm{I}}$) grows up to infinity as $t\to\infty$.

We put $F_\Delta(\lambda)=F(\lambda, \lambda)$. Then our result is the following:

THEOREM 3.2. *Assume that the function $F(\lambda, \mu)$ satisfies the following conditions:*

(F.1) *$F(\lambda, \mu)$ is a nonnegative locally Lipschitz continuous function on $\boldsymbol{R}_+ \times \boldsymbol{R}_+$ with $F(\lambda, \mu)>0$ for $\lambda>0$, $\mu>0$ and nondecreasing in λ for each fixed μ.*

(F.2) *$F(\lambda, \mu)$ is nondecreasing in μ for each fixed λ.*

(F.3*) $\displaystyle\int_0^\delta F_\Delta(\lambda)/\lambda^{2+\frac{2\alpha}{d}}\, d\lambda = \infty$ *for some* $\delta > 0$.

(F.4*) *There exists a positive constant c (≤ 1) such that*

$$\text{(a)}\quad F_\Delta(\lambda_1\lambda_2) \geq c\lambda_2^{1+\frac{2\alpha}{d}} F_\Delta(\lambda_1) \qquad \textit{for} \quad 0 < \lambda_1 \leq \lambda_2,\ \lambda_1 < c,$$

$$\text{(b)}\quad F_\Delta(\lambda_1\lambda_2) \geq c\lambda_2^{2+\frac{2\alpha}{d}} F_\Delta(\lambda_1) \qquad \textit{for} \quad 0 < \lambda_2 \leq \lambda_1 < c.$$

Then, for any initial value $a(t, x)$ satisfying

(a.1) *$a(t, x)$ is a nonnegative, bounded and uniformly continuous function on $[-r_0, 0]\times \boldsymbol{R}^d$ with $a(0, x)\not\equiv 0$,*

the positive minimum solution $\underline{u}(t, x; a, F; r_0)$ of the equation ($\underline{\mathrm{I}}$) blows up in a finite time or grows up to infinity as $t\to\infty$.

For proving the theorem we prepare two lemmas. We note that the minimum solution $\underline{u}(t, x)=\underline{u}(t, x; a, F; r_0)$ satisfies the integral equation

$$(\underline{\mathrm{I}}')\qquad \begin{cases} \underline{u}(t, x) = P_t a(0, x) + \displaystyle\int_0^t ds P_{t-s} F(\underline{u}_*(s, \cdot), \underline{u}(s, \cdot))(x), & t > 0, \\ \underline{u}(t, x) = a(t, x), \quad -r_0 \leq t \leq 0, \ x\in \boldsymbol{R}^d. \end{cases}$$

LEMMA 3.3. *Suppose that $a(t, x)$ is a nonnegative, bounded and continuous function on $[-r_0, 0]\times \boldsymbol{R}^d$ with $a(0, x)\not\equiv 0$ and $F(\lambda, \mu)$ is a nonnegative locally Lipschitz continuous function on $\boldsymbol{R}_+\times\boldsymbol{R}_+$. Then, for any fixed time t_1 later than r_0, there exist positive numbers β and t_0 such that*

$$\underline{u}(t_1+s, x; a, F; r_0) \geq \beta p(s+t_0+r_0, x) \qquad \textit{for} \quad -r_0 \leq s \leq 0, \quad x\in \boldsymbol{R}^d.$$

PROOF. Since $\underline{u}(t_1, x)=\underline{u}(t_1, x; a, F; r_0)$ is the solution of ($\underline{\mathrm{I}}'$), by the nonnegativity of F we have

$$\underline{u}(t_1+s, x) \geq P_{t_1+s}a(0, x), \qquad -r_0 \leq s \leq 0, \quad x \in \boldsymbol{R}^d.$$

By (3.4) in Lemma 3.1, there exist positive numbers β and t_0 such that $P_{t_1-r_0}a(0, x) \geq \beta p(t_0, x)$, and hence we have

$$\underline{u}(t_1+s, x) \geq P_{s+r_0}(\beta p(t_0, \cdot))(x) \geq \beta p(s+r_0+t_0, x)$$

for any $-r_0 \leq s \leq 0$ and $x \in \boldsymbol{R}^d$, completing the proof.

For a fixed time t_1 $(>r_0)$ we put $\tilde{a}(t, x) = \underline{u}(t_1+t, x)$, $-r_0 \leq t \leq 0$. Then by the above lemma there exist positive constants β and t_0 such that $\tilde{a}(t, x) \geq \beta p(t+t_0+r_0, x) \equiv \hat{a}(t, x)$, $-r_0 \leq t \leq 0$, and hence by making use of Theorem 2.2′ we have

$$\begin{aligned}\underline{u}(t, x) &= \underline{u}(t-t_1, x; \tilde{a}, F; r_0) \\ &\geq \underline{u}(t-t_1, x; \hat{a}, F; r_0), \qquad t > t_1.\end{aligned}$$

Therefore, in order to prove the theorem it is enough to consider the case when

$$a(t, x) = \beta p(t+t_0+r_0, x), \qquad -r_0 \leq t \leq 0, \quad x \in \boldsymbol{R}^d. \tag{3.5}$$

Moreover by Theorem 2.2′ it is also enough to prove the theorem for a smaller initial value. Thus we may assume that the initial value is of the form (3.5) with $0 < \beta < c/p(t_0, 0)$, $t_0 > 2r_0$ where c is the constant appearing in (F.4*). Before stating the next lemma we introduce some notations.

$$\begin{aligned}
\theta(t) &= \beta p(t+t_0, 0) = \beta(t+t_0)^{-d/(2\alpha)} p(1, 0), \\
u_0(t, x) &= \beta p(t+t_0, 2^{1+\frac{3}{2\alpha}} x), \\
\varphi(t) &= \int_0^{t/2} F_\Delta(\theta(s))/\theta(s)ds \\
&= \frac{2\alpha(\beta p(1, 0))^{2\alpha/d}}{d} \int_{\theta(t/2)}^{\theta(0)} F_\Delta(\lambda)/\lambda^{2+\frac{2\alpha}{d}} d\lambda, \\
\varphi_n(t) &= 2^n \varphi^{-1}\left\{\left(\varphi\left(\frac{t}{2^{n+1}}\right) - \frac{1}{2^n}\right) \vee 0\right\}, \quad n = 0, 1, 2, \dots, \\
\psi_0(t) &= t, \quad \psi_{n+1}(t) = \psi_n(\varphi_n(t)), \qquad n = 0, 1, 2, \dots.
\end{aligned} \tag{3.6}$$

The following properties can be proved easily.

(3.7) $\varphi(t)$ is a strictly increasing function and $\lim_{t\to\infty} \varphi(t) = \infty$.

$$\psi_n(t) = \varphi^{-1}\left\{\left(\varphi\left(\frac{t}{2^n}\right) - \sum_{k=0}^{n-1} \frac{1}{2^k}\right) \vee 0\right\}, \quad n = 0, 1, 2, \dots, \tag{3.8}$$

where we interprete that $\sum_{k=0}^{n-1}=0$ for $n=0$.

(3.9) If $\psi_n(t) > t_0$, then $t > 2^n t_0$.

In fact, (3.7) is immediate from the assumption (F.3*) since $F_\Delta(\lambda)$ is positive for $\lambda>0$ and $\theta(t)\downarrow 0$ as $t\uparrow\infty$, and (3.8) is also trivial by the definition of $\psi_n(t)$. We show (3.9). Since $\psi_n(t)=\varphi^{-1}\{\varphi(2^{-n}t)-\sum_{k=0}^{n-1}2^{-k}\}>t_0$ by (3.8), the monotonicity of φ implies $\varphi(2^{-n}t)-\sum_{k=0}^{n-1}2^{-k}>\varphi(t_0)$ and then $2^{-n}t>t_0$, that is, $t>2^n t_0$.

LEMMA 3.4. *Suppose that $F(\lambda, \mu)$ satisfies the conditions* (F.1), (F.2) *and* (F.4*), *and let $a(t, x)=\beta p(t+t_0+r_0, x)$ for $-r_0\le t\le 0$ where β and t_0 are positive constants with $0<\beta<c/p(t_0, 0)$ and $t_0>2r_0$. Then we have*

$$(3.10)\quad \underline{u}(t, x; a, F; r_0) > \{1+B_n(t)\}u_0(t+r_0, x) \qquad \textit{for}\quad \psi_n(t) > t_0,\quad n\ge 0,$$

where

$$B_n(t) = A^{1+\gamma+\cdots+\gamma^n}\cdot 2^{-(1+\sigma)\sum_{k=0}^{n-1}k\gamma^{n-k-1}}\cdot\left\{\varphi\left(\frac{t}{2^n}\right)-\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k\right\}^{\gamma^n},\ ^{2)}$$

$$\gamma = 1+\frac{2\alpha}{d},\qquad \sigma=\frac{d}{\alpha},$$

and A is a positive constant.

This lemma will be proved in §9, and here we proceed to the proof of the theorem.

PROOF OF THEOREM 3.2. Let $a(t, x)$ be given by (3.5) with $0<\beta<c/p(t_0, 0)$ and $t_0>2r_0$. Then (3.7) implies

$$\psi_n(t) = \varphi^{-1}\left(\varphi\left(\frac{t}{2^n}\right)-\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k\right) > \varphi^{-1}\left(\varphi\left(\frac{t}{2^n}\right)-2\right),$$

provided the right hand side of the above inequality is positive. For the proof of the theorem it is enough to show that $\underline{u}(t, x; a, F; r_0)$ grows up to infinity assuming that it does not blow up in a finite time. We may consider only the case $A\le 1$ where A is the constant appearing in Lemma 3.4. We put

$$A_0 = 2^{-(1+\sigma)\sum_{k=0}^{\infty}k\gamma^{-k-1}} < \infty.$$

Making use of Lemma 3.4 and then the inequality

$$A^{1+\gamma+\cdots+\gamma^n} > (A^{\gamma/(\gamma-1)})^{\gamma^n},$$

we have for $\psi_n(t)>t_0$

2) $\sum_{k=0}^{n-1}=0$ for $n=0$.

$$(3.11)\qquad \underline{u}(t, x; a, F; r_0) > B_n(t)u_0(t+r_0, x)$$
$$> \left\{A^{\gamma/(\gamma-1)}A_0\left(\varphi\left(\frac{t}{2^n}\right) - 2\right)\right\}^{\gamma^n} u_0(t + r_0, x).$$

Let K be a compact subset of $\boldsymbol{R}^d$ and set

$$T = \max\{\varphi^{-1}(2A^{-\gamma/(\gamma-1)}A_0^{-1} + 2),\ \varphi^{-1}(\varphi(t_0) + 2)\}.$$

