HIROSHIMA MATH. J.
10 (1980), 409–417

Global solvability of the Laplace operator on a non-compact affine symmetric space

Atsutaka KOWATA and Makoto TANAKA

(Received January 14, 1980)

1. Introduction

A. Cerezo and F. Rouvière prove that the Casimir operator on a complex semisimple Lie group G is surjective on $C^\infty(G)$ ([1]). Further J. Rauch and D. Wigner prove the global solvability of the Casimir operator when G is a non-compact semisimple Lie group with finite center ([6]). S. Helgason proves that each invariant differential operator on a symmetric space X of the non-compact type is surjective on $C^\infty(X)$ ([3]).

In this paper, we will show that the Laplace operator on an affine symmetric space induced by the Casimir operator is globally solvable by means of the method given by J. Rauch and D. Wigner ([6]).

2. Notation and preliminaries

Let M be an infinitely differentiable manifold. We denote by $C^\infty(M)$, $C_c^\infty(M)$, $\mathfrak{X}(M)$, and $\Omega^1(M)$ the space of infinitely differentiable functions on M, the space of infinitely differentiable functions on M with compact support, the space of all smooth vector fields on M and the space of all smooth 1-forms on M, respectively.

Let G be a non-compact connected real semisimple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of G, and B the Killing form of $\mathfrak{g}$. Let σ be an involution of G, G_σ the closed subgroup of G consisting of all the elements left fixed by σ, and H a closed subgroup of G lying between G_σ and the identity component of G_σ. Then the homogeneous space G/H is said to be an affine symmetric space and there always exists a G-invariant measure dgH on G/H which is unique except for a strictly positive factor of proportionality. The eigenvalues of the involution of $\mathfrak{g}$ induced by σ are 1 and -1. Let $\mathfrak{h}$ be the eigenspace for 1 and $\mathfrak{q}$ the eigenspace for -1. The direct decomposition of $\mathfrak{g}$ is: $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$. Since there exists a Cartan involution of $\mathfrak{g}$ commuting with σ, $\mathfrak{g}$ decomposes into a vector space direct sum:

$$\mathfrak{g} = \mathfrak{q} \cap \mathfrak{k} + \mathfrak{h} \cap \mathfrak{k} + \mathfrak{q} \cap \mathfrak{p} + \mathfrak{h} \cap \mathfrak{p}$$

where $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is the Cartan decomposition. Thus we can choose a basis X_1,

..., X_r, $Y_1,\dots,Y_s$ of $\mathfrak{g}$ such that $X_i\in\mathfrak{q}\cap\mathfrak{k}$ $(i=1,\dots,p)$, $X_i\in\mathfrak{h}\cap\mathfrak{k}$ $(i=p+1,\dots,r)$, $Y_i\in\mathfrak{q}\cap\mathfrak{p}$ $(i=1,\dots,q)$, $Y_i\in\mathfrak{h}\cap\mathfrak{p}$ $(i=q+1,\dots,s)$, $B(X_i,X_j)=-\delta_{ij}$ $(1\le i,j\le r)$, and $B(Y_i,Y_j)=\delta_{ij}$ $(1\le i,j\le s)$. Then the Casimir operator of G is of the form

$$-\sum_{i=1}^r X_i^2+\sum_{i=1}^s Y_i^2$$

in the universal enveloping algebra $U(\mathfrak{g}_c)$ of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$. Every element X in $\mathfrak{g}$ defines an infinitesimal transformation X^* as follows;

$$(X^*f)(gH)=\frac{d}{dt}f(\exp(-tX)gH)|_{t=0}\quad (gH\in G/H,\ f\in C^\infty(G/H)).$$

Then if we associate to each X in $\mathfrak{g}$ the linear map $f\mapsto X^*f$ of $C^\infty(G/H)$ into itself, we get a representation of $\mathfrak{g}$ on $C^\infty(G/H)$ which can be extended to a representation of $U(\mathfrak{g}_c)$. For the Casimir operator in $U(\mathfrak{g}_c)$, we denote by C the corresponding Laplace operator on $C^\infty(G/H)$ and write that

