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Z-transforms and overrings of a noetherian ring

Shiroh ITOH

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Introduction

Since Nagata had pointed out the importance of the notion of ideal transforms in relation to the 14-th problem of Hilbert, ideal transforms have been studied by many authors. The notion of Z-transforms of a ring A, Z being a subset of Spec(A) which is stable under specialization, is a generalized one of ideal transforms. We can use ideal or Z-transforms as a powerful tool to study overrings B of a noetherian ring A. This is done as follows. Take a suitable chain $Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_0 = \mathrm{Spec}(A)$ of subsets of Spec(A) and consider the overrings $T(Z_i, A) \cap B$ where $T(Z_i, A)$ is the Z_i-transform of A. Then by examining properties of $T(Z_i, A) \cap B$ inductively, we get the knowledge of properties of B. K. Yoshida, in [22], used this technic and showed some properties of overrings B are determined by local properties at prime ideals in $\mathrm{Ass}_A(B/A)$. But the essential point of this technic is that we can reduce a problem on B to a problem on $(A_\mathfrak{p})^g \cap B_\mathfrak{p}$, $\mathfrak{p} \in \mathrm{Ass}_A(B/A)$, where $(A_\mathfrak{p})^g$ is the global transform of $A_\mathfrak{p}$. This motivation follows from two facts: The first one is a characterization of $\mathrm{Ass}_A(B/A)$, i.e. $\mathrm{Ass}_A(B/A) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_\mathfrak{p} \subset (A_\mathfrak{p})^g \cap B_\mathfrak{p}\}$ (Theorem (2.5)). On the other hand, roughly speaking, the difference between $T(Z_i, A) \cap B$ and $T(Z_{i-1}, A) \cap B$ appears in prime ideals belonging to $Z_{i-1} - Z_i$, and if $Z_{i-1} - Z_i$ is discrete, then $(T(Z_i, A) \cap B)_\mathfrak{p} = A_\mathfrak{p}$ and $(T(Z_{i-1}, A) \cap B)_\mathfrak{p} = (A_\mathfrak{p})^g \cap B_\mathfrak{p}$ for every $\mathfrak{p} \in Z_{i-1} - Z_i$. This is the second fact which we wish to point out. In this paper we shall study overrings of a noetherian ring from the above point of view.

Section 1 consists of preliminary results on Z-transforms and global transforms almost all of which are already known (cf. [1], [6], [9], [12], [13], [14] and [15]). We shall frequently use these results in this paper. In section 2, we shall give basic relations between $\mathrm{Ass}_A(B/A)$ and Z-transforms. We remark here that we shall obtain whole results in this section, especially Corollary (2.12), without using completions and the theorem of Mori-Nagata. Corollary (2.12) is a modified form of Theorem (1.6) in [14], and using this corollary we shall give an alternative proof of the theorem of Mori-Nagata in appendix (see [17] for another proof of this theorem by means of global transforms).

In some cases we can prove some known facts in a unified way by means of Z-transforms. In fact, in section 3, we shall generalize J. Nishimura's results [15, (2.6), (3.1) and (3.2)] (see Theorem (3.1)), and in the last part of section 5 we shall give a unified proof of two basic facts concerning seminormal rings. In section 4, we shall treat finite (S_2)-overrings of a noetherian ring. In Theorem (4.7) we shall give a necessary and sufficient condition for a noetherian ring to have a finite (S_2)-overring. We shall also study how M. Brodmann's result in [2], which gives a sufficient condition for existence of finite (S_2)-overrings, can be deduced from our theorem. As we have already known, for a finite overring B of a noetherian ring A, A is seminormal in B if and only if $A_\mathfrak{p}$ is seminormal in $B_\mathfrak{p}$ for every $\mathfrak{p}\in \mathrm{Ass}_A(B/A)$. In section 5, we shall sharpen this result in terms of global transforms of $A_\mathfrak{p}$, $\mathfrak{p}\in \mathrm{Ass}_A(B/A)$.

Notation and terminology

In this paper, we mean by a ring a commutative ring with identity. Let A be a ring. We denote by $Q(A)$ the total quotient ring of A, and denote by $\bar{A}$ the integral closure of A in $Q(A)$. $\mathrm{Max}(A)$ (resp. $\mathrm{Min}(A)$) will denote the set of all maximal ideals (resp. minimal prime ideals) of A. Let I be an ideal of A. Then $V(I)$ is the set of all prime ideals $\mathfrak{p}$ of A with $I\subseteq\mathfrak{p}$. For a prime ideal $\mathfrak{p}$ of A, $k(\mathfrak{p})=A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.

Let M be an A-module. We say that an ideal I of A is M-regular (resp. regular) if I contains M-regular elements (resp. regular elements) of A. $Q_A(M)$ (or simply $Q(M)$) will denote the A-module $S^{-1}M$, where S is the set of all M-regular elements of A. By definition $Q(A)=Q_A(A)$. $\mathrm{Ass}_A(M)$ will denote the set $\{\mathfrak{p}\in \mathrm{Spec}(A)\mid \mathfrak{p}$ is a minimal prime ideal of $0:_A x$ for some $x\in M\}$. Therefore if A is noetherian, then $\mathrm{Ass}_A(M)=\{\mathfrak{p}\in\mathrm{Spec}(A)\mid \mathfrak{p}=0:_A x$ for some $x\in M\}$ as usual.

