Euler integral transformations of hypergeometric functions of two variables

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§0. Introduction

It is known that the hypergeometric functions $F_1$, $F_2$, $F_3$ and $F_4$ of two variables have the following Euler integral representations

(0.0) $F_1(\alpha,\beta,\beta',\gamma;x,y) = C_0 \int_{D_0} u^{s-1}(1-u)^{\gamma-s-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}du$,

(0.1) $F_2(\alpha,\beta,\beta',\gamma;x,y) = C_1 \int_{D_1} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux-\gamma v)^{-s}dudv$,

(0.2) $F_3(\alpha,\beta,\beta',\gamma;x,y) = C_2 \int_{D_2} u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma-\beta'-1}(1-ux-vy)^{-s}dudv$,

(0.3) $F_4(\alpha,\alpha',\beta,\beta';x,y) = C_3 \int_{D_3} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux-vy)^{-s}dudv$,

(0.4) $F_4(\alpha,\beta,\gamma;\gamma';x,y) = C_4 \int_{D_4} u^{\alpha-\gamma}v^{\alpha'-\gamma}(1-u-v)^{\gamma-\alpha-\gamma}(1-ux-vy)^{-\beta}dudv$,

where $C_j$ are some constants and $D_j$ are some cycles. (0.0) is discovered by E. Picard, (0.4) is discovered by K. Aomoto (see [3]) and the others are discovered by P. Appell.

In this paper, we establish a principle of Euler integral representations and give new integral formulae for the hypergeometric functions $F_2$, $G_1$, $G_2$, $H_1$ and $H_2$ (Horn's notation).

THEOREM.

(0.6) $F_2(\alpha,\beta,\beta',\gamma';x,y)$

$$= C_6 \int_{D_6} u^{\alpha-\gamma}v^{\alpha'-\gamma}(u+v-uv)^{\gamma'-\gamma-2}(1-ux)^{-\beta}(1-vy)^{-\beta'}dudv,$$

(0.7) $G_1(\alpha,\beta,\beta';x,y) = C_7 \int_{D_7} u^{\beta'-1}(u+1)^{-\beta-\beta'}(1-ux-yu)^{-\beta}du,$
\( G_2(\alpha,\alpha',\beta,\beta';x,y) = C_8 \int_{D_8} u^{\beta'-1}(u+1)^{-\beta'-\beta'}(1-ux-\frac{y}{u})^{-\beta'}du, \)

(0.9) \( H_1(\alpha,\beta,\gamma;x,y) = C_9 \int_{D_9} u^{\gamma} v^{\gamma-1}(uv+u-1)^{r-s-\gamma-1}(1-ux-vy)^{-\beta}dudv, \)

(0.10) \( H_2(\alpha,\beta',\gamma,\varepsilon;x,y) = C_{10} \int_{D_{10}} u^{\gamma} v^{\gamma-1}(uv+u-1)^{r-s-\gamma-1}(1-ux-vy)^{-\beta'}dudv, \)

where

\[
C_6 = C_4 = (-1)^{r-s} \frac{\sin \pi (\alpha-\gamma)}{\pi} \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(\gamma+\gamma'-\alpha-1)},
\]

\[
C_7 = C_9 = (-1)^{\beta'+1} \frac{\sin \pi \beta}{\pi} \frac{\Gamma(\beta)\Gamma(\beta')}{},
\]

\[
C_8 = C_{10} = \frac{\Gamma(\varepsilon)}{\Gamma(\varepsilon-\alpha-\gamma)\Gamma(\alpha)\Gamma(\gamma)},
\]

and

\[
D_6 = D_4 = \{(u,v) \in \mathbb{R}^2 | 0 \leq u \leq 1, v \leq 0, u+v-uv \geq 0\},
\]

\[
D_7 = D_8 = \{u \in \mathbb{R} | -1 \leq u \leq 0\},
\]

\[
D_9 = D_{10} = \{(u, v) \in \mathbb{R}^2 | 0 \leq u \leq 1, v \geq 0, uv+u-1 \leq 0\}.
\]

**Corollary.** The system of differential equations satisfied by the function \( y^{-\gamma}G_2(\alpha,\alpha',\beta,\beta';-x,-1) \) is identical to that of \( F_4(\alpha'+\beta',\alpha,\alpha',\beta'-\beta+1;x,y) \).

