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Remark on the dual of some Lipschitz spaces

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The present paper is considered as a supplement to the author's paper [1]. Our aim is to prove the following theorem.

THEOREM. *Let α be a real number, $1<p\leqq\infty$ and $1\leqq q\leqq\infty$.*

(a) *The dual of $\boldsymbol{\lambda}^{\alpha}_{\infty,q}$ is isomorphic to $\boldsymbol{\Lambda}^{-\alpha}_{1,q'}$.*

(b) *There exists an isomorphism η: $\boldsymbol{\Lambda}^{\alpha}_{p,\infty}\to(\boldsymbol{\lambda}^{\alpha}_{p,\infty})''$ such that the restriction of η to $\boldsymbol{\lambda}^{\alpha}_{p,\infty}$ is the canonical embedding of $\boldsymbol{\lambda}^{\alpha}_{p,\infty}$ into its second dual $(\boldsymbol{\lambda}^{\alpha}_{p,\infty})''$.*

Notation and related definitions are given in section 1. As in [1], the proof of the Theorem is done by establishing the corresponding results for some spaces of harmonic functions which are isomorphic to the spaces considered in the Theorem. Our result (a) is an n-dimensional and non-periodic version of a result of T. M. Flett [3; Theorem 19], whereas (b) when $p=\infty$ is that of a result of K. de Leeuw [5; Theorem 2.1] (cf. also [3; Theorem 19]).

1. Notation and preliminaries

We use R^n to denote the n-dimensional Euclidean space, and for each point $x=(x_1,\dots, x_n)$ we write $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$.

Unless otherwise stated, all functions are supposed to be complex-valued. As usual we use $\mathscr{S}=\mathscr{S}(R^n)$ to denote the space of all rapidly decreasing functions on R^n; $\mathscr{D}$ stands for its subspace consisting of functions with compact supports.

For any positive integer k let Z_k^+ be the set of all ordered k-tuples of non-negative integers, and for each $\mu=(\mu_1,\dots, \mu_k)$ let

$$|\mu| = \mu_1 + \cdots + \mu_k.$$

An element of Z_k^+ is called a multi-index.

If u is a function defined on an open subset of R^k, we use D_i^m to denote the partial derivative of u of order m with respect to the i-th coordinate. Further, for each multi-index $\mu=(\mu_1,\dots, \mu_k)$ we write

$$D^{\mu}u = D_1^{\mu_1}\cdots D_k^{\mu_k}u.$$

If f is a measurable function defined on R^n, we set

$$\|f\|_p = \left[\int_{R^n} |f(x)|^p dx\right]^{1/p}, \quad 0 < p < \infty,$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in R^n} |f(x)|,$$

and we define $L^p = L^p(R^n)$, where $1 \leqq p \leqq \infty$, as the space of those measurable functions f for which $\|f\|_p < \infty$, equipped with the norm $\|\cdot\|_p$.

For each real number p, $1 \leqq p \leqq \infty$, we use p' to denote its conjugate, i.e., $1/p + 1/p' = 1$, where we set $1/\infty = 0$.

The Fourier transform of a function $f \in L^1$ is given by

$$\hat{f}(\xi) = \int_{R^n} e^{-2\pi i \xi \cdot y} f(y) dy, \quad \xi \in R^n,$$

where $\xi \cdot y = \xi_1 y_1 + \cdots + \xi_n y_n$.

We consider the space R^{n+1} as the Cartesian product $R^n \times R$, so that we can write each $z \in R^{n+1}$ in the form $z = (x, t)$, where $x \in R^n$ and $t \in R$. We denote by Ω the upper half space $R^n \times]0, \infty[$.

We use B to denote a constant, depending on the particular parameters p, $q, \ldots, \alpha, \beta, \ldots$ concerned in the particular problem in which it appears; if we wish to express the dependency, we write B in the form $B(p, q, \ldots, \alpha, \beta, \ldots)$. These constants are not necessarily the same on any two occurrences.

For measurable functions u defined on Ω, let

$$M_p(u; t) = \left[\int_{R^n} |u(x, t)|^p dx\right]^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(u; t) = \operatorname*{ess\,sup}_{x \in R^n} |u(x, t)|.$$

We also let

$$\|u\|_{p,q} = \left[\int_0^\infty M_p(u; t)^q t^{-1} dt\right]^{1/q}, \quad 0 < q < \infty,$$

$$\|u\|_{p,\infty} = \operatorname*{ess\,sup}_{t > 0} M_p(u; t).$$

Moreover, for any $s > 0$ and $y \in R^n$, u_s, $u^{(s)}$ and $u^{(y,s)}$ are the functions given by

$$u_s(x) = u(x, s), \; x \in R^n,$$

$$u^{(s)}(x, t) = u(x, s + t), \quad (x, t) \in \Omega,$$

$$u^{(y,s)}(x, t) = u(y - x, s + t), \quad (x, t) \in \Omega.$$

We use P to denote the Poisson kernel for Ω, i.e.,

$$P(x, t) = c_n t/(|x|^2 + t^2)^{(n+1)/2} \qquad \text{for} \quad x \in R^n \quad \text{and} \quad t > 0,$$

where $c_n = \pi^{-(n+1)/2}\Gamma((n+1)/2)$. The following two properties of the Poisson kernel are frequently used.