$$C=-\sum_{i=1}^r X_i^{*2}+\sum_{i=1}^s Y_i^{*2}.$$

Now we say that elements (x,X) and (y,Y) in $G\times\mathfrak{q}$ are equivalent if there exists an h in H such that $y=xh$ and $Y=\mathrm{Ad}(h^{-1})X$. We denote by $G\underset{H}{\times}\mathfrak{q}$ the quotient space of $G\times\mathfrak{q}$. Then $G\underset{H}{\times}\mathfrak{q}$ is naturally isomorphic to the tangent vector bundle $T(G/H)$ of G/H. Set $D_Xf=\frac{d}{dt}f(\exp(tX)H)|_{t=0}$ $(X\in\mathfrak{q}, f\in C^\infty(G/H))$ and $L_x(gH)=xgH$ $(x\in G,\ gH\in G/H)$. Then the map $X\mapsto D_X$ defines an isomorphism of $\mathfrak{q}$ onto $T_{eH}(G/H)$ and the map $D\mapsto L_{x*}D$ defines an isomorphism of $T_{eH}(G/H)$ onto $T_{xH}(G/H)$ where L_{x*} denotes the differential of L_x. Let $\mathfrak{q}^*$ be the dual space of $\mathfrak{q}$. Similarly the quotient space $G\underset{H}{\times}\mathfrak{q}^*$ of $G\times\mathfrak{q}^*$ is isomorphic to the cotangent vector bundle $T^*(G/H)$ of G/H. Let p be the natural projection of $G\times\mathfrak{q}^*$ onto $G\underset{H}{\times}\mathfrak{q}^*$ and $[x,\lambda]$ the image of (x,λ) by p $((x,\lambda)\in G\times\mathfrak{q}^*)$. For any D in $T_{eH}(G/H)$, we can choose only one X_D in $\mathfrak{q}$ such that $Df=\frac{d}{dt}f(\exp(tX_D)H)|_{t=0}$ $(f\in C^\infty(G/H))$. For any λ in $\mathfrak{q}^*$, we define ξ_λ in $T^*_{eH}(G/H)$ by $\xi_\lambda(D)=\lambda(X_D)$. Then the map $\lambda\mapsto\xi_\lambda$ defines an isomorphism of $\mathfrak{q}^*$ onto $T^*_{eH}(G/H)$ and the map $\xi\mapsto L_x^{*-1}\xi$ defines an isomorphism of $T^*_{eH}(G/H)$ onto $T^*_{xH}(G/H)$ where L^*_x denotes the codifferential of L_x.

Finally, we give general notions. Let M be a C^∞ manifold. We define a 1-form θ on the cotangent vector bundle T^*M of M as follows:

$$\theta_r(v)=\langle r,\pi_*v\rangle\qquad (r\in T^*M,\ v\in T_r(T^*M))$$

where π is the natural projection of T^*M onto M. θ is called the canonical 1-form on T^*M. Set $\Omega=d\theta$. Then Ω is a non-degenerate 2-form on T^*M. Thus the map $X\mapsto\iota(X)\Omega$ is an isomorphism of $\mathfrak{X}(T^*M)$ onto $\Omega^1(T^*M)$ where

$\iota(X)\Omega$ is the interior product of X and Ω. For each f in $C^\infty(T^*M)$ we define a C^∞ vector field H_f on T^*M by $df=\iota(H_f)\Omega$. H_f is called the Hamiltonian vector field on T^*M corresponding to f.

3. Hamiltonian vector field

We denote by c the principal symbol of C. Let π be the natural projection of $T^*(G/H)$ onto G/H. It is clear that

$$c(\xi) = -\sum_{i=1}^r \langle X^*_{ixH}, \xi\rangle^2 + \sum_{i=1}^s \langle Y^*_{ixH}, \xi\rangle^2 \quad (\xi \in T^*(G/H),\ \pi(\xi) = xH).$$

Since the Casimir operator is contained in the center of $U(\mathfrak{g}_c)$, we have

$$-\sum_{i=1}^r X_i^2 + \sum_{i=1}^s Y_i^2 = -\sum_{i=1}^r (\mathrm{Ad}\,(x)X_i)^2 + \sum_{i=1}^s (\mathrm{Ad}\,(x)Y_i)^2$$