We say that R is an overring of a ring A if R is an A-subalgebra of $Q(A)$ (i.e., $A\subseteq R\subseteq Q(A)$). If a noetherian overring R of A satisfies Serre's property (S_2), then we say that R is an (S_2)-overring of A.

For a finitely generated regular ideal I of A, we frequently identify $\mathrm{End}_A(I)$ $(=\mathrm{Hom}_A(I, I))$ with $I:_{Q(A)}I$.

Let B be an A-algebra, $f\colon A\to B$ the corresponding homomorphism. For an ideal I of B, we write $A\cap I$ instead of $f^{-1}(I)$.

§1. Definitions and preliminaries

Let A be a ring. A *topology* on A is a family F of ideals of A with the following properties: (a) *if* $I\in F$, *then* $J\in F$ *for every ideal* J *of* A *with* $I\subseteq J$, and (b) *if* $I, J\in F$, *then* $IJ\in F$. Let F be a topology on A. For an A-module M, $F_{\mathrm{reg}(M)}$ will denote the set of all M-regular ideals I of A with $I\in F$.

We shall first summarize some elementary results on F-transform which are mostly well known.

DEFINITION (1.1) Let A be a ring, and let F be a topology on A. The *F-transform of an A-module M* is defined to be the set

$$T(F, M) = \{z \in Q_A(M) \mid M:_A z \in F\}.$$

$T(F, M)$ is also an A-module such that $M \subseteq T(F, M) \subseteq Q_A(M)$. If B is an A-algebra, then $T(F, B)$ is a B-subalgebra of $Q_A(B)$. Let I be an ideal of A. If F is the set of all ideals of A which contain I^n for some $n \geqq 0$, then $T(F, A)$ is the usual I-transform of A (cf. [13]).

Since $M:_A z$ contains M-regular elements for every $z \in Q_A(M)$, we have $T(F, M) = T(F_{\mathrm{reg}(M)}, M)$. Moreover if F' is another topology on A such that $F \subseteq F'$, then clearly $T(F, M) \subseteq T(F', M)$.

(1.2) *Let F be a topology on a ring A, and let M be an A-module. Then we have*

(1) *$T(F, M)_{\mathfrak{p}} = M_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \notin F_{\mathrm{reg}(M)}$.*

Assume further that $F_{\mathrm{reg}(M)}$ has a cofinal subfamily consisting of finitely generated ideals. Then we have

(2) *$T(F, N) = T(F, M)$ for every A-module N such that $M \subseteq N \subseteq T(F, M)$. In particular $T(F, T(F, M)) = T(F, M)$.*

Let B be an A-algebra, and let F be a topology on A. We denote by FB the set of all ideals J of B such that $J \supseteq IB$ for some $I \in F$. Note that FB is a topology on B. If $F = F_{\mathrm{reg}(B)}$, then $T(FB, B) \subseteq Q_A(B)$. Therefore we have the following assertion:

(1.3) *Let B be an A-algebra, and let F be a topology on A. Assume that $F = F_{\mathrm{reg}(B)}$. Then $T(F, B) = T(FB, B)$.*

The following assertion is an easy generalization of [13, Lemma 2.6].

(1.4) *Let F be a topology on a ring A, and let M be an A-module. Assume that $F = F_{\mathrm{reg}(M)}$ and F has a cofinal subfamily consisting of finitely generated ideals. Then for every flat A-algebra B, $T(F, M) \otimes_A B = T(FB, M \otimes_A B)$.*

As an immediate corollary to (1.4), $Q_A(M)_{\mathfrak{p}} \subseteq Q_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for every A-module M and $\mathfrak{p} \in \mathrm{Spec}(A)$.

Let A be a ring, and let I be an ideal of A. Let M be an A-module. We denote by $\mathrm{Gr}_A(I, M)$ the polynomial grade of I on M (as for the polynomial grade, we refer to [16]). If A is noetherian and M is finitely generated, then $\mathrm{Gr}_A(I, M)$ is equal to the usual depth of I on M. Let $\mathfrak{p}$ be a prime ideal of A. Note that $\mathrm{Gr}_A(I, M) = 0$ (resp. $\mathrm{Gr}_{A_{\mathfrak{p}}}(\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$) if and only if, for each finitely generated ideal J with $J \subseteq I$ (resp. $J \subseteq \mathfrak{p}$), there exists an element $x (\neq 0)$ of M such that $Jx = 0$ (resp. $J \subseteq \mathrm{Ann}_A(x) \subseteq \mathfrak{p}$) (cf. [16, Chap. 5, Lemma 8]).

(1.5) *Let F be a topology on A, and let M be an A-module. Consider the following conditions on M and F:*

(1) $T(F, M)=M$.

(2) $\mathrm{Gr}_A(I, M)\geqq 2$ *for all* $I\in F_{\mathrm{reg}(M)}$.

(3) $\mathrm{Gr}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p})\geqq 2$ *for all prime ideals* $\mathfrak{p}$ *of* A *with* $\mathfrak{p}\in F_{\mathrm{reg}(M)}$.

Then we have (2)$\Rightarrow$(3)$\Rightarrow$(1). *If $F_{\mathrm{reg}(M)}$ has a cofinal subfamily consisting of finitely generated ideals, then* (1)$\Rightarrow$(2).