(P. 1) $$\hat{P}_s(\xi) = e^{-2\pi|\xi|s},\ \xi \in R^n \quad \text{and} \quad s > 0.$$

(P. 2) $$P_s * P_t = P_{s+t} \qquad \text{for all} \quad s,\ t > 0,$$

where $*$ denotes the convolution operation.

Let f be a measurable function on R^n such that $\int_{R^n} |f(x)|(1+|x|)^{-n-1}dx < \infty$. The Poisson integral of f, denoted by u, is the function defined on Ω by

$$u(x,\ t) = P_t * f(x) = f * P_t(x), \quad (x,\ t) \in \Omega.$$

It is well-known that u is harmonic in Ω and satisfies the following relation:

($*$) $$u(x,\ s+t) = P_s * u_t(x),\ x \in R^n, \quad \text{and} \quad s,\ t > 0.$$

The equation ($*$) is called the semigroup formula hereafter.

For the properties of the Poisson kernel and Poisson integrals, we refer to [1] and the references given there.

Definition 1 (cf. [1; § 3, Definition A]). For any real number b, let $\mathscr{H}_b$ denote the linear space of all harmonic functions u in Ω with the property that if $\mu \in Z_{n+1}^+$, $c > 0$, and K is any compact subset of R^n, there is a positive constant B such that

$$|D^\mu u(x,\ t)| \leqq Bt^{-(b+|\mu|)} \qquad \text{for every } x \text{ in } K \text{ and } t \geqq c.$$

Further, let $\mathscr{H}_b^* = \bigcap_{\gamma<b} \mathscr{H}_\gamma$.

Definition 2 (cf. [1; § 3, Definition B]). For any u in $\mathscr{H}_b$ and $\alpha < b$, $R^\alpha u$ is the function defined on Ω by

(i) $R^0 u = u$;

(ii) if $\alpha > 0$,

$$R^\alpha u(x,\ t) = \frac{1}{\Gamma(\alpha)}\int_0^\infty u(x,\ s+t)s^{\alpha-1}ds;$$

(iii) if α is a negative integer, say $\alpha = -m$, then

$$R^\alpha u = R^{-m}u = (-1)^m D_{n+1}^m u;$$

(iv) if $\alpha = -\beta < 0$ and β is not an integer, then

$$R^\alpha u = R^{-\beta}u = R^{m-\beta}(R^{-m}u),$$

where $m = [\beta]+1$ (here $[\gamma]$ stands for the greatest integer not exceeding γ), and

$R^{m-\beta}$, R^{-m} are defined by (ii) and (iii).

REMARK. If $\alpha<b$ and $u\in\mathscr{H}_b$, then $R^\alpha u\in\mathscr{H}_{b-\alpha}$. If $\alpha+\gamma<b$, $\gamma<b$ and $u\in\mathscr{H}_b$, then $R^{\alpha+\gamma}u=R^\alpha(R^\gamma u)$ (cf. [1; Theorem 3.2]).

DEFINITION 3 (cf. [1; §5]). Let α be a real number, and $1\leqq p,\ q\leqq\infty$. Define

$$\mathscr{H}(\alpha;\ p,\ q)=\{u\in\mathscr{H}^*_{n/p-\alpha};\ \mathscr{E}^\alpha_{p,q}(u)=\|tR^{-\alpha-1}u\|_{p,q}<\infty\}.$$

Moreover, let $\mathscr{O}_0$ be the set of infinitely differentiable functions with compact supports not containing the origin and $\mathscr{H}^*$ be the set of Poisson integrals of functions in $\hat{\mathscr{O}}_0$, where $\hat{\mathscr{O}}_0=\{\hat{\psi}:\psi\in\mathscr{O}_0\}$. If p or q is ∞, define $\mathscr{H}\boldsymbol{\lambda}^\alpha_{p,q}$[1)] to be the closure of $\mathscr{H}^*$ in $\mathscr{H}(\alpha;\ p,\ q)$.

We shall list some results obtained in [1] which will be needed in the proof of the Theorem. Let α be a real number and $1\leqq p,\ q\leqq\infty$.

(1.1) Let $\beta>0$ and $\|t^\beta u\|_{p,q}<\infty$. Then $u\in\mathscr{H}_{\beta+n/p}$(cf. [1; Theorem 3.5]).

