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A subspace of Schwartz space on motion groups

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§1. Introduction

In the theory of harmonic analysis on semisimple Lie groups, it is important to consider the space $\mathscr{C}^p$, $0<p\leqq 2$, which is an L^p type subspace of the Schwartz space $\mathscr{C}=\mathscr{C}^2$, and one of the most important problems at present is to determine the image of $\mathscr{C}^p$ by the Fourier transform. For example, if we consider the space $\mathscr{C}^p(X)$ on a symmetric space X, then the image of $\mathscr{C}^p(X)$ is the space of holomorphic functions in the interior of a certain tube domain of a complex space satisfying some boundedness conditions modulo representations of a compact group (see M. Eguchi [1], Theorem 4.8.1). In the present paper we consider the corresponding space to $\mathscr{C}^p$ for the motion groups.

Let K be a compact connected Lie group acting on a finite dimensional real vector space V as a linear group. Let G be the semidirect product group of V and K. We call this group the motion group. Let $\hat{V}$ be the dual space of V and $\hat{V}_c$ the complexification of $\hat{V}$. We fix a K-invariant inner product $(\ ,\ )$ of V, an orthonormal basis of V with respect to this inner product and its dual basis. We identify V and $\hat{V}$ with $\boldsymbol{R}^n$ by these bases. Let $x=(x_1,\dots, x_n)\in V$ and $\xi=(\xi_1,\dots, \xi_n)\in\hat{V}$, where $n=\dim V$. We put $|x|^2=(x, x)$. Then $|x|^2=x_1^2+\cdots+x_n^2$. We also put $|\xi|^2=\xi_1^2+\cdots+\xi_n^2$. For any $\varepsilon>0$ we define the tube domain F^ε by setting

$$F^\varepsilon = \{\zeta = \xi+i\eta\in\hat{V}+i\hat{V}=\hat{V}_c\,;\ |\eta|\leqq\varepsilon\},$$

where $i=(-1)^{1/2}$. We denote by $\mathrm{Int}\,F^\varepsilon$ the interior of F^ε. We put $F^0=\mathrm{Int}\,F^0=\hat{V}$. Then F^ε and $\mathrm{Int}\,F^\varepsilon$ are K-invariant. Let $\mathfrak{H}=L^2(K)$ be the Hilbert space of square integrable functions on K with respect to the normalized Haar measure dk. Let $\boldsymbol{B}(\mathfrak{H})$ be the Banach space of all bounded linear operators on $\mathfrak{H}$. For $\varepsilon>0$ we denote by $\mathscr{Z}(F^\varepsilon)$ the set of all $\boldsymbol{B}(\mathfrak{H})$-valued C^∞ functions T on $\hat{V}$ which satisfy the following conditions:

(i) The function T extends holomorphically to $\mathrm{Int}\,F^\varepsilon$;

(ii) for any $\alpha\in\boldsymbol{N}^n$, $\ell\in\boldsymbol{N}$ and for any right invariant differential operators y, y' on K

$$\sup_{\zeta\in\mathrm{Int}F^\varepsilon}(1+|\zeta|^2)^\ell\|yD_\zeta^\alpha T(\zeta)y'\| < \infty, \tag{1.1}$$

where $D_\zeta^\alpha=\partial^{|\alpha|}/\partial\zeta_1^{\alpha_1}\cdots\partial\zeta_n^{\alpha_n}$ $(\alpha=(\alpha_1,\dots,\alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n)$;

(iii) for all $k\in K$ and for all $\zeta\in\mathrm{Int}\,F^\varepsilon$

$$T(k\zeta) = R_k T(\zeta) R_k^{-1},$$

where R is the right regular representation of K.
Furthermore, we denote by $\mathscr{Z}(F^0)$ the set of all $\boldsymbol{B}(\mathfrak{H})$-valued C^∞ functions on $\hat{V}$ which satisfy the above conditions (ii) and (iii) for $\varepsilon=0$.