We transform the hypergeometric differential equations, by Euler kernels, so that the transformed differential equations have elementary function solutions. This method, though quite simple, has not been applied to hypergeometric functions of two variables. Moreover, A. Erdélyi mentioned in [2] that "... by transformation of the systems of partial differential equations satisfied by the hypergeometric functions. This method, though simple in theory, is rather laborious in practice and not very useful for discovering new transformations".

For other hypergeometric functions \( G_3, H_3, ..., H_7 \), we have not yet succeeded to discover "elementary" Euler integral formulae.

**§ 1. Review**

Since hypergeometric functions and equations of two variables are not very familiar, we list up the series \( \sum A_{\alpha\beta}x^\alpha y^\beta \) and the systems of differential operators \( P_1, P_2 \) which annihilate them.
Here we used the following symbols.

\[(\alpha, m) = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)}, \quad \delta = x \frac{\partial}{\partial x}, \quad \delta' = y \frac{\partial}{\partial y}.\]
This table reads as follows: (e.g. $H_1$)

$$z = H_1(x, \beta, \gamma, \varepsilon; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha, m-n) (\beta, m+n) (\gamma, n)}{(0, m)(1, m)(1, n)} x^m y^n$$

satisfies

$$P_1 z = \frac{1}{x} \delta(x-1+\delta) z - (\alpha + \delta - \delta') (\beta + \delta + \delta') z = 0,$$

$$P_2 z = \frac{1}{y} \delta'(x-1+\delta) z - (\gamma + \delta') (\beta + \delta + \delta') z = 0.$$

§ 2. Euler kernels

Put

$$K(\lambda): = (1-u x-v y)^{-\lambda},$$

$$K(\lambda, \lambda'): = (1-u x-v y)^{-\lambda} (1-v y)^{-\lambda'}$$

which shall be called Euler kernels of the first and the second kind. The following lemma, though obvious, is essential.

**Lemma.** Let

$$\theta = u \frac{\partial}{\partial u}, \quad \theta' = v \frac{\partial}{\partial v}.$$ 

Then, for $K(\lambda)$, we have

$$\delta K(\lambda) = \theta K(\lambda), \quad \delta' K(\lambda) = \theta' K(\lambda),$$

$$\lambda K(\lambda+1) + (a - \lambda) K(\lambda),$$

$$\lambda u(a + b + b \delta + c \delta') K(\lambda+1),$$

$$\lambda v(a + c + b \delta + c \delta') K(\lambda+1),$$

and, for $K(\lambda, \lambda')$, we have

$$\delta K(\lambda, \lambda') = \theta K(\lambda, \lambda'), \quad \delta' K(\lambda, \lambda') = \theta' K(\lambda, \lambda'),$$

$$\lambda K(\lambda+1, \lambda') + (a - \lambda) K(\lambda, \lambda'),$$

$$\lambda' K(\lambda, \lambda'+1) + (a' - \lambda') K(\lambda, \lambda'),$$

$$\lambda u(a + b + b \delta + c \delta') K(\lambda+1, \lambda'),$$

$$\lambda v(a + c + b \delta + c \delta') K(\lambda+1, \lambda'),$$
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(2.9) \[ \frac{1}{y} \delta'(a + b + c\delta')K(\lambda, \lambda') = \lambda' \nu(a + c + b\delta + c\delta')K(\lambda, \lambda' + 1). \]

§ 3. Formal transformations and their principle

Let \( P_1 \) and \( P_2 \) be one of the pairs of differential operators introduced in § 1. We want to represent the solution \( z = z(x, y) \) of the system

\[ P_iz = 0 \quad (i = 1, 2) \]

by the integral

\[ z = \int \int w(u, v) K \, du \, dv \]

where \( K \) is the Euler kernel of the first or second kind and \( w = w(u, v) \) is some elementary function of \( u \) and \( v \). If we can differentiate under the sign of integral, then we have

\[ P_iz = \int \int w(u, v) P_i K \, du \, dv. \quad (i = 1, 2) \]

If there exist a function \( L \) and differential operators \( Q_i (i = 1, 2) \) with respect to \( u \) and \( v \) which do not include \( x \) and \( y \), such that

\[ P_i K = Q_i L, \quad (i = 1, 2) \]

then \( w \) is determined by the system

\[ Q_i^* w = 0, \quad (i = 1, 2) \]

where \( Q_i^* \) is the adjoint operator of \( Q_i \). And if \( Q_i (i = 1, 2) \) are of the first order, since \( Q_i^* (i = 1, 2) \) are also of the first order, \( w \) will be an elementary function of \( u \) and \( v \).