(1.2) $\mathscr{H}(\alpha;\ p,\ q)$ is a Banach space with norm $\mathscr{E}^\alpha_{p,q}$ provided that, if $\alpha\geqq n/p$ then we must identify harmonic functions u satisfying $D^k_{n+1}u=0$ for some non-negative integer k with the zero element (cf. [1; Theorem 5.1]).

(1.3) If k is a non-negative integer greater than α, then $\|t^{k-\alpha}D^k_{n+1}(\cdot)\|_{p,q}$ is an equivalent norm on $\mathscr{H}(\alpha;\ p,\ q)$ (cf. [1; Theorem 5.1]).

(1.4) If $1\leqq p,\ q<\infty$, then $\mathscr{H}^*$ is dense in $\mathscr{H}(\alpha;\ p,\ q)$ (cf. [1; Theorems 7.1 and 7.2]). Note also that $\mathscr{H}^*$ is dense in $\mathscr{H}\boldsymbol{\lambda}^\alpha_{p,q}$ if p or q is ∞ (see Definition 3).

(1.5) For each real number γ, there exists an isometric isomorphism R^γ of $\mathscr{H}(\alpha;\ p,\ q)$ onto $\mathscr{H}(\alpha+\gamma;\ p,\ q)$. Further, R^γ maps $\mathscr{H}\boldsymbol{\lambda}^\alpha_{p,\infty}$ ($\mathscr{H}\boldsymbol{\lambda}^\alpha_{\infty,q}$ resp.) onto $\mathscr{H}\boldsymbol{\lambda}^{\alpha+\gamma}_{p,\infty}$ ($\mathscr{H}\boldsymbol{\lambda}^{\alpha+\gamma}_{\infty,q}$ resp.). If $\gamma<n/p-\alpha$ or $u\in\mathscr{H}^*$, then $R^\gamma u$ is identical to the one defined in Definition 2 (cf. [1; Theorem 5.1 and its remark, Remark to Lemma 7.1]).

(1.6) Let $\boldsymbol{\Lambda}^\alpha_{p,q}$ $(1\leqq p,\ q<\infty)$, $\boldsymbol{\lambda}^\alpha_{p,\infty}$ $(1\leqq p\leqq\infty)$ and $\boldsymbol{\lambda}^\alpha_{\infty,q}$ $(1\leqq q\leqq\infty)$ denote the Lipschitz spaces defined by Herz [4] (cf. [1; §7]). We let also $\boldsymbol{\Lambda}^\alpha_{p,\infty}$ ($\boldsymbol{\Lambda}^\alpha_{\infty,q}$ resp.) be the space of boundary values of functions in $\mathscr{H}(\alpha;\ p,\ \infty)$ ($\mathscr{H}(\alpha;\ \infty,\ q)$ resp.) with the obvious norm. Then $u\mapsto u(\cdot,0)=\lim_{t\to 0}u(\cdot,\ t)$ is an isomorphism of $\mathscr{H}(\alpha;\ p,\ q)$ onto $\boldsymbol{\Lambda}^\alpha_{p,q}$ and of $\mathscr{H}\boldsymbol{\lambda}^\alpha_{p,q}$ onto $\boldsymbol{\lambda}^\alpha_{p,q}$; the limit being taken in $\mathscr{S}'$ if $\alpha<n/p$ and in $\mathscr{S}'/\mathscr{P}$ (the space of tempered distributions modulo polynomials) if $\alpha\geqq n/p$ (cf. [1; Theorems 6.1, 7.1 and 7.2]).

LEMMA 1 (cf. [1; Lemma 6.1]). *Let $1\leqq p,\ q\leqq\infty$, α be a real number and u be in $\mathscr{H}(\alpha;\ p,\ q)$.*

(i) *$u^{(s)}\in\mathscr{H}(\alpha;\ p,\ q)$ and $\mathscr{E}^\alpha_{p,q}(u^{(s)})\leqq\mathscr{E}^\alpha_{p,q}(u)$ for any $s>0$.*

1) The spaces $\mathscr{H}\boldsymbol{\lambda}^\alpha_{p,\infty}$ and $\mathscr{H}\boldsymbol{\lambda}^\alpha_{\infty,q}$ are denoted by $\mathscr{H}\boldsymbol{\Lambda}^\alpha_{p,\infty}$ and $\mathscr{H}\boldsymbol{\Lambda}^\alpha_{\infty,q}$ respectively in [1].

(ii) *If* $q<\infty$ *or* $q=\infty$ *and* $M_p(R^{-\alpha-1}u; t)=o(t^{-1})$ *as* $t\to 0+$ *and* $t\to\infty$, *then* $u^{(s)}\to u$ *in* $\mathscr{H}(\alpha; p, q)$ *as* $s\to 0+$.