Then we have, for $t \geq 2^n T$, $n \geq 1$,

$$(3.12)\qquad \psi_n(t) > t_0 \quad\text{and}\quad A^{\gamma/(\gamma-1)}A_0\left\{\varphi\left(\frac{t}{2^n}\right) - 2\right\} \geq 2.$$

By (3.9) the first inequality in the above implies $t > 2^n t_0$, which combined with $t_0 > 2r_0$ again implies $t + t_0 + r_0 < 2t$. Therefore, for $2^{n+1}T \geq t \geq 2^n T$, $n \geq 1$, and $x \in K$,

$$(3.13)\qquad u_0(t+r_0, x) = \beta(t+t_0+r_0)^{-\frac{d}{2\alpha}} p(1, 2^{1+\frac{3}{2\alpha}}(t+t_0+r_0)^{-\frac{1}{2\alpha}}x)$$
$$> \beta(2t)^{-\frac{d}{2\alpha}} \inf_{x\in K} p(1, 2^{1+\frac{3}{2\alpha}}(3t_0+r_0)^{-\frac{1}{2\alpha}}x)$$
$$\geq A_1 2^{-(n+2)d/(2\alpha)},$$

where $A_1 = \beta T^{-d/(2\alpha)} \inf_{x\in K} p(1, 2^{1+3/(2\alpha)}(3t_0+r_0)^{-1/(2\alpha)}x) > 0$.
Finally we obtain by (3.11), (3.12) and (3.13)

$$\underline{u}(t, x; a, F; r_0) > A_1 2^{\gamma^n - \{(n+2)d/(2\alpha)\}} \quad \text{for } 2^{n+1}T \geq t \geq 2^n T, \quad x \in K,$$

which completes the proof of Theorem 3.2.

§ 4. Some results on the growing-up of minimum solutions

In this section we consider the equation ($\underline{\mathrm{I}}$), and prove the following comparison theorem, which implies that the local behavior of the function F near the origin plays an important role for the asymptotic behavior of the solution $\underline{u}(t, x; a, F; r_0)$ of ($\underline{\mathrm{I}}$) as $t \to \infty$.

THEOREM 4.1. *Suppose that $F(\lambda, \mu)$ and $\tilde{F}(\lambda, \mu)$ satisfy the following conditions*:

(i) *$F(\lambda, \mu)$ is a Lipschitz continuous function on $[0, 1] \times [0, 1]$ with $F(\lambda, 1) = 0$ for any $\lambda \in [0, 1]$ and $F(\lambda, \mu) > 0$ for $\lambda \in (0, 1]$ and $\mu \in (0, 1)$.*

(ii) *$\tilde{F}(\lambda, \mu)$ is a locally Lipschitz continuous function on $\boldsymbol{R}_+ \times \boldsymbol{R}_+ = [0, \infty) \times [0, \infty)$ with $\tilde{F}(\lambda, 0) = \tilde{F}(0, \mu) = 0$ and nondecreasing in μ for each fixed λ.*

(iii) *$F(\lambda, \mu)$ and $\tilde{F}(\lambda, \mu)$ are nondecreasing in λ for each fixed μ.*

(iv) $\liminf_{(\lambda\downarrow 0,\mu\downarrow 0)} \frac{F(\lambda, \mu)}{\tilde{F}(\lambda, \mu)} > 0.$

Moreover, we assume that for any $r_0>0$ *and any initial value* $a(t, x)$ *satisfying* (a.1) *of Theorem* 3.2 *the solution* $\tilde{\underline{u}}(t, x)=\underline{u}(t, x; a, \tilde{F}; r_0)$ *of*

$$\frac{\partial u}{\partial t} = -(-\Delta)^\alpha u + \tilde{F}(u_*(t, x), u(t, x)) \tag{4.1}$$

either blows up in a finite time or satisfies

$$\limsup_{t\to\infty} \|\tilde{\underline{u}}(t, \cdot)\|_\infty = \infty. \tag{4.2}$$

Then any positive solution $\underline{u}(t, x)$ *of the equation* (I) *with the initial value* $a(t, x)$ *satisfying* (a.1) *and* $0\le a(t, x)\le 1$ *grows up to* 1 *as* $t\to\infty$ *for any* $r_0>0$.

Fundamentally the proof of this theorem is similar to that of Theorem 3.3 of [11], but some changes and modifications in details are necessary. First we prepare two lemmas, in which we assume that $\tilde{F}(\lambda, \mu)$ satisfies (ii) and (iii) of Theorem 4.1.

LEMMA 4.2. *If, for any* $r_0>0$, *any positive solution* $\tilde{\underline{u}}(t, x)$ *of* (4.1) *either blows up in a finite time or satisfies*

$$\limsup_{t\to\infty} \|\tilde{\underline{u}}(t, \cdot)\|_\infty = \infty,$$

then the same holds for any positive solution of

$$\frac{\partial u}{\partial t} = -(-\Delta)^\alpha u + \delta\tilde{F}(u_*(t, x), u(t, x)) \tag{4.3}$$

for any $\delta>0$ *and* $r_0>0$.

PROOF. Let $\tilde{\underline{u}}_\delta(t, x)$ be the solution of (4.3) with initial value $\tilde{a}(t, x)$. Then $\tilde{\underline{u}}_\delta(t, x)$ satisfies

$$\begin{aligned} \tilde{\underline{u}}_\delta(\delta^{-1}t, \delta^{-1/(2\alpha)}x) &= \int p(\delta^{-1}t, \delta^{-1/(2\alpha)}x - y)\tilde{a}(0, y)dy \\ &+ \delta\int_0^{\delta^{-1}t} ds\int p(\delta^{-1}t - s, \delta^{-1/(2\alpha)}x - y)\tilde{F}(\min_{s-r_0\le\tau\le s} \tilde{\underline{u}}_\delta(\tau, y), \tilde{\underline{u}}_\delta(s, y))dy. \end{aligned} \tag{4.4}$$

Making a change of variables and using (3.1), we see that the right hand side of (4.4) is equal to

$$\int p(t, x-y)a_\delta(0, y)dy$$

$$+\int_0^t ds\int p(t-s, x-y)\tilde{F}\Big(\min_{(s/\delta)-r_0\le\tau\le s/\delta}\underline{\tilde{u}}_\delta(\tau, \delta^{-1/(2\alpha)}y), \underline{\tilde{u}}_\delta(\delta^{-1}s, \delta^{-1/(2\alpha)}y)\Big)dy,$$

where $a_\delta(t, x)=\tilde{a}(\delta^{-1}t, \delta^{-1/(2\alpha)}x)$. Therefore, noting

$$\min_{(s/\delta)-r_0\le\tau\le s/\delta}\underline{\tilde{u}}_\delta(\tau, \delta^{-1/(2\alpha)}y) = \min_{s-\delta r_0\le\tau\le s}\underline{\tilde{u}}_\delta(\delta^{-1}\tau, \delta^{-1/(2\alpha)}y),$$

we have

$$\underline{\tilde{u}}_\delta(t, x) = \underline{u}(\delta t, \delta^{1/(2\alpha)}x; a_\delta, \tilde{F}; \delta r_0),$$

which implies the statement of Lemma 4.2.

Next we introduce a class $\mathscr{A}$ of nonnegative monotone radial functions:

$$\mathscr{A} = \{a\in C(\boldsymbol{R}^d): a(x)\ge 0, \quad \not\equiv 0;\ a(x)\ge a(y) \qquad \text{for} \quad |x|\le|y|\}.$$

LEMMA 4.3. *If $a(t, x)$ is bounded continuous on $[-r_0, 0]\times\boldsymbol{R}^d$ and belongs to $\mathscr{A}$ for each $-r_0\le t\le 0$, then the solution $\underline{u}(t, x; a, \tilde{F}; r_0)$ of the equation* (4.1) *with initial value $a(t, x)$ belongs to $\mathscr{A}$ for each $0<t<\underline{T}_\infty(a, \tilde{F}; r_0)$.*

This lemma can be proved in a way similar to [11: Lemma 3.2] noting that $p(t, x)$ is a positive monotone decreasing function of $|x|$ for each $t>0$.

PROOF OF THEOREM 4.1. From what we have remarked immediately after the proof of Lemma 3.3 in § 3, it is enough to prove that the solution $\underline{u}(t, x)$ of (I̲) with a special initial value $a(t, x)=\beta p(t+t_0+r_0, x)$, $-r_0\le t\le 0$, grows up to 1 as $t\to\infty$. However, since by Theorem 2.2′ it is also enough to deal with the case of smaller initial value, we may consider only the case when the initial value $a(t, x)$ is continuous on $[-r_0, 0]\times\boldsymbol{R}^d$ and satisfies the following conditions.

(4.5) There exists a compact subset K_0 of $\boldsymbol{R}^d$ such that $a(t, x)=0$ for $x\notin K_0$, $-r_0\le t\le 0$.

(4.6) $a(t, x)\in\mathscr{A}$ for each $t\in[-r_0, 0]$ and $\|a\|_\infty<1$.

Given such an initial value $a(t, x)$, we take an arbitrary positive constant M so that $1>M>\|a\|_\infty$. By the assumptions (i) and (iv) we can take $\delta>0$ so small that $F(\lambda, \mu)>\delta\tilde{F}(\lambda, \mu)$ for $0<\lambda, \mu\le(1+M)/2$. Lemma 4.2 together with the assumption of Theorem 4.1 implies that the solution $\underline{u}(t, x; a, \delta\tilde{F}; r_0)$ of (4.3) either blows up in a finite time or satisfies $\limsup_{t\to\infty}\|\underline{u}(t, \cdot; a, \delta\tilde{F}; r_0)\|_\infty=\infty$, and hence if we define T_δ by

$$T_\delta = \inf\{t>0: \|\underline{u}(t, \cdot; a, \delta\tilde{F}; r_0)\|_\infty > (1+M)/2\},$$

then $T_\delta < \infty$. Moreover, it can be easily proved that $\lim_{\delta \downarrow 0} T_\delta = \infty$. Now the rest of the proof is divided into three steps.