Let U^ξ be the induced representation of G by the representation $\xi \in \hat{V}$ of V: For $g=(x, k) \in G$ and $F \in \mathfrak{H}$

$$(U_g^\xi F)(k_1) = e^{i\langle \xi, k_1^{-1}x\rangle} F(k^{-1}k_1).$$

We put $dx=(2\pi)^{-n/2}dx_1 \cdots dx_n$, the Lebesgue measure on V. We can normalize the Haar measure dg on G so that $dg=dxdk$. The Fourier transform of a complex valued integrable function f on G is a $\boldsymbol{B}(\mathfrak{H})$-valued function $\hat{f}$ on $\hat{V}$ defined by

$$\hat{f}(\xi) = \int_G f(g) U_g^\xi \, dg.$$

Then $\mathscr{Z}(F^0)$ is the image of the space of rapidly decreasing functions and for any $\varepsilon>0$, $\mathscr{Z}(F^\varepsilon)$ is contained in $\mathscr{Z}(F^0)$ (cf. Lemma 1).

In §2 we define a space $\mathscr{S}_\varepsilon$. For $0<p\leqq 2$ we put $\mathscr{C}^p(G)=\mathscr{S}_{2/p-1}$. Then this space $\mathscr{C}^p(G)$ is an analogous one to the Schwartz space $\mathscr{C}^p$ for symmetric spaces. The main theorem (§3) asserts that $\mathscr{S}_\varepsilon$ and $\mathscr{Z}(F^\varepsilon)$ are topologically isomorphic by the Fourier transform. In §4 we consider the dual space of $\mathscr{S}_\varepsilon$, the space of ε-tempered distributions.

§2. The space $\mathscr{S}_\varepsilon$

Let $\mathfrak{k}$ be the Lie algebra of K. We denote by $U(\mathfrak{k}_c)$ the universal enveloping algebra of the complexification $\mathfrak{k}_c$ of $\mathfrak{k}$. We regard any element of $U(\mathfrak{k}_c)$ as a right invariant differential operator on K. We denote by λ and μ the left and the right regular representations of G, respectively, and also denote by the same symbols their differentials. Let $\mathscr{S}_\varepsilon$ be the set of all C^∞ functions f on G satisfying the following condition: For any $\alpha \in \boldsymbol{N}^n$, $\ell \in \boldsymbol{N}$ and $y, y' \in U(\mathfrak{k}_c)$

$$\sup_{(x,k)\in G} e^{\varepsilon|x|} (1+|x|^2)^\ell |(D_x^\alpha \lambda(y)\mu(y')f)(x, k)| < \infty, \tag{2.1}$$

where $D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$.

For $f \in \mathscr{S}_\varepsilon$ we denote by $\gamma_{\alpha,\ell,y,y'}^{(\varepsilon)}(f)$ the left-hand side of (2.1). And for $T \in \mathscr{Z}(F^\varepsilon)$ we denote by $\hat{\gamma}_{\alpha,\ell,y,y'}^{(\varepsilon)}(T)$ the left-hand side of (1.1). We topologize $\mathscr{S}_\varepsilon$ and $\mathscr{Z}(F^\varepsilon)$ by the system of seminorms $\{\gamma_{\alpha,\ell,y,y'}^{(\varepsilon)}\}$ and $\{\hat{\gamma}_{\alpha,\ell,y,y'}^{(\varepsilon)}\}$, respectively. Then both $\mathscr{S}_\varepsilon$ and $\mathscr{Z}(F^\varepsilon)$ are Fréchet spaces.

Let $\mathscr{D}$ be the space of all complex valued C^∞ functions on G with compact supports, having the usual topology. We denote by $\mathscr{Z}$ the Fourier image of $\mathscr{D}$.