LEMMA 2 (cf. [1; Lemma 6.2 and its proof]). *Let* $1\leqq p, q\leqq\infty$, f *be a function in* L^p *and* u *be its Poisson integral. Let* k *be a positive integer such that* $2k>\alpha>0$.

(i) $u^{(s)}\in\mathscr{H}(\alpha; p, q)$ *and* $\mathscr{E}^{\alpha}_{p,q}(u^{(s)})\leqq Bs^{-\alpha}\|f\|_p$ *for all* $s>0$.

(ii) *Furthermore, if all partial derivatives of* f *of order less than* $2k+1$ *exist, are bounded and belong to* L^p, *then* $u\in\mathscr{H}(\alpha; p, q)$ *and* $\mathscr{E}^{\alpha}_{p,q}(u)\leqq B(\|f\|_p+\|\Delta^k f\|_p)$. (Here Δ^k is the Laplace operator iterated k times.)

LEMMA 3 (cf. [1; Lemma 6.3]). *Let* $1\leqq p, q\leqq\infty$, α *be a positive number and* k *be a positive integer such that* $2k>\alpha$. *Define*

$$\langle u, v\rangle_k = \langle u, v\rangle = \frac{1}{\Gamma(2k)}\int_0^\infty\int_{R^n} t^{2k-1}u(x, t/2)R^{-2k}v(x, t/2)dxdt$$

for all u *in* $\mathscr{H}(-\alpha; p', q')$ *and all* v *in* $\mathscr{H}(\alpha; p, q)$.

(i) $\langle\cdot,\cdot\rangle$ *is a continuous bilinear form on* $\mathscr{H}(-\alpha; p', q')\times\mathscr{H}(\alpha; p, q)$.

(ii) *If* $u\in\mathscr{H}(-\alpha; p', q')$ *and* v *is the Poisson integral of a* $\psi\in\mathscr{S}$, *then*

$$\langle u, v\rangle = \lim_{s\to 0+}\int_{R^n} u(x, s)\psi(x)dx.$$

Moreover, if $\langle u, w\rangle=0$ *for every* w *which is the Poisson integral of a function in* $\mathscr{S}$, *then* $u=0$. (Note that $\langle\cdot,\cdot\rangle$ does not depend on k.)

Hereafter, let E' and E'' denote the dual and second dual of the normed vector space E.

LEMMA 4 (cf. [1; Lemmas 8.2 and 8.3]). *Let* $1\leqq p, q\leqq\infty$ *and* α *be a positive number. Let* F *be in* $\mathscr{H}(\alpha; p, q)'$ *and let* $u(y, s)=F(P^{(y,s)})$ *for all* $(y, s)\in\Omega$.

(i) u *is harmonic in* Ω, *and* $u(\cdot, s)$ *is bounded and uniformly continuous on* R^n *for each* $s>0$.

(ii) *If* g *is a* L^∞-*function with compact support and* $s>0$, *then*

$$F(v^{(s)}) = \int_{R^n} u(y, s)g(y)dy,$$

where v *is the Poisson integral of* g.

2. Proof of the Theorem

First, we prepare some lemmas.

LEMMA 5. *Let k be a non-negative integer and $1 \leqq p < \infty$. Let $f \in L^p$ such that $|\cdot|^k f \in L^p$. Then there exists a sequence $\{\psi_i\} \subset \mathcal{O}_0$ with the property that*

$$\|\psi_i - f\|_p \longrightarrow 0 \ \textit{and} \ \| |\cdot|^k(\psi_i - f)\|_p \longrightarrow 0 \ \textit{as} \ i \longrightarrow \infty.$$

PROOF. This follows easily from the well-known regularizing process.

For $\alpha > 0$ and $w \in C_K$, the space of all continuous functions with compact supports in Ω, define

$$T^\alpha(w)(x, s) = \int_0^\infty \int_{R^n} t^{\alpha-1} P(x - y, s + t) w(y, t) dy dt$$

for every $(x, s) \in \Omega$ (cf. [1; §9]). An easy application of Fubini's theorem and property (P. 2) in §1 shows that $T^\alpha(w)$ is the Poisson integral of the function f defined by

$$f(y) = \int_0^\infty \int_{R^n} t^{\alpha-1} P(y - z, t) w(z, t) dz dt, \ y \in R^n.$$

It then follows that $f \in C^\infty$, and $D^\kappa f$ belong to L^p and vanishes at infinity for any p, $1 \leqq p \leqq \infty$, and $\kappa \in Z_n^+$. Further, property (P. 1) in §1 implies that

$$|\hat{f}(\xi)| \leqq B e^{-2\pi|\xi|a}, \ \xi \in R^n,$$

where B and a are positive constants that depend on w.

