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A note on Gruenberg algebras

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1. Let $\rho(L)$, $\mathfrak{e}(L)$ and $\bar{\mathfrak{e}}(L)$ denote respectively the Hirsch-Plotkin radical, the sets of left Engel and bounded left Engel elements of a Lie algebra L over a field $\mathfrak{k}$. The classes of abelian, nilpotent and solvable Lie algebras over $\mathfrak{k}$ are denoted respectively by $\mathfrak{A}$, $\mathfrak{N}$ and $\mathrm{E}\mathfrak{A}$. If $\mathfrak{X}$ is a class of Lie algebras, then $\mathrm{L}\mathfrak{X}$ and $\acute{\mathrm{E}}\mathfrak{X}$ denote respectively the classes of locally $\mathfrak{X}$-algebras and algebras with ascending $\mathfrak{X}$-series.

Simonjan [3] has shown that the class of Gruenberg algebras equals $\acute{\mathrm{E}}\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$ over a field of characteristic 0. Amayo and Stewart have asked the following among "Some open questions" in [1]:

Question 40. *Over a field of characteristic* $p>0$, *suppose that* $L \in \acute{\mathrm{E}}\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$. *Is it true that* $x \in L$ *implies* $\langle x \rangle$ asc L *?*

In this note we shall give an affirmative answer to this question. This will be obtained as a collorary of the following theorem, which is proved over a field of characteristic 0 in [1, Theorem 16.4.2].

THEOREM 1. *Let L be a Lie algebra over a field* $\mathfrak{k}$ *of arbitrary characteristic.*
(a) *If* $L \in \acute{\mathrm{E}}\mathfrak{A}$, *then* $\rho(L) \subseteq \mathfrak{e}(L) = \{x \in L \mid \langle x \rangle \operatorname{asc} L\}$.
(b) *If* $L \in \mathrm{E}\mathfrak{A}$, *then* $\bar{\mathfrak{e}}(L) = \{x \in L \mid \langle x \rangle \operatorname{si} L\}$.

COROLLARY *Let L be a Lie algebra over a field* $\mathfrak{k}$ *of arbitrary characteristic belonging to* $\acute{\mathrm{E}}\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$. *Then* $x \in L$ *implies* $\langle x \rangle \operatorname{asc} L$.

We employ notations and terminology in [1]. All Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified.

2. We show the following lemma on ascending series of a Lie algebra, which is an extension of Lemma 16 in [2].

LEMMA. *Let L be a Lie algebra and* $x \in \mathfrak{e}(L)$. *Assume that L has an ascending* $\mathfrak{X}$*-series where* $\mathfrak{X} = \mathfrak{A}$, $\mathrm{L}\mathfrak{N}$ *or* $\mathrm{LE}\mathfrak{A}$. *Then L has an ascending* $\mathfrak{X}$*-series with terms idealized by x.*

PROOF. Let $(L_\alpha)_{\alpha \leqq \lambda}$ be an ascending $\mathfrak{X}$-series of L with an ordinal λ. Let H_α be the sum of $\langle x \rangle$-invariant subspaces of L_α ($\alpha \leqq \lambda$). Then H_α is the largest $\langle x \rangle$-invariant subalgebra of L_α (cf. [2, Lemma 15]). Clearly $H_0 = L_0 = 0$,

$H_\lambda = L_\lambda = L$ and $H_\alpha \leqq H_\beta$ for $\alpha \leqq \beta \leqq \lambda$. Let $\alpha < \lambda$. Then $[H_\alpha, H_{\alpha+1}] \subseteq [L_\alpha, L_{\alpha+1}] \subseteq L_\alpha$ and $[H_\alpha, H_{\alpha+1}]$ is an $\langle x \rangle$-invariant subspace, whence $[H_\alpha, H_{\alpha+1}] \subseteq H_\alpha$ and so $H_\alpha \lhd H_{\alpha+1}$. Let $\mu \leqq \lambda$ be a limit ordinal and $y \in H_\mu$. Since $x \in \mathfrak{e}(L)$, there exists an integer $n = n(x, y)$ such that $[y, {}_n x] = 0$. Thus $\langle y^{\langle x \rangle} \rangle = \langle y, [y, x], \dots, [y, {}_{n-1}x] \rangle$ is a finitely generated subalgebra of $L_\mu = \cup_{\alpha<\mu} L_\alpha$. Hence $\langle y^{\langle x \rangle} \rangle \leqq L_\alpha$ for some $\alpha < \mu$. Since $\langle y^{\langle x \rangle} \rangle$ is idealized by x, $\langle y^{\langle x \rangle} \rangle \leqq H_\alpha$. Therefore $H_\mu = \cup_{\alpha<\mu} H_\alpha$. Thus $(H_\alpha)_{\alpha \leqq \lambda}$ is an ascending series of L with terms idealized by x.

Let $\mathfrak{X} = \mathfrak{A}$. Then for any $\alpha < \lambda$ $H_{\alpha+1}^2 \leqq L_{\alpha+1}^2 \leqq L_\alpha$ and $H_{\alpha+1}^2$ is idealized by x. Hence $H_{\alpha+1}^2 \leqq H_\alpha$, that is, $H_{\alpha+1}/H_\alpha \in \mathfrak{A}$.

