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On the oscillation of solutions of forced even order nonlinear differential equations

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The oscillation of even order differential equations, both forced and unforced, has been an area of a large amount of interest. For example, see [4] and its bibliography.

In this paper we will give some results concerning the oscillation of solutions to equations of the form

$$(1) \qquad x^{(2n)} + f(t, x, x', \dots, x^{(2n-1)}) = R^{(2n)}(t)$$

where the functions f and R satisfy appropriate conditions. Our conditions on R generalize those found in [2] and [3].

A solution to (1) on an interval $[a, \infty)$ is said to be oscillatory if it has an unbounded set of zeros. A real valued function R is called *strongly bounded* if it assumes its maximum and minimum on every interval of the form $[a, \infty)$, $0<a$. Throughout the remainder of this paper $\mathbf{R}$ and $\mathbf{R}^+$ will denote the reals and nonnegative reals respectively.

LEMMA 1. *Let $R \in C^{2n}[\mathbf{R}^+, \mathbf{R}]$ be strongly bounded and $f \in C^0[\mathbf{R}^+ \times \mathbf{R}^{2n}, \mathbf{R}]$ be such that $x_1 f(t, x_1, \dots, x_{2n}) \geq 0$ for every $t \geq 0$ and $(x_1, \dots, x_{2n}) \in \mathbf{R}^{2n}$. If x is a bounded solution of $x^{(2n)} + f(t, x, x', \dots, x^{(2n-1)}) = R^{(2n)}(t)$ on an interval $[a, \infty)$, then exactly one of the following holds:*

(i) *x is oscillatory,*

(ii) *there is a $b>0$ such that $0<x(t)$ and*

$$(-1)^k[x^{(k)}(t) - R^{(k)}(t)] \leq 0 \text{ for } k = 1, 2, \dots, 2n \text{ on } [b, \infty),$$

(iii) *there is a $b>0$ such that $x(t)<0$ and*

$$(-1)^k[x^{(k)}(t) - R^{(k)}(t)] \geq 0 \text{ for } k = 1, 2, \dots, 2n \text{ on } [b, \infty).$$

If x is any nonoscillatory solution on an interval $[a, \infty)$, then

(iv) *there are c, $C>0$ such that $C<|x(t)|$ whenever $c \leq t$.*

Moreover, if x is an unbounded nonoscillatory solution, then $|x(t)| \to \infty$ as $t \to \infty$.

PROOF. Let x denote a bounded nonoscillatory solution of $x^{(2n)}+f=R^{(2n)}$. We will do the proof for the case $x(t)>0$ on $[a, \infty)$. The proof is similar if $x(t)<0$ on $[a, \infty)$. Since $x(t)>0$ on $[a, \infty)$, we have $x^{(2n)}(t)-R^{(2n)}(t)=-f\leq 0$ on $[a, \infty)$. By standard arguments, we also have $(-1)^k(x^{(k)}(t)-R^{(k)}(t))\leq 0$ on some interval $[b, \infty)$, e.g., see the proof of Theorem 1 in [1]. In particular, $x'(t)-R'(t)\geq 0$ on $[b, \infty)$. We will now show that $x'(t)-R'(t)\geq 0$ on an interval of the form $[b, \infty)$ even if x is an unbounded nonoscillatory solution. Suppose that x is a positive unbounded nonoscillatory solution such that $x'(t)-R'(t)$ assumes nonpositive values on every interval of the form $[b, \infty)$. We can not have $x'(t)-R'(t)\leq 0$ on an interval $[b, \infty)$ because if we did then $x(t)-R(t)\leq x(b)-R(b)$ for $t\geq b$ which is impossible since x is unbounded. Hence, $x'(t)-R'(t)$ assumes both positive and negative values on every interval $[b, \infty)$. It follows easily that $x''(t)-R''(t)$ and each of its derivatives also assume both positive and negative values on every interval $[b, \infty)$. In particular $x^{(2n)}(t)-R^{(2n)}(t)$ assumes positive and negative values on every interval $[b, \infty)$. This is impossible because $x^{(2n)}(t)-R^{(2n)}(t)=-f\leq 0$. Hence, we must have $x'(t)-R'(t)\geq 0$. Thus for any nonoscillatory solution $x(t)$ we have that $x(t)-R(t)$ is a nondecreasing function. If $x(t)-R(t)\to\infty$ as $t\to\infty$, then (iv) occurs since R is bounded. If $x(t)-R(t)\not\to\infty$ as $t\to\infty$, then there is an A such that $x(t)-R(t)\to A$ as $t\to\infty$. Let $c\geq b$ be such that $R(c)\leq R(t)$ for $c\leq t$. Since $x(c)-R(c)\leq A$ we have $0<x(c)\leq A+R(c)$. Set $C=\frac{1}{2}(A+R(c))$. Then for t sufficiently large we have $x(t)\geq A+R(t)-C\geq A+R(c)-C=2C-C=C$. Hence, there is a $c>0$ such that $x(t)>C$ whenever $t>c$. Finally suppose that x is a positive unbounded nonoscillatory solution. If $x(t)\not\to\infty$ as $t\to\infty$ then there are a number $E>0$ and a sequence $\{t_i\}$ such that $t_i\to\infty$ and $x(t_i)\leq E$. Since $x'(t)-R'(t)\geq 0$ on an interval $[b, \infty)$ and $x(t)$ is unbounded we may assume that b was chosen so that $x(b)-R(b)>E-\min\{R(t)\colon 0\leq t\}$. From $x'(t)-R'(t)\geq 0$ we obtain