LEMMA 6. *Let $\alpha > 0$ and k be a positve integer such that $2k > \alpha$. Let f be a function in L^∞ and u be its Poisson integral. If $\hat{f}$ and $|\cdot|^{2k}\hat{f}$ are in L^1, then $u \in \mathscr{H}\boldsymbol{\lambda}_{\infty,q}^\alpha$ for any q, $1 \leqq q \leqq \infty$.*

PROOF. It is obvious that, under the above assumptions, f is almost everywhere equal to a function whose partial derivatives of order less than $2k+1$ exist and are bounded, so that we may assume that f has this property. Hence, to establish the lemma, on account of Lemma 2 it is sufficient to find a sequence $\{\phi_i\} \subset \hat{\mathcal{O}}_0$ such that $\|\phi_i - f\|_\infty \to 0$ and $\|\Delta^k(\phi_i - f)\|_\infty \to 0$ as $i \to \infty$. Now Lemma 5 implies that there is a sequence $\{\psi_i\} \subset \mathcal{O}_0$ so that $\|\psi_i - \hat{f}\|_1 \to 0$ and $\|(2\pi|\cdot|)^{2k}(\psi_i - \hat{f})\|_1 \to 0$ as $i \to \infty$. Set $\hat{\phi}_i = \psi_i$. Then $\phi_i \in \hat{\mathcal{O}}_0$ and

$$\|\phi_i - f\|_\infty \leqq \|\psi_i - \hat{f}\|_1,$$

$$\|\Delta^k(\phi_i - f)\|_\infty \leqq \|(2\pi|\cdot|)^{2k}(\psi_i - \hat{f})\|_1.$$

The proof of the lemma is thus complete.

COROLLARY. *Let $\alpha > 0$, $s > 0$, $y \in R^n$ and $w \in C_K$. Then $P^{(y,s)}$ and $T^\alpha(w)$ belong to $\mathscr{H}\boldsymbol{\lambda}_{\infty,q}^\alpha$ for any q, $1 \leqq q \leqq \infty$. Also, the set of Poisson integrals of* *functions in* $\mathscr{D}$ *is dense in* $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{\infty,q}$.

LEMMA 7. *Let* $\alpha>0$, $1<q\leqq\infty$ *and* u *be in* $\mathscr{H}(-\alpha;\,1,\,\infty)$ *such that*

$$\sup_{\substack{v\in\mathscr{H}^*\\ \mathscr{E}^{\alpha}_{\infty,q}(v)\leqq 1}} |\langle u^{(s)}, v\rangle| \leqq C < \infty \qquad \textit{for all} \quad s>0.$$

Then $u\in\mathscr{H}(-\alpha;\,1,\,q')$ *and* $\mathscr{E}^{-\alpha}_{1,q'}(u)\leqq BC$.

PROOF. The proof can be done exactly in the same way as Lemma 9.4 of [1]. The only new thing to be taken care of is that for $w\in C_K$, $T^{\alpha}(w)$ can be approximated in the $\mathscr{E}^{\alpha}_{\infty,1}$-norm by functions in $\mathscr{H}^*$, and this follows from the corollary to Lemma 6.

As in [1], the Theorem is an easy consequence of the following two statements. Let α, p, q be as in the Theorem.

(S. 1) $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{\infty,q}{}'$ *is isomorphic to* $(\mathscr{H}-\alpha;\,1,\,q')$.

(S. 2) *There exists an isomorphism* $\theta\colon \mathscr{H}(\alpha;\,p,\,\infty)\to\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}{}''$ *such that the restriction of* θ *to* $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}$ *is the canonical embedding of* $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}$ *into its second dual* $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}{}''$.

PROOF OF (S. 1) (cf. [1; Theorems 8.1 and 9.2]). On account of (1.5), we may assume that $\alpha>0$. Let k be a positive integer such that $2k>\alpha$. Let u be in $\mathscr{H}(-\alpha;\,1,\,q')$. Then it follows from Lemma 3 that $F_u=\langle u,\,\cdot\rangle$ is a continuous linear functional on $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{\infty,q}$ with $\|F_u\|\leqq B\mathscr{E}^{-\alpha}_{1,q'}(u)$, and $F_u=0$ implies $u=0$.