for any x in G. Hence it follows from the above remarks that

$$\begin{aligned} c(\xi) &= -\sum_{i=1}^r \langle(\mathrm{Ad}\,(x)X_i)^*_{xH}, \xi\rangle^2 + \sum_{i=1}^s \langle(\mathrm{Ad}\,(x)Y_i)^*_{xH}, \xi\rangle^2 \\ &= -\sum_{i=1}^r \langle L_{x*}D_{X_i}, \xi\rangle^2 + \sum_{i=1}^s \langle L_{x*}D_{Y_i}, \xi\rangle^2 \\ &= -\sum_{i=1}^p \langle L_{x*}D_{X_i}, \xi\rangle^2 + \sum_{i=1}^q \langle L_{x*}D_{Y_i}, \xi\rangle^2 \end{aligned}$$

because $D_X=0$ if X is in $\mathfrak{h}$. Therefore we obtain that

$$\begin{aligned} c([x, \lambda]) &= -\sum_{i=1}^p \langle L_{x*}D_{X_i}, L_x^{*-1}\xi_\lambda\rangle^2 + \sum_{i=1}^q \langle L_{x*}D_{Y_i}, L_x^{*-1}\xi_\lambda\rangle^2 \\ &= -\sum_{i=1}^p \langle D_{X_i}, \xi_\lambda\rangle^2 + \sum_{i=1}^q \langle D_{Y_i}, \xi_\lambda\rangle^2 \\ &= -\sum_{i=1}^p \lambda(X_i)^2 + \sum_{i=1}^q \lambda(Y_i)^2. \end{aligned}$$

Next for each X in $\mathfrak{g}$, we define a C^∞ vector field $\tilde{X}$ on $G\times\mathfrak{q}^*$ by the formula:

$$(\tilde{X}f)(x, \lambda) = \frac{d}{dt}f(x\exp(tX), \lambda)\,|_{t=0} \quad (f\in C^\infty(G\times\mathfrak{q}^*),\ (x, \lambda)\in G\times\mathfrak{q}^*).$$

For each ν in $\mathfrak{q}^*$, we define a C^∞ vector field ∂_ν on $G\times\mathfrak{q}^*$ by the formula:

$$(\partial_\nu f)(x, \lambda) = \frac{d}{dt}f(x, \lambda + t\nu)\,|_{t=0} \quad (f\in C^\infty(G\times\mathfrak{q}^*),\ (x, \lambda)\in G\times\mathfrak{q}^*).$$

Then $T_{(x,\lambda)}(G\times\mathfrak{q}^*)$ is spanned by $\tilde{X}_{(x,\lambda)}$ $(X\in\mathfrak{g})$ and $\partial_{\nu(x,\lambda)}$ $(\nu\in\mathfrak{q}^*)$. We define a C^∞ vector field $\tilde{E}$ on $G\times\mathfrak{q}^*$ by the formula:

$$\tilde{E}_{(x,\lambda)} = 2\sum_{i=1}^p \lambda(X_i)\tilde{X}_{i(x,\lambda)} - 2\sum_{i=1}^q \lambda(Y_i)\tilde{Y}_{i(x,\lambda)} \quad ((x, \lambda)\in G\times\mathfrak{q}^*).$$

Taking account of the G-invariance of the Casimir operator, we obtain the following lemma by elementary computations.

LEMMA 1.

$$-\sum_{i=1}^{p}\lambda(X_i)X_i+\sum_{i=1}^{q}\lambda(Y_i)Y_i=-\sum_{i=1}^{p}\lambda(\mathrm{Ad}(h)X_i)\,\mathrm{Ad}(h)X_i$$
$$+\sum_{i=1}^{q}\lambda(\mathrm{Ad}(h)Y_i)\,\mathrm{Ad}(h)Y_i\quad(h\in H,\ \lambda\in\mathfrak{q}^*).$$

It follows from the above lemma that

$$p_{*(x,\lambda)}\tilde{E}=p_{*(xh,{}^t\mathrm{Ad}(h)\lambda)}\tilde{E}.$$

Therefore we can define a C^∞ vector field E on $G\times_H\mathfrak{q}^*$ by the formula;

$$E_{[x,\lambda]}=p_{*(x,\lambda)}\tilde{E}.$$

Next we shall show that the vector field E is equal to the Hamiltonian vector field H_c. To do so, we require next two lemmas.