PROOF. (2)$\Rightarrow$(3) is clear (cf. [16, Chap. 5, Exercise 10]). (3)$\Rightarrow$(1): Suppose contrarily that $T(F, M)\neq M$, and let z be an element of $T(F, M)-M$. We then put $z=x/s$ where $x\in M$ and s is an M-regular element of A. Let $\mathfrak{p}$ be a minimal prime ideal of $M:_A z=sM:_A x$. Then $\mathfrak{p}\in F_{\mathrm{reg}(M)}$, $\mathrm{Gr}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}, (M/sM)_\mathfrak{p})=0$; and hence $\mathrm{Gr}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p})=1$. This is a contradiction. We now assume that $F_{\mathrm{reg}(M)}$ has a cofinal subfamily consisting of finitely generated ideals, and we shall prove (1)$\Rightarrow$(2). Suppose contrarily that there exists an M-regular ideal I of A such that $I\in F$ and $\mathrm{Gr}_A(I, M)=1$. Let s be an M-regular element in I, and let J be a finitely generated ideal of A such that $J\subseteq I$ and $J\in F_{\mathrm{reg}(M)}$. Since $\mathrm{Gr}_A(I, M/sM)=0$, $Jx\subseteq sM$ for some $x\in M-sM$. Then $x/s\in T(F, M)=M$ because $J\in F$; hence $x\in sM$. This is a contradication. This completes the proof.

Let A be a ring, and let Z be a subset of $\mathrm{Spec}(A)$ which is stable under specialization. We denote by $F(Z)$ the set of all ideals I of A such that $V(I)\subseteq Z$. Then $F(Z)$ is a topology on A. For an A-module M, $Z_{\mathrm{reg}(M)}$ will denote the set of all M-regular prime ideals $\mathfrak{p}$ of A with $\mathfrak{p}\in Z$.

DEFINITION (1.6) The *Z-transform of an A-module M* is the $F(Z)$-transform of M, and we denote it by $T(Z, M)$.

(1.7) (cf. (1.2)) *Let A, Z be the same as above, and let M be an A-module. Then we have the following assertions.*

(1) $T(Z, M)=T(Z_{\mathrm{reg}(M)}, M)=T(F(Z)_{\mathrm{reg}(M)}, M)$.

(2) *For an element z of $Q_A(M)$, $z\in T(Z, M)$ if and only if $z/1\in M_\mathfrak{p}$ for every $\mathfrak{p}\in\mathrm{Spec}(A)-Z$. In particular $T(Z, M)_\mathfrak{p}=M_\mathfrak{p}$ for every $\mathfrak{p}\in\mathrm{Spec}(A)-Z_{\mathrm{reg}(M)}$.*

(3) *$T(Z, N)=T(Z, M)$ for every A-module N such that $M\subseteq N\subseteq T(Z, M)$. In particular $T(Z, T(Z, M))=T(Z, M)$.*

PROOF. The assertions (1) and (2) are obvious. (3): By (1) above, $N_\mathfrak{p}=M_\mathfrak{p}$ for every $\mathfrak{p}\in\mathrm{Spec}(A)-Z$. Therefore the assertion follows from (2).

(1.8) *Let B be an A-algebra, and let Z be a subset of* $\mathrm{Spec}(A)$ *which is stable under specialization. We put $Z'=\{Q\in\mathrm{Spec}(B)\mid Q\cap A\in Z\}$.* Then $F(Z)B=F(Z')$.

PROOF. Let J be an ideal of B such that $J\neq B$ and $V(J)\subseteq Z'$, and let $\mathfrak{p}$ be a minimal prime ideal of $J \cap A$. Since $J \cap (A-\mathfrak{p})=\phi$, $J_{\mathfrak{p}} \neq B_{\mathfrak{p}}$. Therefore there exists a prime ideal P of B such that $J_{\mathfrak{p}} \subseteq P_{\mathfrak{p}} \subset B_{\mathfrak{p}}$. Then $P \in Z'$ and $P \cap A=\mathfrak{p}$; hence $\mathfrak{p} \in Z$. This shows that $F(Z') \subseteq F(Z)B$. The assertion now follows because the opposite inclusion clearly holds.

By virtue of (1.3), (1.7) and (1.8), we have the following assertion:

(1.9) *Let A, B, Z and Z' be the same as in* (1.8). *Assume further that* $F(Z')_{\mathrm{reg}(B)}=F(Z)_{\mathrm{reg}(B)}B$ (*e.g., B is an overring of A*). *Then* $T(Z, B)=T(Z', B)$.

Let A be a ring, and let Z be a subset of $\mathrm{Spec}(A)$ which is stable under specialization. For a prime ideal $\mathfrak{p}$ of A, $Z_{\mathfrak{p}}$ will denote the set $\{\mathfrak{q}A_{\mathfrak{p}} \mid \mathfrak{q} \in Z$ and $\mathfrak{q} \subseteq \mathfrak{p}\}$. We say that an element $\mathfrak{p}$ of Z is a generic point of Z if $\mathfrak{q} \notin Z$ for any prime ideal $\mathfrak{q}$ of A such that $\mathfrak{q} \subset \mathfrak{p}$. Z_{gen} will denote the set of all generic points of Z. For an ideal I of A, $T(V(I), *)$ is denoted simply by $T(I, *)$. If I is finitely generated, then $T(I, A)$ is the I-transform of A in the sense of Nagata (cf. [13]).