Conversely, assume that $F\in\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{\infty,q}{}'$. Let $u(y,\,s)=F(P^{(y,s)})$ for every $(y,\,s)\in\Omega$. Then u is harmonic in Ω by Lemma 4. Denoting by v the Poisson integral of a function $\psi\in\mathscr{D}$, we derive from Lemmas 3 and 4 that

$$F(v^{(s)}) = \int_{R^n} u(y,\,s)\psi(y)dy = \langle u^{(s)}, v\rangle \tag{2.1}$$

which, together with Lemma 2, implies that

$$\left|\int_{R^n} u(y,\,s)\psi(y)dy\right| = |F(v^{(s)})| \leqq \|F\|\mathscr{E}^{\alpha}_{\infty,q}(v^{(s)}) \leqq B\|F\|s^{-\alpha}\|\psi\|_{\infty}.$$

Hence

$$M_1(u;\,s) = \sup_{\substack{\psi\in\mathscr{D}\\ \|\psi\|_{\infty}\leqq 1}} \left|\int_{R^n} u(y,\,s)\psi(y)dy\right| \leqq B\|F\|s^{-\alpha}.$$

It then follows from (1.1) and (1.3) that $u\in\mathscr{H}(-\alpha;\,1,\,\infty)$ and $\mathscr{E}^{-\alpha}_{1,\infty}(u)\leqq B\|F\|$. Assume that $1<q\leqq\infty$ and let v be in $\mathscr{H}^*$. Since v can be approximated in the $\mathscr{E}^{\alpha}_{\infty,q}$-norm by Poisson integrals of functions in $\mathscr{D}$ (see the corollary after

Lemma 6), we derive from (2.1) that $F(v^{(s)})=\langle u^{(s)}, v\rangle$ for every $v\in\mathscr{H}^*$. Therefore

$$\sup_{\substack{v\in\mathscr{H}^* \\ \mathscr{E}^{\alpha}_{\infty,q}(v)\leqq 1}} |\langle u^{(s)}, v\rangle| = \sup_{\substack{v\in\mathscr{H}^* \\ \mathscr{E}^{\alpha}_{\infty,q}(v)\leqq 1}} |F(v^{(s)})|$$

$$\leqq \|F\| \qquad \text{for every} \quad s>0,$$

which, by Lemma 7, implies that $u\in\mathscr{H}(-\alpha;\, 1,\, q')$ and $\mathscr{E}^{-\alpha}_{1,q'}(u)\leqq B\|F\|$.

Now let w be the Poisson integral of $\phi\in\mathscr{D}$. Since $w^{(s)}\to w$ in $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{\infty,q}$ by Lemma 1, the continuity of F, Lemma 3 and (2.1) imply that

$$F(w) = \lim_{s\to 0} F(w^{(s)}) = \lim_{s\to 0}\int_{R^n} u(y, s)\phi(y)dy = \langle u, w\rangle.$$

Observing that the set of Poisson integrals of functions in $\mathscr{D}$ is dense in $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{\infty,q}$, we derive that $F=F_u$.

By combining the above results, we conclude that $u\mapsto F_u=\langle u, \cdot\rangle$ is an isomorphism of $\mathscr{H}(-\alpha;\, 1, q')$ onto $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{\infty,q}{}'$. The proof of (S. 1) is thus complete.

Before proceeding on with the proof of (S. 2), we need two more lemmas.

LEMMA 8. *Let* $1<p\leqq\infty$ *and* $\alpha\geqq 0$. *Define*

$$\langle\!\langle u, v\rangle\!\rangle = \langle R^{-2\alpha}u,\, R^{2\alpha}v\rangle \quad \text{if } \alpha>0$$

$$\langle\!\langle u, v\rangle\!\rangle = \langle R^{-1}u,\, R^{1}v\rangle \quad \text{if } \alpha=0$$

for all $u\in\mathscr{H}(\alpha;\, p,\, \infty)$ *and all* $v\in\mathscr{H}(-\alpha;\, p',\, 1)$. *Then* $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ *is a continuous bilinear form on* $\mathscr{H}(\alpha;\, p,\, \infty)\times\mathscr{H}(-\alpha;\, p',\, 1)$. *Further* $u\mapsto G_u=\langle\!\langle u, \cdot\rangle\!\rangle$ *is an isomorphism of* $\mathscr{H}(\alpha;\, p,\, \infty)$ *onto* $\mathscr{H}(-\alpha;\, p',\, 1)'$, *and* $\langle\!\langle u, v\rangle\!\rangle=\langle v, u\rangle$ *for any* $u\in\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}$ *and* $v\in\mathscr{H}(-\alpha;\, p',\, 1)$. *Consequently,* $v\mapsto H_v=\langle\!\langle\cdot, v\rangle\!\rangle$ *is an isomorphism of* $\mathscr{H}(-\alpha;\, p',\, 1)$ *onto* $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}{}'$.