LEMMA 2. *Let θ be the canonical 1-form on $G\times_H\mathfrak{q}^*$, and p^* the codifferential of p. Then we have the following properties*:

(i)
$$(p^*\theta)_{(x,\lambda)}(\tilde{X})=\begin{cases}\lambda(X) & (X\in\mathfrak{q}),\\ 0 & (X\in\mathfrak{h}).\end{cases}$$

(ii) $(p^*\theta)_{(x,\lambda)}(\tilde{E})=2\sum_{i=1}^{p}\lambda(X_i)^2-2\sum_{i=1}^{q}\lambda(Y_i)^2.$

(iii) $(p^*\theta)_{(x,\lambda)}(\partial_v)=0.$

(iv) $(p^*\theta)_{(x,\lambda)}([\tilde{E},\tilde{X}])=0\quad(X\in\mathfrak{g}).$

(v) $(p^*\theta)_{(x,\lambda)}([\tilde{E},\partial_v])=-2\sum_{i=1}^{p}\lambda(X_i)v(X_i)+2\sum_{i=1}^{q}\lambda(Y_i)v(Y_i).$

(vi) $(\partial_v)_{(x,\lambda)}((p^*\theta)(\tilde{E}))=4\sum_{i=1}^{p}\lambda(X_i)v(X_i)-4\sum_{i=1}^{q}\lambda(Y_i)v(Y_i).$

PROOF. In view of the definition of θ, we have

$$\begin{aligned}(p^*\theta)_{(x,\lambda)}(v)&=\theta_{p(x,\lambda)}(p_*v)\\&=\langle p(x,\lambda),\pi_*p_*v\rangle\\&=\langle L_x^{*-1}\xi_\lambda,\pi_*p_*v\rangle\qquad(v\in T_{(x,\lambda)}(G\times\mathfrak{q}^*)).\end{aligned}$$

Since $\pi_*p_*\tilde{X}_{(x,\lambda)}f=\frac{d}{dt}f(x\exp(tX)H)|_{t=0}$ $(f\in C^\infty(G/H))$, we obtain that $\pi_*p_*\tilde{X}=0$ if $X\in\mathfrak{h}$ and $\pi_*p_*\tilde{X}=L_{x*}D_X$ if $X\in\mathfrak{q}$. Hence

$$(p^*\theta)_{(x,\lambda)}(\tilde{X})=\begin{cases}\lambda(X) & (X\in\mathfrak{q}),\\ 0 & (X\in\mathfrak{h}).\end{cases}$$

Thus (i) follows. Since

$$\pi_* p_* \tilde{E}_{(x,\lambda)} f = 2 \sum_{i=1}^{p} \lambda(X_i) \frac{d}{dt} f(x \exp(tX_i)H)\,|_{t=0}$$

$$- 2 \sum_{i=1}^{q} \lambda(Y_i) \frac{d}{dt} f(x \exp(tY_i)H)\,|_{t=0} \qquad (f \in C^\infty(G/H)),$$

we have

$$\pi_* p_* \tilde{E}_{(x,\lambda)} = 2 \sum_{i=1}^{p} \lambda(X_i) L_{x*} D_{X_i} - 2 \sum_{i=1}^{q} \lambda(Y_i) L_{x*} D_{Y_i}.$$

This completes the proof of (ii). By $\pi_* p_* \partial_{\nu(x,\lambda)} = 0$, (iii) is easily proved. We turn to the proof of (iv). We note that

$$\begin{aligned}[\tilde{E}, \tilde{X}] &= 2 \sum_{i=1}^{p} \lambda(X_i) [\tilde{X}_i, \tilde{X}] - 2 \sum_{i=1}^{q} \lambda(Y_i) [\tilde{Y}_i, \tilde{X}] \\ &= 2 \sum_{i=1}^{p} \lambda(X_i) \widetilde{[X_i, X]} - 2 \sum_{i=1}^{q} \lambda(Y_i) \widetilde{[Y_i, X]}.\end{aligned}$$

Since $[X_i, X] \in \mathfrak{h}$ if $X \in \mathfrak{q}$ and $[X_i, X] \in \mathfrak{q}$ if $X \in \mathfrak{h}$, it follows from (i) that

$$(p^*\theta)_{(x,\lambda)}([\tilde{E}, \tilde{X}]) = \begin{cases} 2 \sum_{i=1}^{p} \lambda(X_i)\lambda([X_i, X]) \\ \qquad - 2 \sum_{i=1}^{q} \lambda(Y_i)\lambda([Y_i, X]) & (X \in \mathfrak{h}), \\ 0 & (X \in \mathfrak{q}). \end{cases}$$