$$\begin{aligned} x(b)-R(b) &\leq x(t_i)-R(t_i) \\ &\leq E-\min\{R(t)\colon 0\leq t\} \end{aligned}$$

for every $t_i>b$. This contradicts our choice of b. Hence $x(t)\to\infty$ whenever $t\to\infty$. This completes the proof.

THEOREM 2. *Let* $p\in C^0[\mathbf{R}^+, \mathbf{R}^+]$, $g\in C^0[\mathbf{R}^{2n}, \mathbf{R}^+]$, $f\in C_0[\mathbf{R}^+\times\mathbf{R}^{2n}, \mathbf{R}]$, *and* $R\in C^{2n}[\mathbf{R}^+, \mathbf{R}]$ *be such that*

(i) *R is strongly bounded,*

(ii) *for every $c>0$ there is a $C>0$ such that $g(x_1, x_2,\dots, x_{2n})\geq C$ whenever $|x_1|\geq c$,*

(iii) *$x_1^{-1}f(t, x_1, x_2,\dots, x_{2n})\geq p(t)g(x_1, x_2,\dots, x_{2n})$ for every $t\geq 0$ and $(x_1, x_2, \dots, x_{2n})\in\mathbf{R}^{2n}$,*

(iv) $\int_0^\infty t^{2n-2}p(t)dt = +\infty$.

Then every solution of

$$x^{(2n)} + f(t, x, x', \ldots, x^{(2n-1)}) = R^{(2n)}(t)$$

on an interval of the form (a, ∞) *is oscillatory.*

PROOF. Suppose that x is a nonoscillatory solution on $[a, \infty)$. It suffices to consider the case that $x(t)>0$ on $[a, \infty)$. Set $W(t)=x(t)-R(t)$. We begin by proving that there is a $C>0$ such that

$$\frac{C}{(k-1)!}\int_t^\infty (u-t)^{k-1}p(u)x(u)du \le (-1)^{k+1}W^{(2n-k)}(t) \tag{2}$$

on some interval of the form $[b, \infty)$ for $k=1, 2, \ldots, 2n-1$. From Lemma 1 (iv) and hypotheses (ii), (iii) we have

$$W^{(2n)}(t) \le -Cp(t)x(t) \le 0 \tag{3}$$

for some $C>0$ and $a \le t$. Thus $W^{(2n-1)}$ is a decreasing function. In particular, there is an $a_1 \ge a$ such that either $W^{(2n-1)}(t) \ge 0$ or $W^{(2n-1)}(t)<0$ on $[a_1, \infty)$. Suppose $W^{(2n-1)}(t)<0$ on $[a_1, \infty)$. Then

$$W^{(2n-1)}(t) \le W^{(2n-1)}(a_1) \tag{4}$$

since $W^{(2n)}(t) \le 0$ on $[a_1, \infty)$. Repeated integrations of each side of (4) shows that

$$x(t)-R(t)$$
$$= W(t) \le W^{(2n-1)}(a_1)\frac{(t-a_1)^{2n-1}}{(2n-1)!} + W^{(2n-2)}(a_1)\frac{(t-a_1)^{2n-2}}{(2n-2)!} + \cdots + W(a_1)$$