PROOF. We shall prove the lemma only for $\alpha>0$, since the case $\alpha=0$ can be similarly treated. First, Lemma 3 and (1.5) imply that

$$|\langle\!\langle u, v\rangle\!\rangle| = |\langle R^{-2\alpha}u,\, R^{2\alpha}v\rangle| \leqq B\mathscr{E}^{-\alpha}_{p,\infty}(R^{-2\alpha}u)\mathscr{E}^{\alpha}_{p',1}(R^{2\alpha}v)$$

$$= B\mathscr{E}^{\alpha}_{p,\infty}(u)\mathscr{E}^{-\alpha}_{p',1}(v).$$

Hence, it follows easily that $u\mapsto G_u=\langle\!\langle u, \cdot\rangle\!\rangle$ is a continuous, 1–1 and linear map of $\mathscr{H}(\alpha;\, p,\, \infty)$ into $\mathscr{H}(-\alpha;\, p',\, 1)'$. To see that this map is onto, let G be in $\mathscr{H}(-\alpha;\, p',\, 1)'$. Let F be the element of $\mathscr{H}(\alpha;\, p',\, 1)'$ defined by

$$F(w) = G(R^{-2\alpha}w) \qquad \text{for} \quad w\in\mathscr{H}(\alpha;\, p',\, 1)$$

or equivalently by

$$G(v) = F(R^{2\alpha}v) \qquad \text{for} \quad v \in \mathscr{H}(-\alpha;\, p',\, 1).$$

The proof of Theorem 8.1 of [1] then implies that there exists $u' \in \mathscr{H}(-\alpha;\, p,\, \infty)$ such that

$$G(v) = F(R^{2\alpha}v) = \langle u',\, R^{2\alpha}v\rangle \qquad \text{for every} \quad v \in \mathscr{H}(-\alpha;\, p',\, 1).$$

Let $u = R^{2\alpha}u' \in \mathscr{H}(\alpha;\, p,\, \infty)$. Then we conclude that

$$G(v) = \langle R^{-2\alpha}u,\, R^{2\alpha}v\rangle = \langle\!\langle u,\, v\rangle\!\rangle \qquad \text{for all} \quad v \in \mathscr{H}(-\alpha;\, p',\, 1).$$

Lastly, to show that $\langle\!\langle u, v\rangle\!\rangle = \langle v, u\rangle$ for all $u \in \mathscr{H}\lambda^{\alpha}_{p,\infty}$ and $v \in \mathscr{H}(-\alpha;\, p',\, 1)$, it is sufficient, on account of (1.4), to verify this relation for any $u, v \in \mathscr{H}^*$. With this restriction on u and v, various applications of Fubini's theorem and integration by parts below are easily justified. Let k be a positive integer such that $2k > \alpha$. Then a repeated application of integration by parts shows that

$$\begin{aligned}\langle\!\langle u,\, v\rangle\!\rangle &= \frac{1}{\Gamma(2k)}\int_0^\infty t^{2k-1}\left\{\int_{R^n} R^{-2\alpha}u(x,\, t/2)R^{-2k}\, R^{2\alpha}v\,(x,\, t/2)\, dx\right\}dt \\ &= \frac{1}{\Gamma(2k)}\int_0^\infty t^{2k-1}\left\{\int_{R^n} R^{-2k}R^{-2\alpha}u(x,\, t/2)R^{2\alpha}v(x,\, t/2)dx\right\}dt.\end{aligned}$$

Denoting the last integral on R^n by I, we derive from Fubini's theorem, Definition 2 and its remark that

$$\begin{aligned} I &= \int_{R^n} R^{-2\alpha}R^{-2k}u(x,\, t/2)\left[\frac{1}{\Gamma(2\alpha)}\int_0^\infty s^{2\alpha-1}\left\{\int_{R^n} P(x-y,\, s)v(y,\, t/2)dy\right\}ds\right]dx \\ &= \int_{R^n} v(y,\, t/2)\left\{\frac{1}{\Gamma(2\alpha)}\int_0^\infty s^{2\alpha-1}R^{-2\alpha}R^{-2k}u(y,\, s+t/2)\, ds\right\}dy \\ &= \int_{R^n} v(y,\, t/2)R^{-2k}u(x,\, t/2)dy. \end{aligned}$$

Consequently, $\langle\!\langle u,\, v\rangle\!\rangle = \langle v,\, u\rangle$, and the proof of the lemma is complete.

The following general lemma is known.

LEMMA 9 (cf. [2; Lemma 30]). *Let X, Y be Banach spaces, and let Z be a closed subspace of X. Let $B(\cdot,\, \cdot)$ be a continuous bilinear form on $X \times Y$. Define*

$$\mu\colon X \longrightarrow Y',\ \mu(x) = B(x,\, \cdot),$$

$$\nu\colon Y \longrightarrow Z',\ \nu(y) = B(\cdot,\, y).$$

Assume that μ and ν are isomorphisms of X onto Y' and of Y onto Z', respectively. Then there exists an isomorphism $\theta\colon X \to Z''$ such that the restriction of θ to Z

is the canonical embedding of Z into its second dual Z''.