Let $\mathfrak{X} = \mathrm{LE}\mathfrak{A}$ and S be any finite subset of $H_{\alpha+1}$ $(\alpha < \lambda)$. Since $x \in \mathfrak{e}(L)$, $K = \langle S^{\langle x \rangle} \rangle$ is a finitely generated subalgebra of $L_{\alpha+1}$. By the hypothesis $L_{\alpha+1}/L_\alpha$ is locally solvable, whence $K^{(m)} \leqq L_\alpha$ for some integer m. Furthermore $K^{(m)}$ is idealized by x, so that $K^{(m)} \leqq H_\alpha$. Thus $H_{\alpha+1}/H_\alpha \in \mathrm{LE}\mathfrak{A}$.

Similarly, if $\mathfrak{X} = \mathrm{L}\mathfrak{N}$, then $H_{\alpha+1}/H_\alpha$ $(\alpha < \lambda)$ is an $\mathrm{L}\mathfrak{N}$-algebra.

PROOF OF THEOREM 1. (a) Let $x \in \mathfrak{e}(L)$. By Lemma L has an ascending $\mathfrak{A}$-series $(H_\alpha)_{\alpha \leqq \lambda}$ with terms idealized by x. Put $H_{\alpha,i} = \{y \in H_{\alpha+1} \mid [y, {}_i x] \in H_\alpha\}$ for any $\alpha < \lambda$ and any $i \in \mathbf{N}$. Then it is easily seen that

$$H_\alpha + \langle x \rangle = H_{\alpha,0} + \langle x \rangle \lhd H_{\alpha,1} + \langle x \rangle \lhd \cdots,$$

$$\cup_{i \geq 0}(H_{\alpha,i} + \langle x \rangle) = H_{\alpha+1} + \langle x \rangle.$$

Therefore $\langle x \rangle$ asc L.

(b) Let $x \in \bar{\mathfrak{e}}(L)$. Then there exists an integer $n = n(x)$ such that $[L, {}_n x] = 0$. By the same argument as above we have

$$L^{(i+1)} + \langle x \rangle \lhd^n L^{(i)} + \langle x \rangle \quad (i \in \mathbf{N}).$$

Therefore $\langle x \rangle$ si L. This completes the proof.

PROOF OF COROLLARY. Since $L \in \acute{\mathrm{E}}\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$,

$$L = \rho(L) \subseteq \{x \in L \mid \langle x \rangle \text{ asc } L\} \subseteq L$$

by Theorem 1.

We note that $\rho(L) \subsetneqq \mathfrak{e}(L)$ in general and that the subsets $\{x \in L \mid \langle x \rangle \text{ asc } L\}$ and $\{x \in L \mid \langle x \rangle \text{ si } L\}$ are not necessarily subalgebras of L over a field of positive characteristic. To see these we consider Hartley's example $L = P + (x, y, z)$ [1, Lemma 3.1.1 and Example 6.3.6]. The following facts are well known: (a) If char $\mathfrak{k} = 0$, then $\rho(L) = P$ and $y \in \mathfrak{e}(L)$. (b) If char $\mathfrak{k} = p > 0$, then $\rho(L) = P$ and $x, y \in \mathfrak{e}(L) = \bar{\mathfrak{e}}(L)$ but $z = [x, y] \notin \mathfrak{e}(L) = \bar{\mathfrak{e}}(L)$. Since $L \in \mathrm{E}\mathfrak{A}$, the assertions follow from Theorem 1.

We remark that Corollary may be obtained from [2, Theorem 17].

3. As usual, let $\{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{A}$ and $\mathfrak{E}$ denote respectively the smallest L-closed and É-closed class containing $\mathfrak{A}$ and the class of Engel algebras. Then it is well known that $\{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{A} \cap \mathfrak{E} \leqq \mathrm{L}\mathfrak{N}$ [1, Corollary 16.3.10]. If $\langle x \rangle \operatorname{asc} L$ for any $x \in L$, then clearly $L \in \mathfrak{E}$. Hence by Corollary we have the following

THEOREM 2. *Let L be a Lie algebra. If $L \in \{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{A}$ and $\langle x \rangle \operatorname{asc} L$ for any $x \in L$, then $L \in \mathrm{L}\mathfrak{N}$. In particular if $L \in \acute{\mathrm{E}}\mathfrak{A}$, then the following conditions are equivalent*: (a) $L \in \mathfrak{Gr}$, *i.e.*, $\langle x \rangle \operatorname{asc} L$ *for any* $x \in L$. (b) $L \in \mathfrak{E}$. (c) $L \in \mathrm{L}\mathfrak{N}$.

Finally we note that over any field there exists a Lie algebra L where for any non-zero $x \in L$ $\langle x \rangle \operatorname{asc} L$ but $\langle x \rangle$ is not a subideal of L. Consider, for example, a Lie algebra L constructed by Simonjan [4, Theorem 4]. It belongs to the class $\acute{\mathrm{E}}\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$, and $\bar{\epsilon}(L)=0$ so that $\{x \in L \mid \langle x \rangle \operatorname{si} L\}=0$. Hence by Theorem 2 we see that this algebra has the above property.

References

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