Step 1 is to prove that the inequality

$$\underline{u}(t, x; a, F; r_0) \geq \underline{u}(t, 0; a, \delta \tilde{F}; r_0) - \delta M_0 t_1 - M_1 |x| t_1^{-1/(2\alpha)} \tag{4.7}$$

holds for $0 < t_1 < t \leq T_\delta$, where M_0 and M_1 are positive constants. Let $\tilde{F}_\Delta(\lambda) = \tilde{F}(\lambda, \lambda)$ and $M_0 = \tilde{F}_\Delta((1+M)/2)$. Then $0 \leq \delta \tilde{F}(\lambda, \mu) \leq \delta M_0$ for $0 \leq \lambda, \mu \leq (1+M)/2$. Since

$$\underline{u}(t, x; a, \delta \tilde{F}; r_0) = \underline{u}(t_1, x; v, \delta \tilde{F}; r_0), \qquad 0 < t_1 < t \leq T_\delta$$

holds with $v(s, x) = \underline{u}(t - t_1 + s, x; a, \delta \tilde{F}; r_0)$ for $-r_0 \leq s \leq 0$, applying Theorem 2.2′ to the equations (I) with nonlinear parts $0, \delta \tilde{F}, \delta M_0$ and the common initial value $v(s, x)$, we have

$$\underline{u}(t_1, x; v, 0; r_0) \leq \underline{u}(t, x; a, \delta \tilde{F}; r_0) \leq \underline{u}(t_1, x; v, \delta M_0; r_0),$$

and hence for $0 < t_1 < t \leq T_\delta$

$$P_{t_1} v(0, x) \leq \underline{u}(t, x; a, \delta \tilde{F}; r_0) \leq P_{t_1} v(0, x) + \delta M_0 t_1. \tag{4.8}$$

Putting $x = 0$ in the second inequality of (4.8) we have

$$P_{t_1} v(0, 0) \geq \underline{u}(t, 0; a, \delta \tilde{F}; r_0) - \delta M_0 t_1, \qquad 0 < t_1 < t \leq T_\delta. \tag{4.9}$$

On the other hand, by the property (3.2) we have for each $t > 0$

$$\frac{\partial p}{\partial x_i}(t, x) \leq 0 \qquad \text{for} \quad x_i \geq 0, \quad 1 \leq i \leq d,$$

$$\frac{\partial p}{\partial x_i}(t, x) > 0 \qquad \text{for} \quad x_i < 0, \quad 1 \leq i \leq d,$$

where $x = (x_1, x_2, \ldots, x_d)$, and hence for each fixed i $(1 \leq i \leq d)$

$$\begin{aligned}
&\left| \frac{\partial}{\partial x_i} \int_{\mathbf{R}^d} p(t_1, x - y) v(0, y) dy \right| \\
&= \left| \int_{\Omega_i} \frac{\partial p}{\partial x_i}(t_1, x - y) v(0, y) dy + \int_{\Omega_i^c} \frac{\partial p}{\partial x_i}(t_1, x - y) v(0, y) dy \right| \\
&\leq \|v(0, \cdot)\|_\infty \max \left(\int_{\Omega_i} \frac{\partial p}{\partial x_i}(t_1, x - y) dy, \ - \int_{\Omega_i^c} \frac{\partial p}{\partial x_i}(t_1, x - y) dy \right) \\
&= \|v(0, \cdot)\|_\infty \int_{\mathbf{R}^{d-1}} p(t_1, y^{(i)}) dy^{(i)},
\end{aligned}$$

where $y = (y_1, y_2, \ldots, y_d) \in \mathbf{R}^d$, $\Omega_i = \{y \in \mathbf{R}^d : y_i \geq x_i\}$, $y^{(i)} = (y_1, \ldots, y_{i-1}, 0, y_{i+1},$

$\dots, y_d) \in \boldsymbol{R}^d$ and $dy^{(i)} = dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_d$. Making use of (3.1), we have

$$\int_{\boldsymbol{R}^{d-1}} p(t_1, y^{(i)}) dy^{(i)} = t_1^{-d/(2\alpha)} \int p(1, t_1^{-1/(2\alpha)} y^{(i)}) dy^{(i)}$$

$$= t_1^{-1/(2\alpha)} \int p(1, y^{(i)}) dy^{(i)},$$

and hence

$$\left| \frac{\partial}{\partial x_i} P_{t_1} v(0, x) \right| \leq t_1^{-1/(2\alpha)} \| v(0, \cdot) \|_\infty \int p(1, y^{(i)}) dy^{(i)}.$$

Therefore

$$\left| \nabla P_{t_1} v(0, x) \right|^2 \leq t_1^{-1/\alpha} \| v(0, \cdot) \|_\infty^2 \sum_{i=1}^{d} \left\{ \int p(1, y^{(i)}) dy^{(i)} \right\}^2$$

$$\leq t_1^{-1/\alpha} \left(\frac{1+M}{2} \right)^2 d \cdot \left\{ \int p(1, y^{(1)}) dy^{(1)} \right\}^2,$$

$$\text{(4.10)} \quad \left| P_{t_1} v(0, x) - P_{t_1} v(0, 0) \right| = \left| \int_0^1 (x, \nabla P_{t_1} v(0, sx)) ds \right|$$

$$\leq |x| \left\{ \int_0^1 \left| \nabla P_{t_1} v(0, sx) \right|^2 ds \right\}^{1/2}$$

$$\leq |x| t_1^{-1/(2\alpha)} \frac{1+M}{2} d^{1/2} \int p(1, y^{(1)}) dy^{(1)}$$

$$= |x| t_1^{-1/(2\alpha)} M_1,$$

where $M_1 = (1+M) 2^{-1} d^{1/2} \int p(1, y^{(1)}) dy^{(1)}$. Combining (4.9) with (4.10) we have

$$P_{t_1} v(0, x) \geq \underline{u}(t, 0; a, \delta\tilde{F}; r_0) - \delta M_0 t_1 - |x| t_1^{-1/(2\alpha)} M_1, \qquad 0 < t_1 < t \leq T_\delta,$$

and this together with the first inequality of (4.8) implies

$$\text{(4.11)} \quad \underline{u}(t, x; a, \delta\tilde{F}; r_0) \geq \underline{u}(t, 0; a, \delta\tilde{F}; r_0) - \delta M_0 t_1 - |x| t_1^{-1/(2\alpha)} M_1,$$

for $0 < t_1 < t \leq T_\delta$. By the assumption $F(\lambda, \mu) \geq \delta\tilde{F}(\lambda, \mu)$ for $0 \leq \lambda$, $\mu \leq (1+M)/2$, Theorem 2.2′ implies that $\underline{u}(t, x; a, F; r_0) \geq \underline{u}(t, x; a, \delta\tilde{F}; r_0)$ for $0 \leq t \leq T_\delta$, and hence we obtain (4.7) noting (4.11).

Step 2. Let K be a compact subset of $\boldsymbol{R}^d$ such that $K_0 \subset K$. We shall prove that there exists a positive constant $T (> 2r_0)$ such that

$$\text{(4.12)} \quad \underline{u}(t, x; a, F; r_0) > M \qquad \text{for} \quad T - 2r_0 \leq t \leq T \quad \text{and} \quad x \in K.$$

Since $\underline{u}(T_\delta - t_2, x; a, \delta\tilde{F}; r_0)$, $0 \leq t_2 < T_\delta$, belongs to $\mathscr{A}$ as a function of x by Lemma 4.3, we have

$$\underline{u}(T_\delta - t_2, 0; a, \delta\tilde{F}; r_0) \geq P_{t_2}\underline{u}(T_\delta - t_2, 0; a, \delta\tilde{F}; r_0)$$
$$\geq \underline{u}(T_\delta, 0; a, \delta\tilde{F}; r_0) - \delta M_0 t_2,$$

for $0<t_2<T_\delta$, using the second inequality of (4.8) with $t=T_\delta$ and $t_1=t_2$. Moreover, since the definition of T_δ implies $\underline{u}(T_\delta, 0; a, \delta\tilde{F}; r_0)=(1+M)/2$, we have for $0\leq t_2<T_\delta$

$$\underline{u}(T_\delta - t_2, 0; a, \delta\tilde{F}; r_0) \geq \frac{1+M}{2} - \delta M_0 t_2. \tag{4.13}$$

Combining (4.13) with (4.7) of Step 1, we have

$$\underline{u}(T_\delta - t_2, x; a, F; r_0) \geq \frac{1+M}{2} - \delta M_0 t_2 - \delta M_0 t_1 - M_1|x|t_1^{-1/(2\alpha)} \tag{4.14}$$

for $0<t_1<T_\delta-t_2\leq T_\delta$. In the above inequality we can choose first large t_1 and then small $\delta>0$ so that

$$\begin{cases} t_1 < T_\delta - 2r_0, \\ \dfrac{1+M}{2} - \delta M_0\cdot 2r_0 - \delta M_0 t_1 - M_1|x|t_1^{-1/(2\alpha)} > M \qquad \text{for any} \quad x\in K. \end{cases} \tag{4.15}$$

From (4.14) and (4.15) we have for any $0\leq t_2\leq 2r_0$

$$\underline{u}(T_\delta - t_2, x; a, F; r_0) > M \quad \text{for } 0 < t_1 < T_\delta - t_2 \leq T_\delta, \quad x\in K,$$

and therefore we obtain (4.12) with $T=T_\delta$.

Step 3. For any fixed t with $T-r_0\leq t\leq T$ we put

$$a_t(s, x) = \underline{u}(t+s, x; a, F; r_0), \qquad -r_0 \leq s \leq 0,$$
$$w(s, x) = \underline{u}(s, x; a_t, F; r_0) = \underline{u}(t+s, x; a, F; r_0), \qquad s > 0.$$

Since $a(s, x)<M$ $(x\in K)$, $a(s, x)=0$ $(x\notin K)$ and $T-2r_0\leq t+s\leq T$ for $-r_0\leq s\leq 0$, (4.12) implies

$$a_t(s, x) = \underline{u}(t+s, x; a, F; r_0) > M > a(s, x), \qquad x\in K,$$
$$a_t(s, x) > 0 = a(s, x), \qquad x\notin K,$$

for any $-r_0\leq s\leq 0$, and hence Theorem 2.2′ and (4.12) imply that

$$\underline{u}(t+s, x; a, F; r_0) = w(s, x) \geq \underline{u}(s, x; a, F; r_0) > M,$$

for any $x\in K$ and $T-2r_0\leq s\leq T$, that is,

$$\underline{u}(t, x; a, F; r_0) > M,$$

for any $x \in K$ and $2T-3r_0 \leq t \leq 2T$. Repeating this argument, we have

$$\underline{u}(t, x; a, F; r_0) > M,$$

for any $x \in K$ and $nT-(n+1)r_0 \leq t \leq nT$, and hence we find $T^*>0$ such that

$$\underline{u}(t, x; a, F; r_0) > M,$$

for any $x \in K$ and $t \geq T^*$ (for example, $T^* = \left(\left[\frac{T}{r_0}\right]-1\right)T - \left[\frac{T}{r_0}\right]r_0$). This completes the proof of the theorem, since $0 \leq \underline{u}(t, x; a, F; r_0) \leq 1$ and M was arbitrary under the condition $\|a\|_\infty < M < 1$.