Since $W^{(2n-1)}(a_1)<0$, we have $\lim_{t\to\infty}(x(t)-R(t)) = -\infty$ which is impossible because $x(t)>0$ and $R(t)$ is bounded. Therefore we must have $W^{(2n-1)}(t) \ge 0$ on $[a, \infty)$. From (3) we obtain

$$C\int_t^s p(u)x(u)du + W^{(2n-1)}(s) \le W^{(2n-1)}(t)$$

for $a_1 \le t \le s$. Since $W^{(2n-1)}(s) \ge 0$ for all $s \ge a_1$ we have

$$C\int_t^\infty p(u)x(u)du \le W^{(2n-1)}(t) \tag{5}$$

on $[a_1, \infty)$ which is (2) in the special case $k=1$. Notice that a_1 may be chosen so that $W^{(2n-1)}(t)>0$ on $[a_1, \infty)$. Hence, $W^{(2n-2)}$ is strictly increasing on

$[a_1, \infty)$ and there is an $a_2 \geq a_1$ such that either $W^{(2n-2)}(t) > 0$ or $W^{(2n-2)}(t) < 0$ on $[a_2, \infty)$. Suppose that $W^{(2n-2)}(t) > 0$ on $[a_2, \infty)$. Then

$$W^{(2n-2)}(a_2) < W^{(2n-2)}(t) \tag{6}$$

since $W^{(2n-1)}(t) > 0$ on $[a_2, \infty)$. Repeated integrations of each side of (6) show that

$$W^{(2n-2)}(a_2)\frac{(t-a_2)^{2n-2}}{(2n-2)!} + W^{(2n-3)}(a_2)\frac{(t-a_2)^{2n-3}}{(2n-3)!} + \cdots + W(a_2)$$
$$< W(t) = x(t) - R(t).$$

Since $R(t)$ is bounded there is a $c > 0$ such that

$$\frac{1}{2(2n-2)!} W^{(2n-2)}(a_2)t^{2n-2} < x(t)$$

on $[c, \infty)$. By (5) we now have

$$\frac{1}{2(2n-2)!} W^{(2n-2)}(a_2)C\int_t^\infty u^{2n-2}p(u)du < W^{(2n-1)}(t)$$

which is impossible by hypothesis (iv). Thus, we must have $W^{(2n-2)}(t) < 0$ on $[a_2, \infty)$. Integrating each side of (5), using integration by parts on the left hand side, yields

$$C(s-t)\int_s^\infty p(u)x(u)du + C\int_t^s (u-t)p(u)x(u)du \leq W^{(2n-2)}(s) - W^{(2n-2)}(t)$$

for $a \leq t \leq s$. Since $W^{(2n-2)}(s) < 0$ for all $s \geq a_2$, we have

$$C\int_t^\infty (u-t)p(u)x(u)du \leq -W^{(2n-2)}(t)$$

on $[a_2, \infty)$, which is (2) in the special case $k=2$. Proceeding as in the cases $k=1$ and $k=2$, it can be shown that if

$$\frac{C}{(j-1)!}\int_t^\infty (u-t)^{j-1}p(u)x(u)du \leq (-1)^{j+1}W^{(2n-j)}(t)$$

on $[a_j, \infty)$, then there is an a_{j+1} such that

$$\frac{C}{j!}\int_t^\infty (u-t)^j p(u)x(u)du \leq (-1)^j W^{(2n-j-1)}(t)$$

on $[a_{j+1}, \infty)$ for $j = 2, 3, \ldots, 2n-2$. In particular

$$\frac{C}{(2n-2)!}\int_t^\infty (u-t)^{2n-2}p(u)x(u)du \leq W'(t)$$

on some interval $[b, \infty)$. Using Lemma 1 (iv) there is a $D>0$ such that

$$D\int_t^\infty (u-t)^{2n-2}p(u)du \le W'(t).$$

This contradicts hypothesis (iv). Thus there is no nonoscillatory solution on an interval of the form $[a, \infty)$.

LEMMA 3. *Let p, g, f, and R be as in Theorem 2. If equation* (1) *has a bounded nonoscillatory solution x on an interval $[a, \infty)$, then there are numbers c, $C>0$ such that*

$$\frac{C}{(j-1)!}\int_t^\infty (u-t)^{j-1}p(u)du \le (-1)^{j+1}[x^{(2n-j)}(t) - R^{(2n-j)}(t)]$$

for $c\le t$ and $j=1, 2,\dots, 2n-1$.