(1.10) *Let M be an A-module, and let Z be a subset of* $\mathrm{Spec}(A)$ *which is stable under specialization. Assume that $F(Z_{\mathrm{reg}(M)})$ has a cofinal subfamily consisting of finitely generated, M-regular ideals. Then we have the following assertions.*

(1) *If $Z=Z_{\mathrm{reg}(M)}$, then $T(Z, M)_{\mathfrak{p}}=T(Z_{\mathfrak{p}}, M_{\mathfrak{p}})$ for every $\mathfrak{p} \in Z$.*

(2) *If $\mathfrak{p} \in Z_{\mathrm{reg}(M)}$, then $T(\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}}) \subseteq T(Z, M)_{\mathfrak{p}}$.*

(3) *If $\mathfrak{p} \in (Z_{\mathrm{reg}(M)})_{\mathrm{gen}}$, then $T(Z, M)_{\mathfrak{p}}=T(\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}})$.*

PROOF. Since $T(Z, M)=T(Z_{\mathrm{reg}(M)}, M)$, the assertions (2) and (3) follow from (1). And (1) follows from (1.4) and (1.8).

Let A be a ring. The global transform of A is the $\mathrm{Max}(A)$-transform of A, and we denote it by A^g. The following results, due to Matijevic, are essential in the study of global transforms of noetherian rings.

(1.11) ([9, Theorem and Corollary]) *Let A be a noetherian ring, and let B be an A-algebra contained in A^g. Then we have the following assertions.*

(1) *B/xB is a finite A-module for each regular element x of A. In particular every regular ideal of B is finitely generated.*

(2) *If A is reduced, then B is noetherian.*

(1.12) *Let A and B be the same as in* (1.11). *Then* $B^g=A^g$.

PROOF. By (1.11), a regular prime ideal P of B lies over a maximal ideal of A if and only if P is maximal. Therefore by (1.7) (1) and (1.9), $B^g=T(\mathrm{Max}(A), B)$. Then it follows from (1.7)(3) that $T(\mathrm{Max}(A), B)=A^g$. This completes the proof.

Remark (1.13) Let t be a regular element of a ring A. Then t is invertible in A^g if and only if every prime ideal of A which contains t is a maximal ideal (of height one if A is noetherian). In particular $A^g = Q(A)$ whenever $\dim A = 1$.

As an immediate consequence of (11.1)(2), B^g is finite over A^g whenever B is a finite overring of a noetherian domain A ([14, Lemma (1.4)]). More generally we have the following assertion.

(1.14) *Let B be a finite overring of a noetherian ring A. Then for every overring R of B, $B^g \cap R$ is finite over $A^g \cap R$.*

Proof. Let t be a regular element of A such that $tB \subseteq A$. Then it is easy to see that $tB^g \subseteq A^g$; hence $I = t(B^g \cap R)$ is an ideal of both $A^g \cap R$ and $B^g \cap R$. Therefore it is sufficient to show that $B^g \cap R/I$ is a finite A-module. Since I contains the B-regular element t, it follows from (1.11) that $B^g \cap R/I$ is a finite B-module; hence it is a finite A-module. This completes the proof.

Let A be a noetherian ring, and let M be an A-module. We put $X = \operatorname{Spec}(A)$, and we denote by $\tilde{M}$ a quasi-coherent $\mathcal{O}_X$-module associated to M. Then for a subset Z of $\operatorname{Spec}(A)$ which is stable under specialization with $Z = Z_{\mathrm{reg}(M)}$, $T(Z, M)$ is canonically isomorphic to $\Gamma(X, \mathcal{H}^0_{X/Z}(\tilde{M}))$ (cf. [6, (5.9)]). In particular if A is local and $\operatorname{depth} A = 1$, then $A^g \cong \Gamma(X - \{\mathfrak{m}\}, \mathcal{O}_X)$ where $\mathfrak{m}$ is the maximal ideal of A. Therefore we have the following assertion.

(1.15) ([6, (5.11.1)]) *Let A be a noetherian local ring with* $\operatorname{depth} A = 1$. *Consider the following conditions on A:*

(1) *A^g is a finite A-module.*

(2) *$(A/\mathfrak{p})^g$ is a finite $A/\mathfrak{p}$-module for every $\mathfrak{p} \in \operatorname{Ass}_A(A)$.*

(3) $\dim A/\mathfrak{p} \geqq 2$ *for every* $\mathfrak{p} \in \operatorname{Ass}_A(A)$.

Then we have (1)$\Leftrightarrow$(2)$\Rightarrow$(3).

(1.16) (cf. [6, (5.11.1) and (7.2.2)] and [4, Proposition 1.11]) *Let A be a residue ring of a Cohen-Macaulay local ring such that* $\operatorname{depth} A = 1$. *Then A^g is a finite A-module if and only if* $\dim A/\mathfrak{p} \geqq 2$ *for every* $\mathfrak{p} \in \operatorname{Ass}_A(A)$.