Then by the Paley-Wiener type theorems (K. Kumahara [2], Theorems 2 and 3), $\mathscr{Z}$ is contained in $\mathscr{Z}(F^{\varepsilon})$ for all $\varepsilon \geqq 0$ and the Fourier transform gives a topological linear isomorphism of $\mathscr{S}_0$ onto $\mathscr{Z}(F^0)$. We can prove the following lemma without difficulty.

LEMMA 1. *If* $0 \leqq \varepsilon < \varepsilon'$, *then*

$$\mathscr{D} \subset \mathscr{S}_{\varepsilon'} \subset \mathscr{S}_{\varepsilon} \subset \mathscr{S}_0,$$

$$\mathscr{Z} \subset \mathscr{Z}(F^{\varepsilon'}) \subset \mathscr{Z}(F^{\varepsilon}) \subset \mathscr{Z}(F^0).$$

Let f and h be two elements of $\mathscr{S}_{\varepsilon}$. We denote by $f*h$ the convolution of f and h as usual. We put $f^*(g)=\overline{f(g^{-1})}$.

LEMMA 2. *For any* $\varepsilon \geqq 0$, $\mathscr{S}_{\varepsilon}$ *is closed under the convolution and the mapping* $f \mapsto f^*$.

PROOF. Let $f, h \in \mathscr{S}_{\varepsilon}$. By the definition of the convolution

$$(f*h)(g) = \int_G f(gg'^{-1})h(g')dg',$$

we have $\lambda(g)(f*h)=(\lambda(g)f)*h$ and $\mu(g)(f*h)=f*(\mu(g)h)$ for all $g \in G$. Hence we have $\lambda(y)\mu(y')(f*h)=(\lambda(y)f)*(\mu(y')h)$ for all $y, y' \in U(\mathfrak{k}_c)$. By the invariance of D_x^{α} under the translation of V, we have $(D_x^{\alpha}(f*h))(x, k)=((D_x^{\alpha}f)*h)(x, k)$. Here we used the rapidly decreasingness of f and h. Let $\alpha \in \boldsymbol{N}^n$, $\ell \in \boldsymbol{N}$ and $y, y' \in U(\mathfrak{k}_c)$. Then by the K-invariance of the norm $|x|$ and the inequality

$$|x+x'|^2 \leqq (1+|x|^2)(1+|x'|^2),$$

we have for any $(x, k) \in G$

$$\begin{aligned} & e^{\varepsilon|x|}(1+|x|^2)^{\ell}|(D_x^{\alpha}\lambda(y)\mu(y')f*h)(x, k)| \\ & \leqq \sum_{j=1}^{\ell}\binom{\ell}{j}\int_V\int_K e^{\varepsilon|x-kk'^{-1}x'|}(1+|x-kk'^{-1}x'|^2)^j \\ & \qquad |(D_x^{\alpha}(\lambda(y)f))(x-kk'^{-1}x', kk'^{-1})e^{\varepsilon|x'|}(1+|x'|^2)^j(\mu(y')h)(x', k')|dx'dk'. \end{aligned}$$

Hence there exists a constant $C>0$ such that

$$\gamma_{\alpha,\ell,y,y'}^{(\varepsilon)}(f*h) \leqq C\int_V (1+|x'|^2)^{-n}dx'.$$

Thus $f*h \in \mathscr{S}_{\varepsilon}$. On the other hand, $f^*(x, k)=\overline{f(-k^{-1}x, k^{-1})}$. As K acts on V as a subgroup of $SO(V)$, there exist finite differential operators D_x^{β} and a positive constant C such that

$$|(D_x^{\alpha}f^*)(x, k)| < C\sum_{\beta}|(D_x^{\beta}f)(-k^{-1}x, k^{-1})|.$$

Moreover, we have $(\lambda(g')f^*)(g)=\overline{(\mu(g')f)(g^{-1})}$ and $(\mu(g')f^*)(g)=\overline{(\lambda(g')f)(g^{-1})}$. From these facts and the K-invariance of the norm $|x|$, we have $f^*\in\mathscr{S}_\varepsilon$. q.e.d.