§5. Growing-up problem: The case $F(\lambda, 1)=0$

In this section we consider the equations (I) and ($\underline{\text{I}}$). Combining Theorem 3.2 with Theorem 4.1 and Theorem 2.5 we shall obtain the following results. For $\delta>0$, we put

$$F_\delta(\lambda) = \inf\{F(\xi, \eta): \lambda \leq \xi \leq \delta, \lambda \leq \eta \leq \delta\}, \qquad 0 \leq \lambda \leq \delta.$$

THEOREM 5.1. *Suppose that the function $F(\lambda, \mu)$ satisfies the following conditions.*

(F.1°) *$F(\lambda, \mu)$ is a nonnegative Lipschitz continuous function on $[0, 1] \times [0, 1]$ with $F(\lambda, 1)=0$ for any $\lambda \in [0, 1]$ and $F(\lambda, \mu)>0$ for $(\lambda, \mu) \in (0, 1] \times (0, 1)$, and nondecreasing in λ for each fixed μ.*

(F.3) $$\int_0^\delta F_\delta(\lambda)/\lambda^{2+\frac{2\alpha}{d}} d\lambda = \infty \qquad \textit{for some} \quad \delta > 0.$$

(F.4) *There exist positive constants δ and c_0 (≤ 1) such that*

$$F_\delta(\lambda_1\lambda_2) \geq c_0 \lambda_2^{1+\frac{2\alpha}{d}} F_\delta(\lambda_1) \quad \textit{for} \ 0<\lambda_1 \leq \lambda_2, \ \lambda_1 < c_0, \ \lambda_1\lambda_2 < c_0.$$

Then, for any initial value $a(t, x)$ satisfying

(a.1°) *$a(t, x)$ is a nonnegative, bounded and uniformly continuous function on $[-r_0, 0] \times \boldsymbol{R}^d$ with $0 \leq a(t, x) \leq 1$ and $a(0, x) \not\equiv 0$,*

and for any nonnegative bounded continuous time-lag $r(t, x)$ with $0 \leq r(t, x) \leq r_0$, the positive solution $u(t, x; a, F; r)$ of the equation (I) *dominated by* 1 *grows up to* 1 *as $t \to \infty$.*

REMARK 5.2. Under the condition (F.1°) (or (F.1)), $F_\delta(\lambda)$ is equal to $\inf_{\lambda \leq \eta \leq \delta} F(\lambda, \eta)$, and moreover if $F(\lambda, \mu)$ is nondecreasing in μ for $0<\mu \leq \delta$, then $F_\delta(\lambda) = F_\Delta(\lambda) \equiv F(\lambda, \lambda)$.

The theorem is the immediate consequence of Theorem 2.5 and the following theorem.

THEOREM 5.3. *Under the conditions* (F.1°), (F.3) *and* (F.4) *of Theorem* 5.1, *for any* $r_0>0$ *and any initial value* $a(t, x)$ *satisfying* (a.1°), *the positive solution* $\underline{u}(t, x; a, F; r_0)$ *of the equation* ($\underline{\mathrm{I}}$) *grows up to* 1 *as* $t\to\infty$.

This theorem follows immediately from the following lemma, Theorem 3.2 and Theorem 4.1.

LEMMA 5.4. *For each function* $F(\lambda, \mu)$ *satisfying the conditions* (F.1°), (F.3) *and* (F.4) *of Theorem* 5.1 *there exists a function* $\tilde{F}(\lambda, \mu)$ *satisfying the conditions* (F.1), (F.2), (F.3*), (F.4*) *of Theorem* 3.2 *and* (iv) *of Theorem* 4.1.

PROOF. In a way similar to Lemma 3.6 of [11], for the function $F_\delta(\lambda)$ we can find a nondecreasing locally Lipschitz continuous function $\tilde{F}_\delta(\lambda)$ satisfying the following conditions (i)$\sim$(iv).

(i) $\tilde{F}_\delta(0) = 0,\ \tilde{F}_\delta(\lambda) > 0 \quad \text{for} \quad \lambda > 0.$

(ii) $\int_0^{\delta'} \tilde{F}_\delta(\lambda)/\lambda^{2+(2\alpha/d)}d\lambda = \infty \quad \text{for some} \quad \delta' > 0.$

(iii) There exists a positive constant c (≤ 1) such that

$$\tilde{F}_\delta(\lambda_1\lambda_2) \geq c\lambda_2^{1+(2\alpha/d)}\tilde{F}_\delta(\lambda_1), \qquad 0 < \lambda_1 \leq \lambda_2,\ \lambda_1 < c,$$

$$\tilde{F}_\delta(\lambda_1\lambda_2) \geq c\lambda_2^{2+(2\alpha/d)}\tilde{F}_\delta(\lambda_1), \qquad 0 < \lambda_2 \leq \lambda_1 < c.$$

(iv) $\liminf_{\lambda\downarrow 0} F_\delta(\lambda)/\tilde{F}_\delta(\lambda) > 0.$

Then, $\tilde{F}(\lambda, \mu)=\tilde{F}_\delta(\lambda\wedge\mu)$ has the desired properties.

Next we consider the following equation without time-lag.

$$\text{(III)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^\alpha u + f(u), & t > 0, \\ u(0, x) = a(x), & x \in \boldsymbol{R}^d. \end{cases}$$

Then we have the next theorem in a way similar to the case with time-lag. In this case we can replace the conditions (F.3) and (F.4) by (f.3) and (f.4) which are slightly weaker.

THEOREM 5.5. *Suppose that* $f(\mu)$ *satisfies the following conditions*:

(f.1°) $f(\mu)$ *is a Lipschitz continuous function on* [0, 1] *with* $f(1)=0$ *and* $f(\mu)>0$ *for* $0<\mu<1$.

(f.3) $\int_0^{\delta} f(\mu)/\mu^{2+\frac{2\alpha}{d}}\,d\mu = \infty$ *for some* $\delta > 0$.

(f.4) *There exists a positive constant* c_1 (≤ 1) *such that*

$$f(\mu_1\mu_2) \geq c_1\mu_2^{1+\frac{2\alpha}{d}} f(\mu_1) \quad \text{for } 0 < \mu_1 \leq \mu_2,\ \mu_1 < c_1,\ \mu_1\mu_2 < c_1.$$

Then, for any continuous initial value $a(x)$ *with* $0 \leq a(x) \leq 1$ *and* $a(x) \not\equiv 0$, *the solution* $u(t, x; a, f)$ *of the equation* (III) *grows up to* 1 *as* $t \to \infty$.

§6. Growing-up problem: The case $F(\lambda, \mu) > 0$ for $\lambda > 0, \mu > 0$

This case can be treated by modifying the results of the preceding section.

THEOREM 6.1. *Suppose that the function* $F(\lambda, \mu)$ *satisfies the conditions* (F.1) *of Theorem* 3.2 *and* (F.3), (F.4) *of Theorem* 5.1. *Then, for any initial value* $a(t, x)$ *satisfying* (a.1) *of Theorem* 3.2, *for any* $r_0 > 0$ *and for any bounded continuous time-lag* $r(t, x)$ *with* $0 \leq r(t, x) \leq r_0$, *the solutions* $u(t, x; a, F; r)$ *of the equation* (I) *and* $\underline{u}(t, x; a, F; r_0)$ *of* ($\underline{\text{I}}$) *blow up in a finite time or grow up to infinity as* $t \to \infty$.

PROOF. As in § 5, by virtue of Theorem 2.5 it is enough to prove this theorem for the solution $\underline{u}(t, x; a, F; r_0)$ of ($\underline{\text{I}}$). We assume that $\underline{u}(t, x; a, F; r_0)$ is a global solution of ($\underline{\text{I}}$). Let $g_n(\mu)$, $n \geq 1$, be Lipschitz continuous functions on $[0, n]$ satisfying the following conditions:

(i) $g_n(\mu) = 1$ for $0 \leq \mu \leq n/2$.

(ii) $0 < g_n(\mu) \leq 1$ for $n/2 < \mu < n$.

(iii) $g_n(n) = 0$.

Since $F_n(\lambda, \mu) \equiv F(\lambda, \mu) g_n(\mu)$ satisfies the conditions (F.3), (F.4) and the condition

(F.1′) $F_n(\lambda, \mu)$ is a nonnegative Lipschitz continuous function on $[0, n] \times [0, n]$ with $F_n(\lambda, n) = 0$ for $\lambda \in [0, n]$ and $F_n(\lambda, \mu) > 0$ for $(\lambda, \mu) \in (0, n] \times (0, n)$ and nondecreasing in λ for each fixed μ.

Therefore by Theorem 5.3 (with an obvious modification), $\underline{u}(t, x; a, F_n; r_0)$ grows up to n as $t \to \infty$. On the other hand, since $F(\lambda, \mu) \geq F_n(\lambda, \mu)$, $n \geq 1$, Theorem 2.2′ implies

$$\underline{u}(t, x; a, F; r_0) \geq \underline{u}(t, x; a, F_n; r_0) \quad \text{for } n \geq 1,$$

from which the theorem follows.

In case without time-lag, we have the following result in a similar way.

THEOREM 6.2. *Let $f(\mu)$ be a locally Lipschitz continuous function on* $[0, \infty)$ *with $f(\mu)>0$ for $\mu>0$ and satisfy the conditions* (f.3) *and* (f.4) *of Theorem* 5.5. *Then for any nonnegative bounded continuous initial value $a(x)$ with $a(x)\not\equiv 0$ the solution $u(t, x; a, f)$ of* (III) *blows up in a finite time or grows up to infinity as $t\to\infty$.*

§7. Condition for non growing-up

In this section we consider the equations (I) and ($\bar{\text{I}}$), and seek a sufficient condition in order that some positive solutions of these equations die out as $t\to\infty$. Our result is

THEOREM 7.1. *Suppose that the function $F(\lambda, \mu)$ satisfies the following conditions*:

(F.1a) *$F(\lambda, \mu)$ is a nonnegative locally Lipschitz continuous function on $\boldsymbol{R}_+\times\boldsymbol{R}_+$ with $F(0, 0)=0$.*

(F.1b) *There exists a positive constant c_2 (≤ 1) such that $F(\lambda, \mu)$ is nondecreasing in λ and μ for $0\leq\lambda, \mu<c_2$.*

(F.3$^{\#}$) $\displaystyle\int_0^\delta F_\Delta(\lambda)/\lambda^{2+\frac{2\alpha}{d}}\,d\lambda<\infty$ *for some* $\delta>0$.