Proof. Assume that A is a domain such that $\dim A \geqq 2$. Choose a Cohen-Macaulay local ring R and a prime ideal P of R so that $A = R/P$. Since R is Cohen-Macaulay, there exists a regular sequence $x_1, \dots, x_r$ in P where $r = \operatorname{ht}(P)$. We put $B = R/(x_1, \dots, x_r)R$. Then $\operatorname{depth} B = \dim R - r \geqq 2$ and $P \in \operatorname{Ass}_R(B)$. Since $B^g = B$, A^g is a finite A-module by (1.15). Therefore again by (1.15), we have the desired conclusion.

(1.17) *Let A be a noetherian local ring with the maximal ideal $\mathfrak{m}$. Assume* *that* $\dim A \geqq 2$ *and* $\operatorname{depth} A=1$. *Let* $(0)=\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$ *be an irredundant primary decomposition of* (0) *in* A. *Let* $\mathfrak{p}_i=\sqrt{\mathfrak{q}_i}\,(i=1,\dots, n)$. *If* $\mathfrak{p}_1$ *is a minimal prime ideal of* A *such that* $\dim A/\mathfrak{p}_1=1$, *then* $A^g \cong A_{\mathfrak{p}_1} \times (A/\mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n)^g$.

Proof. Note that $\{\mathfrak{p}_1\}$ is an open and closed subset of $U=\operatorname{Spec}(A)-\{\mathfrak{m}\}$, and that $U-\{\mathfrak{p}_1\}=\operatorname{Spec}(A/\mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n)-\{\mathfrak{m}/\mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n\}$. Since depth $A/\mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n \geqq 1$, $A^g \cong \Gamma(U, \mathcal{O}_U)=\Gamma(\{\mathfrak{p}_1\}, \mathcal{O}_U) \times \Gamma(U-\{\mathfrak{p}_1\}, \mathcal{O}_U) \cong A_{\mathfrak{p}_1} \times (A/\mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n)^g$.

§2. $\mathrm{Ass}_A(B/A)$ and Z-transforms

We shall first make some general remarks on Ass.

Lemma (2.1) *Let* A *be a noetherian ring, and let* M *be an* A*-module. Let* N *be an* A*-submodule of* $Q_A(M)$ *containing* M. *Then we have the following assertions.*

(1) $\mathrm{Ass}_A(N/M) \subseteq \mathrm{Ass}_A(Q_A(M)/M)$.

(2) *For every* $\mathfrak{p} \in \mathrm{Ass}_A(Q_A(M)/M)$, $\mathfrak{p}$ *contains* M*-regular elements.*

(3) $\mathrm{Ass}_A(Q_A(M)/M) \cap V(sA)=\mathrm{Ass}_A(M/sM)$ *whenever* s *is an* M*-regular element.*

Proof. The assertion (1) is clear. The assertion (2) is also clear because $M:_A z$ contains M-regular elements for every $z \in Q_A(M)$. (3): Let s be an M-regular element, and let $\mathfrak{p}$ be an element of $V(sA)$. Then $\mathfrak{p}=sM:_A x$ for some $x \in M$ if an only if $\mathfrak{p}=M:_A z$ for some $z \in Q_A(M)$; hence $\mathfrak{p} \in \mathrm{Ass}_A(M/sM)$ if and only if $\mathfrak{p} \in \mathrm{Ass}_A(Q_A(M)/M)$.

Corollary (2.2) *Let* A, M *and* N *be the same as in* (2.1). *Assume further that* M *is finitely generated. Then we have the following assertions.*

(1) *For every* $\mathfrak{p} \in \mathrm{Ass}_A(N/M)$, $\mathfrak{p}$ *contains an* M*-regular element and* $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})=1$.

(2) $\mathrm{Ass}_A(N/M) \cap V(sA)$ *is a finite set for every* M*-regular element* s *of* A.

(3) $\mathrm{Ass}_A(N/M)$ *is a finite set if an only if* $M_s=N_s$ *for some* M*-regular element* s *of* A. *In particular, if* B *is an* A*-algebra of finite type contained in* $Q(A)$, *then* $\mathrm{Ass}_A(B/A)$ *is a finite set.*

Proof. The assertions (1) and (2) follow from (2.1). (3): Suppose first that $\mathrm{Ass}_A(N/M)$ is a finite set, and choose an M-regular element s so that $s \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_A(N/M)$. Then $\mathrm{Ass}_{A_s}(N_s/M_s)=\phi$; hence $M_s=N_s$ (cf. [10, Lemma, p. 50]). Conversely suppose that $M_s=N_s$ for some M-regular element s. Then every $\mathfrak{p} \in \mathrm{Ass}_A(N/M)$ contains s. Therefore by (2.1) (3), we have $\mathrm{Ass}_A(N/M) \subseteq \mathrm{Ass}_A(M/sM)$; hence $\mathrm{Ass}_A(N/M)$ is a finite set.

Proposition (2.3) *Let* A, M *and* N *be the same as in* (2.1). *Let* Z *be a* *subset of* Spec(A) *which is stable under specialization. Then* $\mathrm{Ass}_A(T(Z, M) \cap N/M) = \mathrm{Ass}_A(N/M) \cap Z$.

PROOF. Since $\mathrm{Supp}(T(Z, M) \cap N/M) \subseteq Z$ (cf. (1.7)), we have $\mathrm{Ass}_A(T(Z, M) \cap N/M) \subseteq \mathrm{Ass}_A(N/M) \cap Z$. Let $\mathfrak{p}$ be an element of $\mathrm{Ass}_A(N/M) \cap Z$. Then $\mathfrak{p} = M :_A z$ for some $z \in N$. Since $\mathfrak{p} \in Z$, we have $z \in N \cap T(Z, M)$. Therefore $\mathfrak{p} \in \mathrm{Ass}_A(T(Z, M) \cap N/M)$. This completes the proof.