PROOF OF (S. 2). Again we shall give a proof only for the case $\alpha>0$, because the case $\alpha\leqq 0$ can be handled in a similar way. Apply Lemma 9 with $X=\mathscr{H}(\alpha;\ p,\infty)$, $Y=\mathscr{H}(-\alpha;\ p',1)$, $Z=\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}$ and $B(\cdot,\cdot)=\langle\!\langle\cdot,\cdot\rangle\!\rangle$. Then the assumptions of Lemma 9 are satisfied by Lemma 8. Hence, the statement (S. 2) follows.

3. Remarks

(i) The method used here can also be adopted to show the following two results. Let α be a real number, $1<p\leqq\infty$ and $1\leqq q\leqq\infty$.

(S. 3) *The dual of $\boldsymbol{\partial}(\alpha;\ \infty, q)$ is isomorphic to $\boldsymbol{\Lambda}(-\alpha;\ 1, q')$.*

(S. 4) *There exists an isomorphism ρ: $\boldsymbol{\Lambda}(\alpha;\ p,\infty)\to\boldsymbol{\partial}(\alpha;\ p,\infty)''$ such that the restriction of ρ to $\boldsymbol{\partial}(\alpha;\ p,\infty)$ is the canonical embedding of $\boldsymbol{\partial}(\alpha;\ p,\infty)$ into its second dual.*

Here for a real number β and $1\leqq p,\ q\leqq\infty$, $\boldsymbol{\Lambda}(\beta;\ p, q)$ is the Lipschitz space defined by M. H. Taibleson [10] (see also [2]), and $\boldsymbol{\partial}(\beta;\ p,\infty)$ is the closure of $\mathscr{D}$ (or $\mathscr{S}$) in $\boldsymbol{\Lambda}(\beta;\ p,\infty)$. Note that if $1<p<\infty$, then $\boldsymbol{\partial}(\beta;\ p,\infty)=\boldsymbol{\lambda}(\beta;\ p,\infty)$, and hence (S. 4) for this case was obtained earlier by T. M. Flett [2; Theorem 26].

(ii) There is another method of studying Lipschitz (Besov) spaces which is based on Mihlin-Hörmander's multiplier theorem, Plancherel-Polya-type inequality for entire functions of exponential type and the abstract theory of interpolation. This method was initially developed by J. Peetre ([8], [9]) and has also been extensively studied by H. Triebel and others (see [11] for a comprehensive bibliography on the development of this method). There are duality results but mostly for non-homogeneous spaces of Taibleson (see e.g., [11], [12]); Professor Triebel indicated to the author that there should be no serious difficulty in extending some of them to the homogeneous case.

We conclude this paper by stating two problems that should be worth studying.

PROBLEM 1. This problem was raised to the author by Professor R. Johnson (private communication). He asked what one can say about the dual of $\boldsymbol{\Lambda}^{\alpha}_{p,q}$ when either p or q is ∞. He also suggested that $(\boldsymbol{\Lambda}^{\alpha}_{p,\infty})'=\boldsymbol{\Lambda}^{-\alpha}_{p',1}\oplus\boldsymbol{S}^{\alpha}_{p,\infty}$, where $\boldsymbol{S}^{\alpha}_{p,\infty}$ is the set of "singular elements" in a sense to be specialized. For simplicity assume $\alpha>0$. By considering the corresponding space of harmonic functions and taking $F\in\mathscr{H}(\alpha;\ p,\infty)'$, we see that there is a unique $u\in\mathscr{H}(-\alpha;\ p', 1)$ such that $F(v)=F_u(v)=\langle u, v\rangle$ for all $v\in\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}$. Using this fact and identifying F_u with u, we see that $\mathscr{H}(\alpha;\ p,\infty)'=\mathscr{H}(-\alpha;\ p', 1)\oplus\mathscr{H}\boldsymbol{S}^{\alpha}_{p,\infty}$, where $\mathscr{H}\boldsymbol{S}^{\alpha}_{p,\infty}$ is the closed subspace of $\mathscr{H}(\alpha; p, \infty)'$ consisting of all linear functionals that vanish on $\mathscr{H}\boldsymbol{\lambda}^{\alpha}_{p,\infty}$. The problem is then reduced to that of finding as much information as possible about $\mathscr{H}\boldsymbol{S}^{\alpha}_{p,\infty}$ or $\boldsymbol{S}^{\alpha}_{p,\infty}$.

PROBLEM 2. The second problem is of a more general nature. Can the method used in [1], [2] and [3] be adopted to treat general domains in R^n? For a domain in R^n with the cone property, the analogue of $\boldsymbol{\Lambda}(\alpha; p, q)$ was defined by T. Muramatu [6] and the dual of this space in many extreme cases was investigated in [7] by a different method.

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