From Lemma 2, $\mathscr{S}_\varepsilon$ is a topological $*$-algebra. In fact, if we reread the proof of Lemma 2, we can see that the convolution and the involution * are continuous.

§3. The main theorem

THEOREM. *For any $\varepsilon\geqq 0$, the Fourier transform gives a topological linear isomorphism of $\mathscr{S}_\varepsilon$ onto $\mathscr{Z}(F^\varepsilon)$.*

PROOF. Since $\mathscr{S}_\varepsilon$ and $\mathscr{Z}(F^\varepsilon)$ are Fréchet spaces, it is sufficient to prove that the Fourier transform gives a continuous bijection between $\mathscr{S}_\varepsilon$ and $\mathscr{Z}(F^\varepsilon)$. On the other hand, we know that the Fourier transform gives a topological isomorphism of $\mathscr{S}_0$ onto $\mathscr{Z}(F^0)$ (see [2], Theorem 3) and that $\mathscr{S}_\varepsilon$ and $\mathscr{Z}(F^\varepsilon)$ are contained in $\mathscr{S}_0$ and $\mathscr{Z}(F^0)$, respectively. Hence it is sufficient to prove that the Fourier transform gives a continuous surjection of $\mathscr{S}_\varepsilon$ to $\mathscr{Z}(F^\varepsilon)$ for $\varepsilon>0$.

Let f be an element of $\mathscr{S}_\varepsilon$. Then the function $\hat{f}$ on V defined by

$$\hat{f}(\xi)=\int_G f(g)U_g^\xi\,dg$$

is C^∞ ([2], Theorem 3). For $\zeta\in\operatorname{Int}F^\varepsilon$ we put

$$T(\zeta)=\int_G f(g)U_g^\zeta\,dg,$$

that is, it is an operator on $\mathfrak{H}$ defined as follows: For $F\in\mathfrak{H}$

$$(T(\zeta)F)(k_1)=\int_V\int_K f(x,k)\,e^{i\langle\zeta,k_1^{-1}x\rangle}F(k^{-1}k_1)dxdk.$$

Since $\zeta\in\operatorname{Int}F^\varepsilon$, $|\operatorname{Im}\zeta|<\varepsilon$ and $e^{-\langle\operatorname{Im}\zeta,x\rangle}\leqq e^{\varepsilon|x|}$ for all $x\in V$. We have, therefore,

$$\|T(\zeta)\|^2\leqq\int_K\left\{\int_V|f(x,k)|e^{\varepsilon|x|}\,dx\right\}^2dk.$$

There is a constant $C>0$ such that

$$e^{\varepsilon|x|}(1+|x|^2)^n|f(x,k)|\leqq C$$

for all $k\in K$ and $x\in V$. Then

$$\|T(\zeta)\|\leqq C\int_V(1+|x|^2)^{-n}dx<\infty.$$

Hence $T(\zeta)\in B(\mathfrak{H})$.

We next see the holomorphy of $T(\zeta)$ in the tube domain. For any $\alpha=(\alpha_1,\dots,\alpha_n)\in \boldsymbol{N}^n$

$$|D_\zeta^\alpha e^{i\langle\zeta,x\rangle}| \leqq |i^{|\alpha|}x_1^{\alpha_1}\cdots x_n^{\alpha_n}|e^{-\langle \mathrm{Im}\zeta,x\rangle}$$
$$\leqq (1+|x|^2)^{|\alpha|}e^{-\langle \mathrm{Im}\zeta,x\rangle}$$