(F.4′) *There exists a positive constant c_2' (≤ 1) such that*

$$F_\Delta(\lambda_1\lambda_2)\geq c_2'\lambda_2F_\Delta(\lambda_1) \quad \text{for } 0<\lambda_1<c_2',\ \lambda_2\geq 1,\ \lambda_1\lambda_2<c_2'.$$

Then, for some small initial value $a(t, x)$ satisfying the condition (a. 1) *of Theorem* 3.2, *the positive solutions $u(t, x; a, F; r)$ of the equation* (I) *and $\bar{u}(t, x; a, F; r_0)$ of* ($\bar{\text{I}}$) *converge to* 0 *uniformly in x as $t\to\infty$.*

We may assume that the constants c_2 and c_2' are the same by taking the smaller one. By Theorem 2.5 we may consider only the equation ($\bar{\text{I}}$). Moreover it is sufficient to prove the theorem for ($\bar{\text{I}}$) replacing the local conditions (F.1b), (F.4′) by the following global conditions.

(F.1b′) $F(\lambda, \mu)$ is nondecreasing in λ and μ on $\boldsymbol{R}_+\times\boldsymbol{R}_+$.

(F.4″) There exists a positive constant c_2 (≤ 1) such that

$$F_\Delta(\lambda_1\lambda_2)\geq c_2\lambda_2F_\Delta(\lambda_1) \qquad \text{for} \quad \lambda_2\geq 1,\ \lambda_1>0.$$

Let $h(s, x)=P_sa(0, x)$ for $s>0$ and $h(s, x)=a(s, x)$ for $-r_0\leq s\leq 0$ and set

$$b=\sup_{t\geq 0, x\in\boldsymbol{R}^d} h^*(t, x)/P_ta(0, x), \quad h^*(t, x)=\max_{t-r_0\leq s\leq t} h(s, x).$$

Assuming $b<\infty$, we consider the equation

$$(7.1)\qquad \begin{cases} \dfrac{dv}{dt}(t) = \dfrac{F_\Delta(bk(t)v(t))}{c_2k(t)}, & t>0,\\ v(0) = 1, \end{cases}$$

where $k(t)=\|P_ta(0,\cdot)\|_\infty$.

LEMMA 7.2. *Suppose that $F(\lambda, \mu)$ satisfies the conditions* (F.1a), (F.1b′), (F.4″) *and $a(t, x)$ satisfies the condition* (a.1) *of Theorem* 3.2, *and let $v(t)$ be the solution of the above equation* (7.1). *Then, we have*

$$\bar{u}(t, x; a, F; r_0) \le v(t)P_ta(0, x)$$

whenever $v(\cdot)$ exists till time t.

This lemma corresponds to Lemma 5.2 of [11] and the proof is similar, as sketched here. We put

$$\begin{cases} v_0(t) = 1,\\ v_n(t) = 1 + \dfrac{1}{c_2}\displaystyle\int_0^t F_\Delta(bk(s)v_{n-1}(s))/k(s)ds, & n = 1, 2,\dots, \end{cases}$$

$$\begin{cases} u_0(t, x) = P_ta(0, x), & t>0,\\ u_0(t, x) = a(t, x), & -r_0 \le t \le 0, \end{cases}$$

$$\begin{cases} u_n(t, x) = P_ta(0, x) + \displaystyle\int_0^t dsP_{t-s}F(u^*_{n-1}(s,\cdot), u_{n-1}(s,\cdot))(x), & t>0,\\ u_n(t, x) = a(t, x), \quad -r_0 \le t \le 0, \quad n = 1, 2,\dots. \end{cases}$$

Then we can prove by induction that

$$u_n(t, x) \le v_n(t)P_ta(0, x) \qquad \text{for} \quad t>0, \quad n = 0, 1, 2,\dots.$$

The conclusion of the lemma follows since $v_n(t)\to v(t)$ and $u_n(t, x)\to\bar{u}(t, x; a, F; r_0)$ as $n\to\infty$ whenever $v(\cdot)$ exists till time t.

PROOF OF THEOREM 7.1. Let $0<\beta<1$, $t_0>r_0$ and $a(t, x)=\beta p(t+t_0, x)$, $-r_0\le t\le 0$. Then we have

$$\begin{cases} k(t) = \|\beta p(t+t_0,\cdot)\|_\infty = \beta p(t+t_0, 0), & t>0,\\ b = \{(t_0-r_0)/t_0\}^{-d/(2\alpha)}. \end{cases}$$

Now we consider the solution $v(t)$ of the equation (7.1) and the solution $w(t)$ of

$$(7.2)\qquad \frac{dw}{dt}(t) = \frac{w(t)}{c_2^2k(t)}F_\Delta(b\beta^{-1/2}k(t)), \quad w(0) = 1.$$

Using the condition (F.4″) to $F_\Delta(b\beta^{-1/2}k(t))$ with $\lambda_1=bk(t)w$ and $\lambda_2=\beta^{-1/2}w^{-1}\geq 1$ (for $1\leq w\leq\beta^{-1/2}$), we have

$$\frac{w}{c_2^2k(t)}F_\Delta(b\beta^{-1/2}k(t))\geq\frac{\beta^{-1/2}}{c_2k(t)}F_\Delta(bk(t)w)\geq\frac{1}{c_2k(t)}F_\Delta(bk(t)w),$$

for $1\leq w\leq\beta^{-1/2}$. Therefore

$$v(t)\leq w(t)\quad\text{whenever}\quad w(t)\leq\beta^{-1/2}.$$

$w(t)$ can be solved explicitly; in fact we have

$$\begin{aligned}w(t)&=\exp\left\{\frac{1}{c_2^2}\int_0^t\frac{F_\Delta(b\beta^{-1/2}k(s))}{k(s)}ds\right\}\\&\leq\exp\left\{\frac{2\alpha b^{(d+2\alpha)/d}\beta^{(2\alpha-d)/(2d)}}{c_2^2d}p(1,0)^{2\alpha/d}\int_0^{b\beta^{-1/2}k(0)}F_\Delta(\lambda)\lambda^{-2-(2\alpha/d)}d\lambda\right\},\end{aligned}$$

which converges to 1 uniformly in t as $t_0\to\infty$, because $k(0)=\beta p(t_0,0)\to 0$ as $t_0\to\infty$. Therefore $w(t)\leq\beta^{-1/2}$ for all t if t_0 is sufficiently large, and hence $v(t)$ is bounded. Thus $\bar{u}(t,x;a,F;r_0)\leq v(t)P_ta(0,x)\to 0$ as $t\to\infty$.

EXAMPLE. If $F(\lambda,\mu)$ is nondecreasing in λ and μ for small $\lambda>0$ and $\mu>0$, then $F_\delta(\lambda)=F_\Delta(\lambda)=F(\lambda,\lambda)$ for smaller $\delta>0$. We consider the case when $F_\Delta(\lambda)$ $(=F_\delta(\lambda))$ is given by

$$F_\Delta(\lambda)=\lambda^{1+\frac{2\alpha}{d}}\left\{\log\frac{1}{\lambda}\cdot\log_{(2)}\frac{1}{\lambda}\cdots\log_{(n-1)}\frac{1}{\lambda}\cdot\left(\log_{(n)}\frac{1}{\lambda}\right)^\sigma\right\}^{-1}$$

near the origin, $F(\lambda,\mu)$ is smooth on $[0,1]\times[0,1]$, positive in $(0,1]\times(0,1)$, nondecreasing in $\lambda\in[0,1]$ for each fixed μ $(0\leq\mu\leq 1)$, nondecreasing in μ (for small $\mu>0$) for each fixed λ $(0\leq\lambda\leq 1)$, and $F(\lambda,1)=0$ for $0\leq\lambda\leq 1$, where $\sigma\geq 0$, $n\geq 1$ and $\log_{(k)}\mu=\log\log\cdots\log\mu$ (k-times). (For example $F(\lambda,\mu)=\lambda^\xi\mu^\eta\cdot\left(\log\frac{1}{\mu}\right)^{-\sigma}$, where $\xi\geq 0$, $\eta>0$, $\sigma>0$ and $\xi+\eta=1+(2\alpha/d)$.)

(a) If $0\leq\sigma\leq 1$, then we can prove that $F(\lambda,\mu)$ satisfies the conditions (F.1°), (F.3) and (F.4) of Theorem 5.1, and hence any positive solution of (I) with the initial value $a(t,x)$ satisfying (a.1°) grows up to 1 as $t\to\infty$.

(b) If $\sigma>1$, then we can prove that $F(\lambda,\mu)$ satisfies the conditions (F.1a), (F.1b), (F.3#) and (F.4′) of Theorem 7.1. Therefore some positive solution of (I) dominated by 1 converges to 0 uniformly in x as $t\to\infty$ (cf. [11]).

§8. Remarks to the blowing-up problem

In Theorem 6.1 we have found a sufficient condition under which the solution $u(t,x;a,F;r)$ of the equation (I) either blows up in a finite time or grows up to infinity as $t \to \infty$. Here we seek a sufficient condition under which the solution $u(t, x; a, F; r)$ of (I) blows up in a finite time. In this problem, of course, the behavior of $F(\lambda, \mu)$ near $\mu = \infty$ for large λ plays an important role.