When $X \in \mathfrak{h}$, according to Lemma 1 we have

$$\begin{aligned}&\sum_{i=1}^{p} \lambda(\mathrm{Ad}(\exp tX)X_i)^2 - \sum_{i=1}^{q} \lambda(\mathrm{Ad}(\exp tX)Y_i)^2 \\ &= \sum_{i=1}^{p} \lambda(X_i)^2 - \sum_{i=1}^{q} \lambda(Y_i)^2.\end{aligned}$$

Differentiating the above equation at $t=0$, we obtain that

$$2 \sum_{i=1}^{p} \lambda(X_i)\lambda([X, X_i]) - 2 \sum_{i=1}^{q} \lambda(Y_i)\lambda([X, Y_i]) = 0,$$

which completes the proof of (iv). It is clear that

$$\begin{aligned}[\tilde{E}, \partial_\nu] &= -2 \sum_{i=1}^{p} \partial_\nu(\lambda(X_i))\tilde{X}_i + 2 \sum_{i=1}^{q} \partial_\nu(\lambda(Y_i))\tilde{Y}_i \\ &= -2 \sum_{i=1}^{p} \nu(X_i)\tilde{X}_i + 2 \sum_{i=1}^{q} \nu(Y_i)\tilde{Y}_i.\end{aligned}$$

Thanks to (i), (v) follows. According to (ii), we have

$$\begin{aligned}(\partial_\nu)_{(x,\lambda)}(p^*\theta(\tilde{E})) &= 2 \sum_{i=1}^{p} \partial_\nu(\lambda(X_i)^2) - 2 \sum_{i=1}^{q} \partial_\nu(\lambda(Y_i)^2) \\ &= 4 \sum_{i=1}^{p} \lambda(X_i)\nu(X_i) - 4 \sum_{i=1}^{q} \lambda(Y_i)\nu(Y_i),\end{aligned}$$

which completes the proof of (vi).

LEMMA 3. *Let Ω define the non-degenerate 2-form on $T^*(G/H)$ by*

$\Omega=d\theta$. *Then we have*

(i) $(p^*\iota(E)\Omega)_{(x,\lambda)}(\tilde{X})=0 \qquad (x\in \mathfrak{g})$,
(ii) $(p^*\iota(E)\Omega)_{(x,\lambda)}(\partial_v)=-2\sum_{i=1}^{p}\lambda(X_i)v(X_i)+2\sum_{i=1}^{q}\lambda(Y_i)v(Y_i)$.

PROOF. For any v in $T_{(x,\lambda)}(G\times\mathfrak{q}^*)$, we have

$$\begin{aligned}(p^*\iota(E)\Omega)_{(x,\lambda)}(v) &= (\iota(E)\Omega)_{p(x,\lambda)}(p_*v)\\ &= \Omega_{p(x,\lambda)}(E_{p(x,\lambda)}, p_*v)\\ &= (p^*\Omega)_{(x,\lambda)}(\tilde{E}_{(x,\lambda)}, v)\\ &= (d(p^*\theta))_{(x,\lambda)}(\tilde{E}, v)\\ &= \tilde{E}_{(x,\lambda)}((p^*\theta)(v))-v((p^*\theta)(\tilde{E}))-(p^*\theta)_{(x,\lambda)}([\tilde{E}, v]).\end{aligned}$$

Therefore the present lemma is an immediate consequence of Lemma 2.

Now since $p^*dc=d(c\circ p)$ and $(c\circ p)(x,\lambda)=-\sum_{i=1}^{p}\lambda(X_i)^2+\sum_{i=1}^{q}\lambda(Y_i)^2$, we have $p^*dc=-2\sum_{i=1}^{p}\lambda(X_i)d\lambda(X_i)+2\sum_{i=1}^{q}\lambda(Y_i)d\lambda(Y_i)$, and so $(p^*dc)(\tilde{X})=0$ $(X\in\mathfrak{g})$ and

$$(p^*dc)(\partial_v)=-2\sum_{i=1}^{p}\lambda(X_i)v(X_i)+2\sum_{i=1}^{q}\lambda(Y_i)v(Y_i) \qquad (v\in\mathfrak{q}^*).$$

Hence according to Lemma 3, we have

$$p^*\iota(E)\Omega=p^*dc.$$

Therefore regarding that the map p^* of $\Omega^1(G\times_H\mathfrak{q}^*)$ into $\Omega^1(G\times\mathfrak{q}^*)$ is injective, we obtain the following proposition.