PROPOSITION (2.4) *Let R be an overring of a noetherian ring A, and let Z and Z' be subsets of* Spec(A) *which are stable under specialization. Assume that $Z \subseteq Z'$. We put $B = T(Z, A) \cap R$ and $B' = T(Z', A) \cap R$. Then there is a bijection* $\mathrm{Ass}_B(B'/B) \simeq \mathrm{Ass}_A(B'/A) \cap (Z' - Z)$.

PROOF. (i) Let P be an element of $\mathrm{Ass}_B(B'/B)$, and put $\mathfrak{p} = P \cap A$. We shall show that $\mathfrak{p} \in Z' - Z$. Suppose that $\mathfrak{p} \notin Z'$. Then by (1.7) (2), $B_\mathfrak{p} = B'_\mathfrak{p} = A_\mathfrak{p}$; hence $B'_P = B_P$. This contradicts the fact $P \in \mathrm{Ass}_B(B'/B)$. Therefore $\mathfrak{p} \in Z'$. Suppose next that $\mathfrak{p} \in Z$. We can choose an element x of $B' - B$ so that P is a minimal prime ideal of $B :_B x$. Since $\mathfrak{p}$ is finitely generated, this implies that $\sqrt{B :_B sx} \supseteq \mathfrak{p}$ for some $s \in B - P$. Since $\mathfrak{p} \in Z$, $sx \in T(Z, A) \cap R = B$; hence $x/1 \in B_P$. This is a contradiction. Therefore $\mathfrak{p} \notin Z$.

(ii) Let $\mathfrak{p}$ be an element of $Z' - Z$. Since $B_\mathfrak{p} = A_\mathfrak{p}$ (cf. (1.2)(1)), there exists a unique prime ideal P of B which lies over $\mathfrak{p}$. Since $A_\mathfrak{p} = B_P$, it follows from [10, Lemma, p. 50] that $\mathfrak{p} \in \mathrm{Ass}_A(B'/A)$ if and only if $P \in \mathrm{Ass}_B(B'/B)$. This completes the proof.

We shall now give a characterization of $\mathrm{Ass}_A(N/M)$ which is a key point in our study of overrings of a noetherian ring.

THEOREM (2.5) *Suppose that A is a noetherian ring. Let M be an A-module, and let N be an A-submodule of $Q_A(M)$ containing M. Then we have*

$$\mathrm{Ass}_A(N/M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_\mathfrak{p} \subset T(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}) \cap N_\mathfrak{p}\}.$$

PROOF. Let $\mathfrak{p}$ be a prime ideal of A such that $M_\mathfrak{p} \subset T(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}) \cap N_\mathfrak{p}$. Then we can choose an element x of N so that $x/1 \in T(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}) - M_\mathfrak{p}$. Since $M_\mathfrak{p} :_{A_\mathfrak{p}} x/1$ contains $(\mathfrak{p}A_\mathfrak{p})^n$ for some n, $\mathfrak{p}$ is a minimal prime ideal of $M :_A x$; hence $\mathfrak{p} \in \mathrm{Ass}_A(N/M)$. Conversely let $\mathfrak{p}$ be an element of $\mathrm{Ass}_A(N/M)$. Then $\mathfrak{p} = M :_A x$ for some $x \in N$. It is easy to see that $x/1 \in T(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}) \cap N_\mathfrak{p} - M_\mathfrak{p}$. This completes the proof.

As an example we show the following proposition due to K. Yoshida [22].

PROPOSITION (2.6) *Let A be a noetherian ring. Then*

$$\mathrm{Ass}_A(\bar{A}/A) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{ht}(\mathfrak{p}) = 1,\ A_\mathfrak{p} \subset \overline{A_\mathfrak{p}} \text{ and } \mathfrak{p} \text{ is regular}\}$$

$\cup \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \operatorname{ht}(\mathfrak{p}) \geqq 2, \operatorname{depth} A_{\mathfrak{p}} = 1$ *and* $\mathfrak{p}$ *is regular*$\}$.

(2.6) follows from (2.5) and the following two lemmas.

LEMMA (2.7) *Let* $\mathfrak{p}$ *be a regular prime ideal of a noetherian ring* A. *Then* $\overline{A_{\mathfrak{p}}} \cap (A_{\mathfrak{p}})^g = \bar{A}_{\mathfrak{p}} \cap (A_{\mathfrak{p}})^g$.

PROOF. By [3, Chap. 5, Proposition 16], $Q(A)_{\mathfrak{p}} \cap \overline{A_{\mathfrak{p}}} = \bar{A}_{\mathfrak{p}}$. Since $(A_{\mathfrak{p}})^g \subseteq Q(A)_{\mathfrak{p}}$ (cf. (1.10) (2)), $\overline{A_{\mathfrak{p}}} \cap (A_{\mathfrak{p}})^g = (\overline{A_{\mathfrak{p}}} \cap Q(A)_{\mathfrak{p}}) \cap (A_{\mathfrak{p}})^g = \bar{A}_{\mathfrak{p}} \cap (A_{\mathfrak{p}})^g$.