THEOREM 8.1. *Suppose that the function $F(\lambda, \mu)$ satisfies the conditions of Theorem* 6.1 *and the following* (F.5).
(F.5) *There exist positive constants λ_0, μ_0 and c_3 such that*

$$\text{(a)} \quad F(\lambda_0, \mu_2) \geq c_3 F(\lambda_0, \mu_1) \qquad \textit{for} \quad \mu_0 \leq \mu_1 \leq \mu_2,$$

$$\text{(b)} \quad \int^{\infty} \frac{d\mu}{F(\lambda_0, \mu)} < \infty.$$

Then, for any initial value $a(t, x)$ satisfying (a.1) *of Theorem* 3.2 *and for any $r_0 > 0$, the solution $\underline{u}(t, x; a, F; r_0)$ of* ($\underline{\mathrm{I}}$) (*and hence, for any continuous time-lag $r(t, x)$ with $0 \leq r(t, x) \leq r_0$, the solution $u(t, x; a, F; r)$ of* (I)) *blows up in a finite time.*

PROOF. Assuming that $u(t, x) = \underline{u}(t, x; a, F; r_0)$ does not blow up in a finite time, we derive a contradiction. By Theorem 6.1 $u(t, x)$ grows up to infinity as $t \to \infty$ and hence for any $M > \lambda_0$ there exists $t_M > r_0$ such that $u(t, x) > M$ for $|x| \leq 1$ and $t \geq t_M - r_0$. We put

$$\rho_M(t) = \min_{|x| \leq 1} u(t + t_M, x) > M, \qquad t \geq 0,$$

$$\eta = \inf_{0 \leq t \leq 1} \min_{|x| \leq 1} \int_{|y| \leq 1} p(t, x-y) dy > 0.$$

Now $u(t, x)$ satisfies the following equation

$$\text{(8.1)} \quad u(t + t_M, x) = P_t u(t_M, x) + \int_0^t ds P_{t-s} F(u_*(s + t_M, \cdot), u(s + t_M, \cdot))(x),$$

for any $0 \leq t < \infty$, and then

$$\rho_M(t) \geq \inf_{0 \leq t \leq 1} \min_{|x| \leq 1} P_t u(t_M, x) \geq \eta \rho_M(0) \geq \eta M > \mu_0, \qquad 0 \leq t \leq 1,$$

provided $M > \mu_0/\eta$. Therefore by the assumption (a) of (F.5) we have

$$\rho_M(t) \geq \eta M + c_3 \eta \int_0^t F(\lambda_0, \rho_M(s)) ds,$$

for $0 \leq t \leq 1$ and $M > \mu_0/\eta$. Let $\varphi(t)$ be the solution of

$$\text{(8.2)} \qquad \varphi(t) = \eta M + c_3^2 \eta \int_0^t F(\lambda_0, \varphi(s)) ds.$$

Then we have, for $0 \leq t \leq 1$,

$$(8.3) \qquad \rho_M(t) \geq \varphi(t).$$

In fact, (8.3) can be proved as follows. Let $\varphi_\varepsilon(t)$ be the solution of (8.2) with the first term ηM (in the right hand side) replaced by $\eta M - \varepsilon\ (>\mu_0)$. First we show that $\rho_M(t) \geq \varphi_\varepsilon(t)$. Assume the contrary and define

$$\tau = \inf\{t > 0 : \rho_M(t) < \varphi_\varepsilon(t)\}.$$

Then we have

$$(8.4) \quad 0 = \rho_M(\tau) - \varphi_\varepsilon(\tau) \geq \varepsilon + c_3\eta\int_0^\tau \{F(\lambda_0, \rho_M(s)) - c_3F(\lambda_0, \varphi_\varepsilon(s))\}ds \geq \varepsilon,$$

since the integrand in the above is nonnegative by (a) of (F.5). (8.4) is absurd. Since $0 < \varepsilon\ (< \eta M - \mu_0)$ is arbitrary, we have (8.3). On the other hand, since $\varphi(t)$ satisfies the equation

$$\int_{\eta M}^{\varphi(t)} \frac{d\mu}{F(\lambda_0, \mu)} = c_3^2\eta t,$$

the assumption (b) of (F.5) implies $\varphi(1) = \infty$ provided M is large enough. Therefore $\rho_M(1) = \infty$, which contradicts the assumption that $u(t, x)$ does not blow up in a finite time.

In the case without time-lag we have the following result.

THEOREM 8.2. *Suppose that the function $f(\mu)$ satisfies the conditions of Theorem* 6.2 *and the following* (f.5).
(f.5) *There exist positive constants μ_0 and c_3 such that*

$$f(\mu_2) \geq c_3 f(\mu_1) \qquad \text{for} \quad \mu_0 \leq \mu_1 \leq \mu_2.$$

Then, for any nonnegative bounded continuous initial value $a(x)\ (\not\equiv 0)$, the solution $u(t, x; a, f)$ of (III) *blows up in a finite time if and only if*

$$(8.5) \qquad \int^\infty \frac{d\mu}{f(\mu)} < \infty.$$

PROOF. We prove "only if" part. Suppose that $u(t, x) = u(t, x; a, f)$ blows up at time $T_\infty < \infty$, take $t_0 < T_\infty$ so that $\|u(t, \cdot)\|_\infty > \mu_0$ for any $t \geq t_0$ and set $a_0 = \|u(t_0, \cdot)\|_\infty$. The assumption (f.5) implies the existence of a constant $c_4 > 0$ such that

$$(8.6) \qquad f(\mu_2) \geq c_4 f(\mu_1) \qquad \text{for} \quad 0 < \mu_1 \leq \mu_2,\ \mu_0 \leq \mu_2.$$

Since, $u(t+t_0, x) = P_t u(t_0, x) + \int_0^t ds P_{t-s} f(u(s+t_0, \cdot))(x)$, $t < T_\infty - t_0$, an application of (8.6) yields

$$\|u(t+t_0,\cdot)\|_\infty \le a_0 + \frac{1}{c_4}\int_0^t f(\|u(s+t_0,\cdot)\|_\infty)ds, \qquad t < T_\infty - t_0.$$

Let $\varphi(t)$ be the solution of the equation

$$\varphi(t) = a_0 + c_4^{-2}\int_0^t f(\varphi(s))ds.$$

Then $\|u(t+t_0,\cdot)\|_\infty \le \varphi(t)$ for $t<t_\infty$ $(\le T_\infty - t_0 < \infty)$, where t_∞ is the blowing-up time of $\varphi(t)$. Since $\varphi(t)$ satisfies the equation

$$\int_{a_0}^{\varphi(t)} \frac{d\mu}{f(\mu)} = c_4^{-2}t, \quad t < t_\infty < \infty,$$

the integral of left hand side of (8.5) is finite. This completes the proof of the lemma.

REMARK 8.3. As in Theorem 3.2 we can prove the following fact. Any positive solution of (III) either blows up in a finite time or grows up to infinity as $t\to\infty$, if $f(\mu)$ satisfies the following conditions.

(f.1) $f(\mu)$ is a nonnegative locally Lipschitz continuous function on $\boldsymbol{R}_+=[0, \infty)$ with $f(\mu)>0$ for $\mu>0$.

(f.2) $f(\mu)$ is nondecreasing.

(f.3) The same as in Theorem 5.5 in § 5.

(f.4*) There exists a positive constant c (≤ 1) such that

$$\text{(a)}\quad f(\mu_1\mu_2) \ge c\mu_2^{1+(2\alpha/d)}f(\mu_1) \qquad \text{for} \quad 0<\mu_1\le\mu_2,\ \mu_1<c,$$

$$\text{(b)}\quad f(\mu_1\mu_2) \ge c\mu_2^{2+(2\alpha/d)}f(\mu_1) \qquad \text{for} \quad 0<\mu_2\le\mu_1<c.$$

However, this fact combined with Theorem 8.2 implies that only the blowing-up case occurs, because $\int^\infty f(\mu)^{-1}d\mu<\infty$ follows from (a) of (f.4*). For the case $\alpha=1$, see Theorem 2.1 of [11].

§ 9. Proof of Lemma 3.4

In this section we prove Lemma 3.4 stated in § 3. We adopt the notations of § 3. Especially, we must recall the notations of (3.6) and the properties (3.7), (3.8) and (3.9). In addition, we must notice that the followings hold.

(9.1) $\qquad \psi_n(t) > t_0$ implies $\varphi_{n-1}(t) > 2^{n-1}t_0, \qquad n = 1, 2,\dots.$

(9.2) For any constants $\kappa\ge 1$ and $t\ge s\ge 0$, we have

$$\varphi(\kappa t) - \varphi(\kappa s) \ge c\kappa^{-\sigma}\{\varphi(t) - \varphi(s)\}, \qquad \sigma = d/\alpha.$$

In fact, (9.1) is immediate from (3.9), that is, $\psi_n(t)=\psi_{n-1}(\varphi_{n-1}(t))>t_0$ implies $\varphi_{n-1}(t)>2^{n-1}t_0$. (9.2) is proved as follows. By the definition of $\varphi(t)$, we have

$$\varphi(\kappa t) - \varphi(\kappa s) = \int_{\kappa s/2}^{\kappa t/2} \frac{F_\Delta(\theta(\tau))}{\theta(\tau)} d\tau \tag{9.3}$$

$$= \kappa \int_{s/2}^{t/2} \frac{F_\Delta(\theta(\kappa\xi))}{\theta(\kappa\xi)} d\xi.$$

On the other hand, by the monotonicity of F_Δ, we have

$$F_\Delta(\theta(\kappa\xi)) = F_\Delta(\beta(\kappa\xi + t_0)^{-d/(2\alpha)}p(1, 0)) \geq F_\Delta(\kappa^{-d/(2\alpha)}\theta(\xi)).$$

Here we can apply the assumption (F.4*) in Theorem 3.2 to $F_\Delta(\kappa^{-d/(2\alpha)}\theta(\xi))= F_\Delta(\lambda_1\lambda_2)$ with $\lambda_1=\theta(\xi)$ and $\lambda_2=\kappa^{-d/(2\alpha)}$ because we have assumed that β is so small that $\theta(\xi)<c$ by $0<\beta<c/p(t_0, 0)$. In case $\lambda_1<\lambda_2$ we have from (a) of (F.4*)

$$F_\Delta(\kappa^{-d/(2\alpha)}\theta(\xi)) \geq c\lambda_2^{1+(2\alpha/d)}F_\Delta(\lambda_1)$$

$$= c\kappa^{-1-(d/(2\alpha))}F_\Delta(\theta(\xi))$$

$$\geq c\kappa^{-1-(d/\alpha)}F_\Delta(\theta(\xi)),$$

while in case $\lambda_1 \geq \lambda_2$ we have from (b) of (F.4*)

$$F_\Delta(\kappa^{-d/(2\alpha)}\theta(\xi)) \geq c\lambda_2^{2+(2\alpha/d)}F_\Delta(\lambda_1)$$

$$= c\kappa^{-1-(d/\alpha)}F_\Delta(\theta(\xi)).$$

Therefore, in both cases we have

$$F_\Delta(\theta(\kappa\xi)) \geq c\kappa^{-1-(d/\alpha)}F_\Delta(\theta(\xi)),$$

and hence, noting $\theta(\kappa\xi)\leq\theta(\xi)$, we obtain from (9.3)

$$\varphi(\kappa t) - \varphi(\kappa s) \geq c\kappa^{-d/\alpha}\int_{s/2}^{t/2} \frac{F_\Delta(\theta(\xi))}{\theta(\xi)} d\xi$$

$$= c\kappa^{-\sigma}\{\varphi(t) - \varphi(s)\}, \quad \sigma = d/\alpha.$$

We now proceed to the proof of Lemma 3.4.

We shall prove (3.10) by induction in n.