PROOF. We will indicate how to do the proof by induction. Suppose that $x^{(2n)}+f=R^{(2n)}$ has a bounded nonoscillatory solution x on $[a, \infty)$. Without loss of generality we may assume that $x(t)>0$ on $[a, \infty)$. By Lemma 1 (iv), and hypotheses (ii), and (iii) of Theorem 2 there are c, $C>0$ such that $f(t, x(t), x'(t), \dots, x^{(2n-1)}(t))\ge Cp(t)$ on $[c, \infty)$. By Lemma 1 (ii) we may also assume that $(-1)^k[x^{(k)}(t)-R^{(k)}(t)]\le 0$ for $k=1, 2,\dots, 2n$ on $[c, \infty)$. Set $W(t)=x(t)-R(t)$. Then $(-1)^k W^{(k)}(t)\le 0$ for $k=1, 2,\dots, 2n$. From the differential equation we obtain $W^{(2n)}(t)=-f\le -Cp(t)$ on $[c, \infty)$. Hence,

$$0\le W^{(2n-1)}(s)\le W^{(2n-1)}(t) - C\int_t^s p(u)du$$

for $c\le t\le s$ so that $C\int_t^s p(u)du\le W^{(2n-1)}(t)$ for $c\le t\le s$. If we now let $s\to\infty$, we obtain

$$C\int_t^\infty p(u)du \le W^{(2n-1)}(t)$$

which is the desired result when $j=1$. If we now integrate each side of this inequality using integration by parts on the left hand side, we obtain

$$C(s-t)\int_s^\infty p(u)du + C\int_t^s (u-t)p(u)du \le W^{(2n-2)}(s) - W^{(2n-2)}(t).$$

Since the function $W^{(2n-2)}(s)$ is nonpositive on $[c, \infty)$ we have

$$C\int_t^\infty (u-t)p(u)du \le W^{(2n-2)}(t)$$

which is the desired result when $j=2$. Proceeding as in the cases $j=1$ and $j=2$

the desired result can be established by induction.

THEOREM 4. *Let p, g, f, and R be as in Theorem 2. If* $\int_0^\infty t^{2n-1}p(t)dt=\infty$, *then any solution x of* $x^{(2n)}+f(t, x, x',\dots, x^{(2n-1)})=R^{(2n)}(t)$ *on an interval of the form* $[a, \infty)$ *is either oscillatory or* $|x(t)|\to\infty$ *as* $t\to\infty$.

PROOF. Suppose that x is bounded nonoscillatory and $x(t)>0$ on $[a, \infty)$. From Lemma 3 we have $\frac{C}{(2n-2)!}\int_t^\infty (u-t)^{2n-2}p(u)du \le x'(t)-R'(t)$. If we integrate each side of this inequality, using integration by parts on the left hand side, we obtain

$$\frac{C(s-t)}{(2n-1)!}\int_s^\infty (u-t)^{2n-2}p(u)du+\frac{C}{(2n-1)!}\int_t^s (u-t)^{2n-1}p(u)du$$
$$\le x(s)-R(s)-x(t)+R(t).$$

Since $\int_t^s u^{2n-1}p(u)du\to\infty$ as $s\to\infty$ and R is bounded, we must have $x(s)\to\infty$ as $s\to\infty$. This contradicts our assumption that x is bounded. Therefore x must be unbounded and by Lemma 1 we have $|x(t)|\to\infty$ as $t\to\infty$.

The results in [2] and [3], which are related to those given above, require that R satisfy one of the following conditions:

(I) $R(t)$ is oscillatory and $R^{(k)}(t)\to 0$ as $t\to\infty$ for $k=0, 1, 2,\dots, 2n-1$.

(II) There exist numbers λ_1, $\lambda_2>0$ and sequences $\{t_n\}$, $\{s_n\}$ such that $t_n\to\infty$ as $n\to\infty$, $s_n\to\infty$ as $n\to\infty$, $R(t_n)=\lambda_1$, $R(s_n)=-\lambda_2$, and $-\lambda_2\le R(t)\le\lambda_1$ for all t sufficiently large.

In [2] and [3] it is shown that with (I) or (II) and assumptions of f equivalent to those assumed here, the condition $\int_0^\infty t^{2n-1}p(t)dt=\infty$ is sufficient for any bounded solution of $x^{(2n)}+f=R^{(2n)}$ on an interval $[b, \infty)$ to oscillate.

Clearly any function which satisfies (I) or (II) is strongly bounded. Hence, Theorem 4 generalizes the corresponding results in [2] and [3]. Moreover, a strongly bounded function need not satisfy (I) or (II). For example, in the case $n=1$ the function $R(t)=(1+t^{-1})\sin t$ is strongly bounded, but satisfies neither (I) or (II). None of the theorems presented here are true if R is merely assumed to be bounded. For example, $x(t)=t^{-3}$ is a nonoscillatory solution of $x''+2x=2t^{-3}+12t^{-5}$. Here $f(t, x, x')=2x$ satisfies the hypotheses of the theorems and $R(t)=t^{-1}+t^{-3}$ is bounded on any interval $[a, \infty)$, $0<a$.

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