LEMMA (2.8) *Let* A *be a noetherian local ring with* depth $A=1$ *and* dim $A \geqq 2$. *Then* depth $A \geqq 2$ *if and only if* $A = A^g \cap \bar{A}$.

PROOF. Note that a noetherian local ring $(R, \mathfrak{m})$ with depth $R=1$ is a DVR if and only if the canonical map $R \to \operatorname{Hom}_R(\mathfrak{m}, \mathfrak{m})$ is an isomorphism. Suppose first that depth $A=1$. Since A is not a DVR, $A \subset \operatorname{Hom}_A(\mathfrak{p}, \mathfrak{p})$ where $\mathfrak{p}$ is the maximal ideal of A. Therefore $A \subset \operatorname{Hom}_A(\mathfrak{p}, \mathfrak{p}) \subseteq A^g \cap \bar{A}$. Suppose next that depth $A \geqq 2$. Then $A = A^g$; hence $A = A^g \cap \bar{A}$. This completes the proof.

The following lemma is already known (cf. [11, (33.11)]), but we can prove it without using completions and the theorem of Mori-Nagata.

LEMMA (2.9) *Let* C *be an integral overring of a noetherian ring* A, *and let* P *be a regular prime ideal of* C *such that* $\operatorname{ht}(P)=1$ *and* $A_{P \cap A} \neq C_{P \cap A}$. *Then* $P \cap A \in \operatorname{Ass}_A(C/A)$.

PROOF. By Lemma (2.10) below, there exists a finite A-subalgebra B of C such that $\operatorname{ht}(P \cap B)=1$ and $A_{P \cap A} \neq B_{P \cap A}$. We shall show that $P \cap A \in \operatorname{Ass}_A(B/A)$. We put $Q = P \cap B$ and $\mathfrak{p} = P \cap A$. Since $Q(A)_{\mathfrak{p}} \subseteq Q(A_{\mathfrak{p}})$, $B_{\mathfrak{p}}$ is a finite overring of $A_{\mathfrak{p}}$; therefore we may assume that A is a local ring with the maximal ideal $\mathfrak{p}$. Let a be a regular element of A such that $aB \subseteq A$. Since $a \in \mathfrak{p}$ and $\operatorname{ht}(Q)=1$, we have $sQ^n \subseteq aB (\subseteq A)$ for some positive integer n and $s \in B - Q$. If $s \notin A$, then $A :_A s \supseteq \mathfrak{p}^n$; hence $\mathfrak{p} \in \operatorname{Ass}_A(B/A)$. If $s \in A$, then s is invertible in A; hence $A :_A B \supseteq \mathfrak{p}^n$; therefore $\mathfrak{p} \in \operatorname{Ass}_A(B/A)$. This completes the proof.

LEMMA (2.10) *Let* C *be an integral overring of a noetherian ring* A, *and let* P *be a minimal prime ideal of* aC *where* a *is a regular element of* A. *Then there exists a finite overring* B *of* A *such that* $A \subseteq B \subseteq C$ *and* $\operatorname{ht}(P \cap B)=1$.

PROOF. Let $\{b_1, \dots, b_n\}$ be a set of generators of $P \cap A$. Since P is a minimal prime ideal of aC, $\sqrt{aC_P} = PC_P$. Therefore $sb_i^e = ax_i (i=1, \dots, n)$ for some positive integer e, $s \in C-P$ and $x_1, \dots, x_n \in C$. We put $B = A[s, x_1, \dots, x_n]$ and $Q = P \cap B$. Since $(P \cap A)B_Q \subseteq \sqrt{aB_Q}$, there exists a height one prime ideal Q' of B such that $(P \cap A)B \subseteq Q' \subseteq Q$. Since $Q' \cap A = Q \cap A = P \cap A$, we have

$Q=Q'$. This completes the proof.

THEOREM (2.11) *Assume that A is a noetherian ring. Let C be an integral overring of A, and let Z be a subset of* Spec(A) *which is stable under specialization. We put $B=T(Z, A)\cap C$ and $Z'=\{Q\in \mathrm{Spec}(B) \mid Q\cap A\in Z\}$. Let V be the set of all regular prime ideals N of C such that* ht$(N)=1$ *and $N\cap A\in Z$. Then we have the following assertions.*

(1) $B=T(Z', B)\cap C$.

(2) $\{N\cap B \mid N\in V\}$ *coincides with the set of all regular prime ideals M of B such that* ht$(M)=1$ *and $M\cap A\in Z$.*

(3) *If $N\in V$, then $B_{N\cap B}=C_{N\cap B}=C_N$.*

PROOF. (1): Since $B\subseteq T(Z, B)\cap C\subseteq T(Z, T(Z, A))\cap C=T(Z, A)\cap C=B$, we have $B=T(Z, B)\cap C$. Then by (1.9), $B=T(Z', B)\cap C$. (2) follows from (3). To prove (3), by virtue of (2.10), we may assume that C is finite over A. Then it follows from the assertion (1) and (2.3) that $\mathrm{Ass}_B(C/B)\cap Z'=\phi$. Let now $N\in V$. Then $N\cap B\in Z'$; hence $N\cap B\notin \mathrm{Ass}_B(C/B)$. Therefore by (2.9), $B_{N\cap B}=C_{N\cap B}=C_N$. This completes the proof.