Step 1. We consider the case $n=0$. Assume that $\psi_0(t)\equiv t>t_0>2r_0$ in this step. First we note that $\underline{u}(t, x)=\underline{u}(t, x; a, F; r_0)$ satisfies the integral equation

$$(\underline{\mathrm{I}}') \quad \begin{cases} \underline{u}(t, x) = P_t a(0, x) + \int_0^t ds \int p(t-s, x-y) F(\underline{u}_*(s, y), \underline{u}(s, y)) dy, & t > 0, \\ \underline{u}(t, x) = a(t, x), \quad -r_0 \le t \le 0, \quad x \in \boldsymbol{R}^d, \end{cases}$$

where $a(t, x)$ is given by (3.5) and

$$\underline{u}_*(s, y) = \min_{s-r_0 \le \tau \le s} \underline{u}(\tau, y).$$

We note that by (3.2)

$$(9.4) \qquad \begin{aligned} P_t a(0, x) &= \beta p(t+t_0+r_0, x) \\ &\ge \beta p(t+t_0+r_0, 2^{1+3/(2\alpha)} x) \\ &= u_0(t+r_0, x). \end{aligned}$$

Let $0 \le s \le t$. We first estimate $\underline{u}(\tau, x)$ for $s-r_0 \le \tau \le s$. By the nonnegativity of F we have

$$\underline{u}(\tau, y) \begin{cases} \ge P_\tau a(0, y) = \beta p(\tau+t_0+r_0, y), & \tau > 0, \\ = a(\tau, y) = \beta p(\tau+t_0+r_0, y), & -r_0 \le \tau \le 0. \end{cases}$$

Using (3.1) and then (3.2), we have for $s-r_0 \le \tau \le s$

$$\begin{aligned} \underline{u}(\tau, y) &\ge \beta(\tau+t_0+r_0)^{-d/(2\alpha)} p(1, (\tau+t_0+r_0)^{-1/(2\alpha)} y) \\ &\ge \left(\frac{\tau+t_0+r_0}{s+t_0}\right)^{-d/(2\alpha)} \beta(s+t_0)^{-d/(2\alpha)} p(1, (s+t_0)^{-1/(2\alpha)} 2^{1+3/(2\alpha)} y) \\ &> 3^{-d/(2\alpha)} u_0(s, y), \end{aligned}$$

where we have used that $(\tau+t_0+r_0)/(s+t_0) \le (t_0+r_0)/t_0 < 3$ for $\tau \le s$ and $t_0 > 2r_0$. Hence

$$\min \{\underline{u}_*(s, y), \underline{u}(s, y)\} > 3^{-d/(2\alpha)} u_0(s, y).$$

Therefore, by the monotonicity of F_Δ, we have from $(\underline{\mathrm{I}}')$ and (9.4)

$$(9.5) \qquad \underline{u}(t, x) > u_0(t+r_0, x) + \int_0^t ds \int p(t-s, x-y) F_\Delta(3^{-d/(2\alpha)} u_0(s, y)) dy.$$

Next, let us assume that $y \in \Omega \equiv \{|y| \le (s+t_0)^{1/(2\alpha)}\}$. Then we have by (3.2)

$$\begin{aligned} u_0(s, y) &= \theta(s) p(1, (s+t_0)^{-1/(2\alpha)} 2^{1+3/(2\alpha)} y)/p(1, 0) \\ &\ge \theta(s) p(1, 2^{1+3/(2\alpha)})/p(1, 0). \end{aligned}$$

Now we apply (F.4*) to $F_\Delta(\theta(s) \cdot 3^{-d/(2\alpha)} p(1, 2^{1+3/(2\alpha)})/p(1, 0)) = F_\Delta(\lambda_1 \lambda_2)$ with $\lambda_1 = \theta(s) < c$ and $\lambda_2 = 3^{-d/(2\alpha)} p(1, 2^{1+3/(2\alpha)})/p(1, 0) < 1$. Then we have, for $y \in \Omega$,

(9.6)
$$\begin{aligned} F_\Delta(3^{-d/(2\alpha)}u_0(s, y)) &\geq F_\Delta(\lambda_1\lambda_2) \\ &\geq \min\{c\lambda_2^\gamma, c\lambda_2^{1+\gamma}\}F_\Delta(\lambda_1) \\ &= c\{3^{-d/(2\alpha)}p(1, 2^{1+3/(2\alpha)})/p(1, 0)\}^{1+\gamma}F_\Delta(\theta(s)) \\ &= ca_1F_\Delta(\theta(s)), \end{aligned}$$

where $a_1 = \{3^{-d/(2\alpha)}p(1, 2^{1+3/(2\alpha)})/p(1, 0)\}^{1+\gamma} > 0$ and $\gamma = 1 + (2\alpha/d)$. On the other hand by (3.1) and (3.3) we have

$$p(t-s, x-y) \geq p(1, 0)^{-1}(t-s)^{-d/(2\alpha)}p(1, 2x(t-s)^{-1/(2\alpha)})p(1, 2y(t-s)^{-1/(2\alpha)}),$$

and therefore from (9.5) and (9.6) we obtain

(9.7)
$$\begin{aligned} &\underline{u}(t, x) - u_0(t+r_0, x) \\ &\geq \int_0^t ds\int_\Omega p(t-s, x-y)F_\Delta(3^{-d/(2\alpha)}u_0(s, y))dy \\ &\geq ca_1\int_0^t dsF_\Delta(\theta(s))p(1, 0)^{-1}(t-s)^{-d/(2\alpha)} \\ &\qquad \times p(1, 2x(t-s)^{-1/(2\alpha)})\int_\Omega p(1, 2y(t-s)^{-1/(2\alpha)})dy \\ &= ca_1\int_0^t dsF_\Delta(\theta(s))(s+t_0)^{d/(2\alpha)}p(1, 0)^{-1}(t-s)^{-d/(2\alpha)} \\ &\qquad \times p(1, 2x(t-s)^{-1/(2\alpha)})\int_{|y|\leq 1} p\left(1, 2y\left(\frac{s+t_0}{t-s}\right)^{1/(2\alpha)}\right)dy \\ &> ca_1\int_0^{t/2} dsF_\Delta(\theta(s))\theta(s)^{-1}\beta(t-s)^{-d/(2\alpha)} \\ &\qquad \times p(1, 2x(t-s)^{-1/(2\alpha)})\int_{|y|\leq 1} p\left(1, 2y\left(\frac{s+t_0}{t-s}\right)^{1/(2\alpha)}\right)dy. \end{aligned}$$

Let $0 < s < t/2$. Since we have assumed that $t > t_0 > 2r_0$, we have

$$t + t_0 + r_0 > t - s > (t + t_0 + r_0)/8,$$

$$\frac{s+t_0}{t-s} < \frac{t+2t_0}{t} < 3.$$

Hence we have, using (3.1) and (3.2),

(9.8)
$$\begin{aligned} &\beta(t-s)^{-d/(2\alpha)}p(1, 2x(t-s)^{-1/(2\alpha)}) \\ &> \beta(t+t_0+r_0)^{-d/(2\alpha)}p(1, 2x\{(t+t_0+r_0)/8\}^{-1/(2\alpha)}) \\ &= u_0(t+r_0, x), \end{aligned}$$

and

$$(9.9)\quad \int_{|y|\le 1} p\left(1,\, 2y\left(\frac{s+t_0}{t-s}\right)^{1/(2\alpha)}\right)dy > \int_{|y|\le 1} p(1,\, 2y\cdot 3^{1/(2\alpha)})dy \equiv a_2 > 0.$$

Therefore we obtain, from (9.7), (9.8) and (9.9), for $\psi_0(t)\equiv t>t_0$,

$$\begin{aligned} \underline{u}(t, x) - u_0(t+r_0, x) &> ca_1a_2u_0(t+r_0, x)\int_0^{t/2} F_\Delta(\theta(s))/\theta(s)ds \\ &= ca_1a_2u_0(t+r_0, x)\varphi(t) \\ &\ge A\varphi(t)u_0(t+r_0, x), \end{aligned}$$

where $A=c^2a_1a_2>0$.

Step 2. We shall prove that (3.10) holds also for $n+1$ under the assumption that (3.10) holds for n. By the definition of $\varphi_n(t)$ we have

$$\varphi\left(\frac{\varphi_n(t)}{2^n}\right) = \left\{\varphi\left(\frac{t}{2^{n+1}}\right) - \frac{1}{2^n}\right\} \vee 0 < \varphi\left(\frac{t}{2^{n+1}}\right),$$

and hence by the monotonicity of φ

$$(9.10)\qquad \varphi_n(t) < \frac{t}{2}.$$

Now we note that $\underline{u}(t, x)=\underline{u}(t-\varphi_n(t), x;\, a_t,\, F;\, r_0)$ where $a_t(s, x)$, $-r_0\le s\le 0$, is equal to $\underline{u}(\varphi_n(t)+s, x)$ in case $\varphi_n(t)+s>0$ and to $a(\varphi_n(t)+s, x)$ in case $-r_0\le\varphi_n(t)+s\le 0$, and so $\underline{u}(t, x)$ satisfies the following integral equation:

$$(9.11)\quad \begin{cases} \underline{u}(t, x) = P_{t-\varphi_n(t)}\underline{u}(\varphi_n(t), x) + \int_0^{t-\varphi_n(t)} ds\int p(t-\varphi_n(t)-s,\, x-y) \\ \qquad\qquad \times F(\underline{u}_*(\varphi_n(t)+s, y),\, \underline{u}(\varphi_n(t)+s, y))dy, \quad t>\varphi_n(t), \\ \underline{u}(t, x) = a_t(t-\varphi_n(t), x), \quad \varphi_n(t)-r_0\le t\le\varphi_n(t). \end{cases}$$

Since $\underline{u}(t, x)$ is also the solution of the integral equation ($\underline{\mathrm{I}}'$), we have $\underline{u}(\varphi_n(t), x)\ge P_{\varphi_n(t)}a(0, x)$, and hence by (9.4)

$$(9.12)\qquad P_{t-\varphi_n(t)}\underline{u}(\varphi_n(t), x) \ge P_t a(0, x) \ge u_0(t+r_0, x).$$

Assume that

$$\psi_{n+1}(t) > t_0.$$

Then (3.9) and (9.1) imply that

$$(9.13)\qquad t>2^{n+1}t_0 \quad\text{and}\quad \varphi_n(t)>2^n t_0.$$

Assuming $s\ge\varphi_n(t)/2$, we shall estimate $F(\underline{u}_*(\varphi_n(t)+s, y),\, \underline{u}(\varphi_n(t)+s, y))$ in (9.11).