PROPOSITION 4. *The vector field E on $T^*(G/H)$ is equal to the Hamiltonian vector field H_c corresponding to the principal symbol c of the Laplace operator C.*

4. Global solvability of the Laplace operator

In this section we assume that G/H is non-compact. To show that the Laplace operator C is surjective on $C^\infty(G/H)$, owing to [6], it suffices to demonstrate the following conditions:

(I) *The principal symbol c of C is real and ${}^tC=C$ where ^{t} is transpose with respect to the G-invariant measure dgH.*
(II) *No null bicharacteristic curve of C lies over a compact subset of G/H.*
(III) *If u is a distribution on G/H with compact support and $Cu=0$, then $u=0$.*

(IV) *For any compact set Γ of G/H, there exists a compact set $\hat{\Gamma}$ satisfying the following properties*;

(a) *Γ is included in the set* $\mathrm{Int}(\hat{\Gamma})$ *of all interior points of $\hat{\Gamma}$.*

(b) *If u is a distribution on G/H with compact support and the support* Supp Cu *of Cu is included in Γ, then* Supp u *is included in $\hat{\Gamma}$.*

PROOF OF (I). It is evident that the principal symbol c of C is real. For any φ and ψ in $C_c^\infty(G/H)$, set

$$(\varphi, \psi) = \int_{G/H} \varphi(gH)\psi(gH)dgH.$$

Then since $(X^*\varphi, \psi) = -(\varphi, X^*\psi)$ $(X \in \mathfrak{g})$ and

$$C = -\sum_{i=1}^{r} X_i^{*2} + \sum_{i=1}^{s} Y_i^{*2},$$

we have $(C\varphi, \psi) = (\varphi, C\psi)$ $(\varphi, \psi \in C_c^\infty(G/H))$.

PROOF OF (II). Recall that the bicharacteristic strips of C are defined as the integral curves of the Hamiltonian vector field H_c in $T^*(G/H)\backslash 0$. Fix $[x, \lambda]$ in $G \times_H \mathfrak{q}^*$ such that $c([x, \lambda]) = 0$. Set

$$\tilde{\gamma}(t) = [x \exp(2t(\sum \lambda(X_i)X_i - \sum \lambda(Y_i)Y_i)), \lambda] \qquad (t \in \boldsymbol{R}).$$

Then by Proposition 4 it is obvious that $\tilde{\gamma}(t)$ is an integral curve of H_c which passes through $[x, \lambda]$ in $G \times_H \mathfrak{q}^*$ and $\pi \circ \tilde{\gamma}(t)$ $(t \in \boldsymbol{R})$ is a bicharacteristic curve which passes through xH in G/H. To show that no null bicharacteristic curve of C lies over a compact set in G/H, it suffices to consider the bicharacteristic curve passing through eH in G/H.

Suppose that $\tilde{\gamma} \subset T^*(G/H)\backslash 0$ is the bicharacteristic strip of C passing through $[e, \lambda_0]$ $(\lambda_0 \neq 0)$ and that $\{\pi \circ \tilde{\gamma}(t);\, t \in \boldsymbol{R}\}$ is relatively compact in G/H. Since the map $gH \mapsto g\sigma(g^{-1})$ of G/H into G is continuous, the image

$$\{\exp(4t(\sum \lambda_0(X_i)X_i - \sum \lambda_0(Y_i)Y_i));\, t \in \boldsymbol{R}\} \quad \text{of} \quad \{\pi \circ \tilde{\gamma}(t);\, t \in \boldsymbol{R}\}$$

is also relatively compact. Set

$$\gamma(t) = \exp(4t(\sum \lambda_0(X_i)X_i - \sum \lambda_0(Y_i)Y_i)) \qquad (t \in \boldsymbol{R}).$$

Let K' be a maximal compact subgroup of G which contains $\{\gamma(t);\, t \in \boldsymbol{R}\}$. Then

$$\frac{d}{dt}\gamma(t)|_{t=0} = 4(\sum_{i=1}^{p} \lambda_0(X_i)X_i - \sum_{i=1}^{q} \lambda_0(Y_i)Y_i)$$

belongs to the Lie algebra $\mathfrak{k}'$ of K'. Consequently $c([e, \lambda_0]) = -\sum_{i=1}^{p} \lambda_0(X_i)^2 + \sum_{i=1}^{q} \lambda_0(Y_i)^2 < 0$ because the Killing form B is negative definite on $\mathfrak{k}'$ and $\lambda_0 \neq 0$. This contradicts the fact that $c([e, \lambda_0]) = 0$.