The integral

$$\int_V\int_K f(k_1x,k)D_\zeta^\alpha e^{i\langle\zeta,x\rangle}F(k^{-1}k_1)dxdk, \tag{3.1}$$

therefore, converges absolutely and uniformly in $\mathrm{Int}\,F^\varepsilon$. Hence for any $F\in\mathfrak{H}$, $T(\zeta)F$ is infinitely differentiable and $D_\zeta^\alpha(T(\zeta)F)$ equals to (3.1). For any fixed j, $1\leqq j\leqq n$, and fixed $\zeta_1,\dots,\zeta_{j-1},\zeta_{j+1},\dots,\zeta_n\in\boldsymbol{C}$, we regard $T(\zeta)$ as a function of ζ_j and denote it by $T_j(\zeta_j)$. Then for $\zeta=(\zeta_1,\dots,\zeta_n)\in\mathrm{Int}\,F^\varepsilon$ and for $t\in\boldsymbol{C}$ such that $(\zeta_1,\dots,\zeta_{j-1},\zeta_j+t,\zeta_{j+1},\dots,\zeta_n)\in\mathrm{Int}\,F^\varepsilon$, we have

$$\|\{T_j(\zeta_j+t)-T_j(\zeta_j)\}/t-dT_j(\zeta_j)/d\zeta_j\|^2$$
$$\leqq \int_K\int_K\left\{\int_V |f(k_1x,k)|e^{-\langle \mathrm{Im}\zeta,x\rangle}|(e^{itx_j}-1)/t-ix_j|dx\right\}^2 dkdk_1.$$

We choose t so that $0<|t|<\varepsilon-|\mathrm{Im}\,\zeta|$. Then

$$|(e^{itx_j}-1)/t-ix_j|e^{-\langle \mathrm{Im}\zeta,x\rangle}\leqq e^{(|t|+|\mathrm{Im}\zeta|)|x|}<e^{\varepsilon|x|}.$$

Hence by the condition (2.1) of f and by Lebesgue's convergence theorem, $T(\zeta)$ is differentiable in the norm of $\boldsymbol{B}(\mathfrak{H})$ and $\partial(T(\zeta)F)/\partial\zeta_j=(\partial T(\zeta)/\partial\zeta_j)F$ for all $F\in\mathfrak{H}$. By repetition of the same arguments we have that $T(\zeta)$ is infinitely differentiable and $D_\zeta^\alpha(T(\zeta)F)=(D_\zeta^\alpha T(\zeta))F$. Hence $T(\zeta)$ is a holomorphic extension of $\hat{f}$ to $\mathrm{Int}\,F^\varepsilon$.

We next prove the continuity of the Fourier transform. For any $\alpha\in\boldsymbol{N}^n$, $\ell\in\boldsymbol{N}$ and $y,y'\in U(\mathfrak{k}_c)$ we can find, by some simple computations, $\alpha^{(1)},\dots,\alpha^{(\nu)}\in\boldsymbol{N}^n$, $\ell^{(1)},\dots,\ell^{(\nu)}\in\boldsymbol{N}$, $y^{(1)},\dots,y^{(\nu)},y'^{(1)},\dots,y'^{(\nu)}\in U(\mathfrak{k}_c)$ and positive constants $C^{(1)},\dots,C^{(\nu)}$ such that

$$\hat{\gamma}^{(\varepsilon)}_{\alpha,\ell,y,y'}(T)^2$$
$$\leqq \textstyle\sum_{j=1}^{\nu} C^{(j)}\displaystyle\int_K\int_K\left\{\int_V e^{\varepsilon|x|}(1+|x|^2)^{\ell^{(j)}}|D_x^{\alpha^{(j)}}\lambda(y^{(j)})\mu(y'^{(j)})f(k'x,k)|dx\right\}^2 dkdk'.$$