First we estimate $\underline{u}_*(\varphi_n(t)+s, y)=\min_{s-r_0\leq\tau\leq s}\underline{u}(\varphi_n(t)+\tau, y)$ and $\underline{u}(\varphi_n(t)+s, y)$. Since $s\geq\varphi_n(t)/2>t_0/2>r_0$ (by (9.13)), we have by the monotonicity of $\psi_n(t)$

$$\psi_n(\varphi_n(t)+\tau) > \psi_n(\varphi_n(t)) \equiv \psi_{n+1}(t) > t_0 \quad \text{for } (0<)s-r_0\leq\tau\leq s.$$

Therefore the induction hypothesis implies that, for $(0<)s-r_0\leq\tau\leq s$,

$$\begin{aligned}(9.14)\qquad \underline{u}(\varphi_n(t)+\tau, y) &> \{1+B_n(\varphi_n(t)+\tau)\}u_0(\varphi_n(t)+\tau+r_0, y)\\ &> \{1+B_n(\varphi_n(t))\}u_0(\varphi_n(t)+\tau+r_0, y),\end{aligned}$$

because $B_n(t)$ is increasing in t. Now we see that by (3.1) and (3.2), for any τ with $s-r_0\leq\tau\leq s$,

$$\begin{aligned}(9.15)\quad u_0(\varphi_n(t)+\tau+r_0, y) &= \beta p(\varphi_n(t)+\tau+r_0+t_0, 2^{1+3/(2\alpha)}y)\\ &= \theta(s)\left(\frac{s+t_0}{\varphi_n(t)+\tau+r_0+t_0}\right)^{d/(2\alpha)}\cdot\frac{p(1, 2^{1+3/(2\alpha)}(\varphi_n(t)+\tau+r_0+t_0)^{-1/(2\alpha)}y)}{p(1, 0)}\\ &> \theta(s)\left(\frac{s+t_0}{\varphi_n(t)+s+r_0+t_0}\right)^{d/(2\alpha)}\cdot\frac{p(1, 2^{1+3/(2\alpha)}(\varphi_n(t)+s+t_0)^{-1/(2\alpha)}y)}{p(1, 0)},\end{aligned}$$

and hence by (9.14) and (9.15)

$$\begin{aligned}&\min\{\underline{u}_*(\varphi_n(t)+s, y), \underline{u}(\varphi_n(t)+s, y)\}\\ &\quad > \{1+B_n(\varphi_n(t))\}\theta(s)\left(\frac{s+t_0}{\varphi_n(t)+s+r_0+t_0}\right)^{d/(2\alpha)}\\ &\qquad\qquad \times\frac{p(1, 2^{1+3/(2\alpha)}(\varphi_n(t)+s+t_0)^{-1/(2\alpha)}y)}{p(1, 0)}.\end{aligned}$$

Since

$$\frac{s+t_0}{\varphi_n(t)+s+r_0+t_0} > \frac{1}{3} \qquad \text{for } s\geq\frac{\varphi_n(t)}{2} > \frac{t_0}{2} > r_0,$$

and

$$p(1, 2^{1+3/(2\alpha)}(\varphi_n(t)+s+t_0)^{-1/(2\alpha)}y) > p(1, 2^{1+3/(2\alpha)}) \quad \text{for } y\in\Omega,$$

we have

$$\begin{aligned}&\min\{\underline{u}_*(\varphi_n(t)+s, y), \underline{u}(\varphi_n(t)+s, y)\}\\ &\quad > \{1+B_n(\varphi_n(t))\}\theta(s)3^{-d/(2\alpha)}p(1, 2^{1+3/(2\alpha)})/p(1, 0),\end{aligned}$$

provided that $\psi_{n+1}(t)>t_0>2r_0$, $s\geq\varphi_n(t)/2$ and $y\in\Omega$. Hence, putting

$$\begin{aligned}\lambda_1 &= \theta(s)\ (<c),\\ \lambda_2 &= \{1+B_n(\varphi_n(t))\}3^{-d/(2\alpha)}p(1, 2^{1+3/(2\alpha)})/p(1, 0),\end{aligned}$$

we have

$$F(\underline{u}_*(\varphi_n(t)+s,\ y),\ \underline{u}(\varphi_n(t)+s,\ y)) \geq F_\Delta(\lambda_1\lambda_2),$$

since $F(\lambda, \mu)$ is nondecreasing in λ and μ. Now we apply (F.4*) to $F_\Delta(\lambda_1\lambda_2)$. In case $\lambda_1 < \lambda_2$ we have from (a) of (F.4*)

$$\begin{aligned} F_\Delta(\lambda_1\lambda_2) &\geq c\lambda_2^\gamma F_\Delta(\lambda_1) \\ &= c[\{1+B_n(\varphi_n(t))\}3^{-d/(2\alpha)}p(1,\ 2^{1+3/(2\alpha)})/p(1,\ 0)]^\gamma F_\Delta(\theta(s)) \\ &> cB_n(\varphi_n(t))^\gamma a_1 F_\Delta(\theta(s)) \end{aligned}$$

with the same constant a_1 appearing in the proof of Step 1, while in case $\lambda_1 \geq \lambda_2$ we have from (b) of (F.4*)

$$\begin{aligned} F_\Delta(\lambda_1\lambda_2) &\geq c\lambda_2^{1+\gamma} F_\Delta(\lambda_1) \\ &= c[\{1+B_n(\varphi_n(t))\}3^{-d/(2\alpha)}p(1,\ 2^{1+3/(2\alpha)})/p(1,\ 0)]^{1+\gamma} F_\Delta(\theta(s)) \\ &> cB_n(\varphi_n(t))^\gamma a_1 F_\Delta(\theta(s)). \end{aligned}$$

Consequently in both cases we have

$$F(\underline{u}_*(\varphi_n(t)+s,\ y),\ \underline{u}(\varphi_n(t)+s,\ y)) > ca_1 B_n(\varphi_n(t))^\gamma F_\Delta(\theta(s))$$

under the conditions $\psi_{n+1}(t) > t_0 > 2r_0$, $s \geq \varphi_n(t)/2$ and $y \in \Omega$, and hence from (9.11) and (9.12) we have

(9.16)

$$\begin{aligned} &\underline{u}(t,\ x) - u_0(t+r_0,\ x) \\ &\geq \int_{\varphi_n(t)/2}^{(t-\varphi_n(t))/2} ds \int p(t-\varphi_n(t)-s,\ x-y) F(\underline{u}_*(\varphi_n(t)+s,\ y),\ \underline{u}(\varphi_n(t)+s,\ y)) dy \\ &> ca_1 B_n(\varphi_n(t))^\gamma \int_{\varphi_n(t)/2}^{(t-\varphi_n(t))/2} ds F_\Delta(\theta(s)) \int_\Omega p(t-\varphi_n(t)-s,\ x-y) dy. \end{aligned}$$

Since $(t-\varphi_n(t))/2 \geq s$ and $t/2 > \varphi_n(t) > t_0 > 2r_0$ (by (9.10) and (9.13)), we have

$$t + t_0 + r_0 > t - \varphi_n(t) - s > (t + t_0 + r_0)/8,$$

and hence, using (3.1) and (3.3), we have

$$\begin{aligned} (9.17) \qquad & p(t-\varphi_n(t)-s,\ x-y) \\ &\geq (t-\varphi_n(t)-s)^{-d/(2\alpha)} p(1,\ 2x(t-\varphi_n(t)-s)^{-1/(2\alpha)}) \\ &\qquad \times p(1,\ 2y(t-\varphi_n(t)-s)^{-1/(2\alpha)})/p(1,\ 0) \end{aligned}$$

$$> \beta(t+t_0+r_0)^{-d/(2\alpha)}p\left(1,\ 2x\left(\frac{t+t_0+r_0}{8}\right)^{-1/(2\alpha)}\right)$$

$$\times\ p(1,\ 2y(t-\varphi_n(t)-s)^{-1/(2\alpha)})/\beta p(1,\ 0)$$

$$= u_0(t+r_0,\ x)p(1,\ 2y(t-\varphi_n(t)-s)^{-1/(2\alpha)})/\beta p(1,\ 0).$$

Making a change of variable, we have for $s \leq (t-\varphi_n(t))/2$

$$(9.18)\qquad \int_\Omega p(1,\ 2y(t-\varphi_n(t)-s)^{-1/(2\alpha)})dy$$

$$= (s+t_0)^{d/(2\alpha)}\int_{|y|\leq 1} p\left(1,\ 2y\left(\frac{s+t_0}{t-\varphi_n(t)-s}\right)^{1/(2\alpha)}\right)dy$$

$$> (s+t_0)^{d/(2\alpha)}\int_{|y|\leq 1} p(1,\ 2y\cdot 3^{1/(2\alpha)})dy \equiv (s+t_0)^{d/(2\alpha)}a_2,$$

where we have used

$$\frac{s+t_0}{t-\varphi_n(t)-s} \leq \frac{t-\varphi_n(t)+2t_0}{t-\varphi_n(t)} < 1 + \frac{4t_0}{t} < 3.$$

Combining (9.16) with (9.17) and (9.18), we have

$$\underline{u}(t,\ x) - u_0(t+r_0,\ x)$$

$$> ca_1a_2B_n(\varphi_n(t))^\gamma u_0(t+r_0,\ x)\int_{\varphi_n(t)/2}^{(t-\varphi_n(t))/2} \frac{F_A(\theta(s))}{\theta(s)}ds.$$

Next, we estimate the integral in the above inequality. Using $t-\varphi_n(t) > t/2$ (by (9.10)) and (9.2), we have

$$\int_{\varphi_n(t)/2}^{(t-\varphi_n(t))/2} \frac{F_A(\theta(s))}{\theta(s)}ds = \varphi(t-\varphi_n(t)) - \varphi(\varphi_n(t))$$

$$> \varphi\left(2^n\cdot\frac{t}{2^{n+1}}\right) - \varphi\left(2^n\cdot\varphi^{-1}\left\{\varphi\left(\frac{t}{2^{n+1}}\right) - \frac{1}{2^n}\right\}\right)$$

$$\geq c(2^n)^{-\sigma}\left\{\varphi\left(\frac{t}{2^{n+1}}\right) - \varphi\left(\frac{t}{2^{n+1}}\right) + \frac{1}{2^n}\right\}$$

$$= c2^{-(1+\sigma)n}.$$

Therefore, recalling the relation $A = c^2a_1a_2$ and the definition of $B_n(t)$, we finally obtain

$$\underline{u}(t,\ x) - u_0(t+r_0,\ x) > A2^{-(1+\sigma)n}B_n(\varphi_n(t))^\gamma u_0(t+r_0,\ x)$$

$$= A^{1+\gamma+\cdots+\gamma^{n+1}}2^{-(1+\sigma)\sum_{k=0}^{n} k\gamma^{n-k}}\left\{\varphi\left(\frac{t}{2^{n+1}}\right) - \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\right\}^{\gamma^{n+1}} u_0(t+r_0,\ x)$$

provided $\psi_{n+1}(t) > t_0$. Thus (3.10) is proved for $n+1$. This completes the proof of Lemma 3.4.

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