PROOF OF (III). Let u be a distribution on G/H. In view of Theorem 7.3 in [4], $WF_A(u)\backslash WF_A(Cu)$ is invariant under the Hamiltonian vector field H_c of C where $WF_A(u)$ is the analytic wave front set of u defined by L. Hörmander.

Now let u be a distribution on G/H with compact support such that $Cu=0$. Then $WF_A(u)=\phi$. In fact, if $[x_0, \lambda_0]\in WF_A(u)$, the null bicharacteristic strip $\tilde{\gamma}(t)$ $(t\in \boldsymbol{R})$ passing through $[x_0, \lambda_0]$ is included in $WF_A(u)$. Hence, by (II) $\pi\circ\tilde{\gamma}(t)$ $(t\in \boldsymbol{R})$ is not bounded, which contradicts the fact that $\pi(WF_A(u))$ is compact. Therefore u is analytic on G/H, and so $u=0$ because G/H is connected.

PROOF OF (IV). Taking a maximal abelian subspace $a_{\mathfrak{q}}$ of $\mathfrak{p}\cap\mathfrak{q}$, we extend a maximal abelian subspace a of $\mathfrak{p}$ such that $a\supset a_{\mathfrak{q}}$. Then $a=a\cap\mathfrak{q}+a\cap\mathfrak{h}$ (direct sum) and $a_{\mathfrak{q}}=a\cap\mathfrak{q}$. Set $A=\exp a$. Then as is easily seen the map φ of $K\times A/(A\cap H)$ into G/H defined by

$$\varphi(k, a\,A\cap H) = kaH \qquad ((k, a\,A\cap H)\in K\times A/(A\cap H))$$

is surjective (see [2] and [5]). For any g in G, write (uniquely)

$$g = k\exp X \qquad (k\in K,\ X\in\mathfrak{p})$$

and set $\|g\|=B(X, X)^{1/2}$, and so $\|\cdot\|$ is continuous. Given gH in G/H, set $d(gH)=\|g\sigma(g^{-1})\|$. Then d is invariant under K, that is

$$d(kgH) = d(gH) \qquad (k\in K,\ gH\in G/H).$$

Now, for any compact subset Γ of G/H, we can find a compact subset Γ_0 such that $\Gamma\subset\mathrm{Int}(\Gamma_0)$. Set $r=\max\{d(gH); gH\in\Gamma_0\}$ and $B_r=\{a\,A\cap H\in A/(A\cap H); d(aH)\leq r\}$. Then $\Gamma_0\subset\varphi(K\times B_r)$, and moreover

$$G/H = \varphi(K\times B_r)\sqcup\varphi(K\times(A/(A\cap H)\backslash B_r)) \qquad \text{(disjoint union)}.$$

Evidently $\varphi(K\times B_r)$ is compact and $\varphi(K\times(A/(A\cap H)\backslash B_r))$ has at most two connected components which are unbounded open subsets. Set $\hat{\Gamma}=\varphi(K\times B_r)$. Then by arguments similar to the proof of (III), one can easily show (IV). The proofs of (I), (II), (III) and (IV) are now complete. Therefore we have the following theorem.

THEOREM 5. *The Laplace operator C is surjective on $C^\infty(G/H)$.*

References

[1] A. Cerezo and F. Rouvière, Operateurs differentiels invariants sur un groupe de Lie, Séminaire Goulaouic-Schwartz, 1972–1973, École Polytechnique Paris.

[2] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, Academic Press, New York, 1978.

[3] ————, The surjectivity of invariant differential operators on symmetric spaces, I, Ann. of Math., **98** (1973), 451–479.
[4] L. Hörmander, Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, Comm. Pure Appl. Math., **24** (1971), 671–703.
[5] O. Loos, Symmetric spaces, I, II, W. A. Benjamin, Inc, New York, 1969.
[6] J. Rauch and D. Wigner, Global solvability of the Casimir operator, Ann. of Math., **103** (1976), 229–236.

Department of Mathematics,
Faculty of Science,
Hiroshima University

Added in Proof: After this paper was written, the authors have found that Weita Chang has obtained the same result; J. Functional Anal., **34** (1979), 481–491.