Since for every $x\in V$ and $k\in K$

$$e^{\varepsilon|x|}(1+|x|^2)^{\ell^{(j)}}|D_x^{\alpha^{(j)}}\lambda(y^{(j)})\mu(y'^{(j)})f(x,k)|$$
$$\leqq \gamma^{(\varepsilon)}_{\alpha^{(j)},\ell^{(j)}+n,y^{(j)},y'^{(j)}}(f)(1+|x|^2)^{-n},$$

we have

$$\hat{\gamma}^{(\varepsilon)}_{\alpha,\ell,y,y'}(T) \leqq \sum_{j=1}^{\nu} C^{(j)} \left\{\int_V (1+|x|^2)^{-n} dx\right\} \gamma^{(\varepsilon)}_{\alpha^{(j)},\ell^{(j)}+n,y^{(j)},y'^{(j)}}(f).$$

The relation $T(k\zeta)=R_k T(\zeta) R_k^{-1}$ can be easily checked. Thus T is a holomorphic extension of $\hat{f}$ to Int F^ε satisfying the conditions (ii) and (iii) in the definition of $\mathscr{Z}(F^\varepsilon)$. Hence $\hat{f} \in \mathscr{Z}(F^\varepsilon)$. And we have proved that the Fourier transform is continuous.

Conversely, let us assume $T \in \mathscr{Z}(F^\varepsilon)$. Then we know that the function f on G defined by

$$f(g) = \int_{\hat{V}} \operatorname{Tr}(T(\xi) U^{\xi}_{g^{-1}}) d\xi$$

is an element of $\mathscr{S}_0$ and that $\hat{f}=T$ (see [2], Theorem 3), where $d\xi=(2\pi)^{-n/2} d\xi_1 \cdots d\xi_n$. Let $\{\phi_j\}_{j\in J}$ be the complete orthonormal basis of $\mathfrak{H}$ chosen in [2], § 3. Then by the conditions in the definition of $\mathscr{Z}(F^\varepsilon)$ and by Theorem 1 of [2], $T(\zeta)$ ($\zeta \in \operatorname{Int} F^\varepsilon$) has a C^∞ kernel function $\kappa(\zeta; k_1, k_2)$:

$$\kappa(\zeta; k_1, k_2) = \sum_{i,j\in J} (T(\zeta)\phi_j, \phi_i) \phi_i(k_1) \overline{\phi_j(k_2)}, \tag{3.2}$$

and

$$(T(\zeta)F)(k_1) = \int_K \kappa(\zeta; k_1, k_2) F(k_2) dk_2, \quad (F \in \mathfrak{H}).$$

Moreover, the series (3.2) converges absolutely and uniformly on $\operatorname{Int} F^\varepsilon \times K \times K$. If we adopt the similar computations in § 3 of [2] to $(1+|\zeta|^2)^\ell y D^\alpha_\zeta T(\zeta) y'$, we can prove that there exists a constant $C_{\alpha,\ell,y,y'}$ such that

$$|(1+|\zeta|^2)^\ell (D^\alpha_\zeta y_{k_1} y'_{k_2} \kappa)(\zeta; k_1, k_2)| \leqq C_{\alpha,\ell,y,y'}$$

for every $\zeta \in \operatorname{Int} F^\varepsilon$ and $k_1, k_2 \in K$, where y_{k_j} ($y \in U(\mathfrak{k}_c)$, $j=1, 2$) denotes differentiation of κ by y with respect to k_j. And the relation $T(k\zeta)=R_k T(\zeta) R_k^{-1}$ corresponds to the relation $\kappa(k\zeta; k_1, k_2)=\kappa(\zeta; k_1 k, k_2 k)$. The function $f(g)$ can be represented by means of κ:

$$f(x, k) = \int_{\hat{V}} \kappa(\xi; 1, k^{-1}) e^{-i\langle \xi, x\rangle} d\xi.$$