We shall determine \( K \) so that a function \( L \) and operators \( Q_i (i = 1, 2) \) of the first order exist. By virtue of the lemma in § 2, we conclude that if the right hand sides of \( P_1 \) and \( P_2 \) contain a term of the form \( (a + \delta + \delta') \) in common, then we let \( K = K(a) \), and if the right hand sides of \( P_1 \) and \( P_2 \) contain terms of the form \( (a + \delta) \) and \( (a' + \delta') \) respectively, then we let \( K = K(a, a') \). Remark that every system \( P_1, P_2 \) in § 1 satisfies one or both of the above conditions.

For example, we shall apply this principle to \( H_1 \). Since the right hand sides of \( P_1 \) and \( P_2 \) have a term \( (\beta + \delta + \delta') \) in common, we let \( K = K(\beta) \). Then by (2.1), ..., (2.4), we have

\[ Q_1 = \beta \{ u(e + \theta) - (a + \theta - \theta') \}, \]

\[ Q_2 = \beta \{ v(\alpha - 1 + \theta - \theta') - (y - \theta') \} \]
and \( L = K(\beta + 1) \).

Since \( \theta^* = -\theta - 1 \) and \( (\theta')^* = -\theta' - 1 \), we have

\[
Q^*_1 = \beta((\epsilon - \theta - 1)u - (x - \theta + \theta')),
\]

\[
Q^*_2 = \beta((x - 1 - \theta + \theta')v - (\gamma - \theta' - 1)) \cdot
\]

An easy calculation shows that the system \( Q^*_1 w = Q^*_2 w = 0 \) has a solution

\[
w = u^{x+y-1}v^{y-1}(uv+u-1)^{-x+y+1}.
\]

For \( F_1 (K = K(\beta, \beta')), G_1 (K = K(\alpha)) \) and \( G_2 (K = K(\alpha, \alpha')) \), the supports of any solutions of the corresponding equations \( Q^*_1 w = Q^*_2 w = 0 \) are on the varieties \( \{u = v\}, \{uv = 1\} \) and \( \{uv = 1\} \) respectively. That is, in the usual sense, the system \( Q^*_1 w = Q^*_2 w = 0 \) is incompatible. In these cases, we modify the respective Euler kernels as follows:

\[
K(\beta, \beta') \longrightarrow (1 - ux)^{-\theta}(1 - vy)^{-\theta'},
\]

\[
K(\alpha) \longrightarrow (1 - ux - \frac{v}{u})^{-s},
\]

\[
K(\alpha, \alpha') \longrightarrow (1 - ux)^{-s}(1 - \frac{v}{u})^{-s'}.
\]

Then, if we replace the double integral by the simple integral, the argument in the beginning of this section leaves valid and moreover we have \( Q^*_1 = Q^*_2 \).

In this way, we obtained (formal) Euler integral representations for the solutions of the systems in § 1.

§ 4. Proof of the theorem

Once the formal integral representations are obtained, the definite integrals (0.1),..., (0.10) are obtained by routine method. That is, by making use of the power series expansion of the Euler kernels

\[
(1 - ux)^{-\lambda}(1 - vy)^{-\lambda'} = \sum_{m,n=0}^\infty \frac{(\lambda, m)(\lambda', n)}{(1, m)(1, n)} u^m v^n x^m y^n,
\]

\[
(1 - ux - vy)^{-\lambda} = \sum_{m,n=0}^\infty \frac{(\lambda, m + n)}{(1, m)(1, n)} u^m v^n x^m y^n
\]

and the formulae

\[
\int_0^1 u^{p-1}(1 - u)^{q-1} du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}
\]
and

\[ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \]

The corollary to the theorem can be deduced from a comparison of the integrals (0.0) and (0.8), or by a transformation of dependent and independent variables in the system of \( G_2 \).

References


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