Then for any $\alpha, \beta \in \boldsymbol{N}^n$ and $y, y' \in U(\mathfrak{k}_c)$, $x^\beta (D^\alpha_x \lambda(y)\mu(y')f)(x, k)$ is a linear combination of integrals of the form

$$\int_{\hat{V}} \xi^{\tilde{\beta}} (D^{\tilde{\alpha}}_\xi \tilde{y}_{k_1} \tilde{y}'_{k_2} \kappa)(\xi; 1, k^{-1}) e^{-i\langle \xi, x\rangle} d\xi,$$

where $\tilde{\alpha}, \tilde{\beta} \in \boldsymbol{N}^n$ and $\tilde{y}, \tilde{y}' \in U(\mathfrak{k}_c)$ and $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$, $\xi^\beta = \xi_1^{\beta_1} \cdots \xi_n^{\beta_n}$. We fix $(x, k) \in G$. Now we put for $\zeta \in \operatorname{Int} F^\varepsilon$

$$\Phi(\zeta) = \zeta^{\beta}(D_{\zeta}^{\tilde{\alpha}}\tilde{y}_{k_1}\tilde{y}'_{k_2}\kappa)(\zeta;\, 1,\, k^{-1})\,.$$

Then $\Phi(\zeta)$ is holomorphic in $\operatorname{Int} F^{\varepsilon}$ and it is rapidly decreasing when $\operatorname{Re}\zeta \to \infty$. Let δ be any real number such that $0<\delta<\varepsilon$. We assume that $x\neq 0$ and put $\eta = -\delta x/|x|$. Then $\xi + i\eta \in \operatorname{Int} F^{\varepsilon}$. Shifting the path of integral, we get

$$\int_{V} \Phi(\xi)\, e^{-i\langle \xi, x\rangle}\, d\xi = \int_{V} \Phi(\xi + i\eta)\, e^{-i\langle \xi + i\eta, x\rangle}\, d\xi.$$

As we can choose a constant C depending on $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{y}$, $\tilde{y}'$ but independent of η and k so that

$$|\Phi(\xi + i\eta)| \leqq C(1 + |\xi|^2)^{-n},$$

we can find a constsnt C' depending on α, ℓ, y and y' but independent of η, k and x such that

$$e^{\varepsilon|x|}(1 + |x|^2)|^{\ell}(D_x^{\alpha}\lambda(y)\mu(y')f)(x,\, k)| \leqq C' e^{\varepsilon|x| + \langle \eta, x\rangle}.$$

Here $\varepsilon|x| + \langle \eta, x\rangle = (\varepsilon - \delta)|x|$. Let δ tend to ε. Then the left-hand side is dominated by C' which is independent of x and k. Hence we have

$$\gamma^{(\varepsilon)}_{\alpha, \ell, y, y'}(f) \leqq C'.$$

Therefore, $f \in \mathscr{S}_{\varepsilon}$. This completes the proof of the theorem. q.e.d.

§4. ε-tempered distributions

Let $\varepsilon>0$. A distribution on G is said to be ε-tempered if it extends to a continuous linear functional on $\mathscr{S}_{\varepsilon}$. It is not difficult to see that $\mathscr{D}$ is dense in $\mathscr{S}_{\varepsilon}$ and that the inclusion mapping of $\mathscr{D}$ to $\mathscr{S}_{\varepsilon}$ is continuous. Hence we can regard the space of ε-tempered distributions as the space of continuous linear functionals on $\mathscr{S}_{\varepsilon}$. Let $\mathscr{S}'_{\varepsilon}$ and $\mathscr{Z}(F^{\varepsilon})'$ be the set of all continuous linear functionals on $\mathscr{S}_{\varepsilon}$ and $\mathscr{Z}(F^{\varepsilon})$, respectively. They become locally convex linear topological spaces when equipped with the weak topology.

Let $\mathscr{F}^{*}$ be the transpose of the Fourier transform of $\mathscr{S}_{\varepsilon}$ onto $\mathscr{Z}(F^{\varepsilon})$. Then we have the following proposition as a corollary of the main theorem.

PROPOSITION. *$(\mathscr{F}^{*})^{-1}$ is a topological linear isomorphism of $\mathscr{S}'_{\varepsilon}$ onto $\mathscr{Z}(F^{\varepsilon})'$.*

References

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