The following corollary can be considered as another form of [14, Theorem (1.6)]. But our proof of it does not depend on the theorem of Mori-Nagata.

COROLLARY (2.12) *Let A be a noetherian local ring with the maximal ideal $\mathfrak{m}$. We put $B=A^g\cap \bar{A}$. Then we have the following assertions.*

(1) *B has only a finite number of maximal ideals, and $k(M)$ is finite over $k(\mathfrak{m})$ for every maximal ideal M of B.*

(2) *If M is a regular maximal ideal of B such that* ht$(M)\geqq 2$, *then $M\notin \mathrm{Ass}_B(B/sB)$ for any regular element s of A.*

(3) *If M is a regular maximal ideal of B such that* ht$(M)=1$, *then MB_M is generated by a single element and $(B_M)_{\mathrm{red}}$ is a* DVR, *i.e., B_M is a quasi-v-ring* (cf. [18]).

(4) *If N is a regular maximal ideal of $\bar{A}$ such that* ht$(N)=1$, *then* ht$(N\cap B)=1$.

PROOF. The assertion (1) follows from (1.11), and the assertion (4) follows from (2.11). Let M be a regular maximal ideal of B. Suppose that $M\in \mathrm{Ass}_B(B/sB)$ for some regular element s of A. Since B/sB is noetherian (cf, (1.11)), we can write $M=sB:_Bx$ for some $x\in B$. Then by (1.12), $w=x/s\in B^g=A^g$. If $wM\subseteq M$, then $w\in A^g\cap \bar{A}=B$ because M is finitely generated (cf. (1.11)); hence $M=B:_Bw=B$; this is a contradiction. Therefore $wM\nsubseteq M$; hence $wMB_M=B_M$. Choose an element z of MB_M so that $wz=1$. Then $MB_M=wzMB_M=zB_M$. In particular it follows from (2.10) that ht$(M)=1$. Therefore by (2.11), $B_M=\bar{A}_M=\bar{A}_N$ where N is the maximal ideal of $\bar{A}$ such that $N\cap B=M$. Since N is regular and $\mathrm{ht}(N)=1$, $Q(\bar{A}_N)=Q(\bar{A})_N$. It follows from [3, Chap. 5, Proposition 16] that $\bar{A}_N$ is integrally closed in $Q(\bar{A}_N)$. Therefore by [18, Proposition 2.7], $(B_M)_{\mathrm{red}}$ is a DVR. Thus the assertions (2) and (3) are proved.

REMARK (2.13) If A is a domain, we can prove (2.12) more easily. In fact B is noetherian (cf. (1.11)) and is integral over A; hence by (1.9), $T(\mathrm{Max}(A), B)=T(\mathrm{Max}(B), B)=B^g$. Therefore the assertions (2), (3) and (4) follow from (2.6) and (2.11).

Assume for a moment that A is a noetherian local ring such that $\dim A\geqq 2$ and depth $A=1$. We put $B=A^g\cap\bar{A}$. By (2.12) (1), $\mathrm{Max}(B)$ is a finite set. On the other hand, since $Q(A)=Q(B)$, we have $\mathrm{Min}(A)\cong\mathrm{Min}(B)$; in particular $\mathrm{Min}(B)$ is a finite set. Let now $\{M_1,\ldots, M_n\}$ (resp. $\{M_{n+1},\ldots, M_m\}$) be the set of maximal ideals M of B such that $\mathrm{ht}(M)\geqq 2$ (resp. $\mathrm{ht}(M)=1$), and let $\{P_1,\ldots, P_r\}$ be the set of minimal prime ideals P of B such that $\dim B/P=1$.

COROLLARY (2.14) ([14, (1.6.5.)]) *Let the situation be as described as above, and let t be an element of $\bigcap_{i=n+1}^m M_i-(\bigcup_{i=1}^n M_i)\cup(\bigcup_{i=1}^r P_i)$ (where $\bigcap_{i=n+1}^m M_i=B$ if the set $\{M_{n+1},\ldots, M_m\}$ is empty). Then t is invertible in A^g and $A^g=B_t$.*

PROOF. Since every prime ideal of B which contains t is maximal, t is regular and $1/t\in B^g=A^g$. Therefore by (2.12) (2) and (1.5), $(B_t)^g=B_t$; hence $A^g=(B_t)^g=B_t$.

We shall conclude this section by making a remark on noetherian rings satisfying Serre's property (S_2).

Let A be a noetherian ring satisfying (S_1), and let Z be the set of all prime ideals $\mathfrak{p}$ of A such that $\mathrm{ht}(\mathfrak{p})\geqq 2$. Then it follows from (1.5) that A satisfies (S_2) if and only if $T(Z, A)=A$. As an application of this fact, we obtain the following

PROPOSITION (2.15) (cf. [21, Theorem 2]) *Let B be a noetherian overring of a noetherian ring A. Assume that B is integral over A and satisfies (S_2). Then A also satisfies (S_2) if and only if* $\mathrm{ht}(\mathfrak{p})=1$ *for every* $\mathfrak{p}\in\mathrm{Ass}_A(B/A)$.

PROOF. Since depth $A_{\mathfrak{p}}=1$ for all $\mathfrak{p}\in\mathrm{Ass}_A(B/A)$, the 'only if' part is trivial. Conversely suppose that $\mathrm{ht}(\mathfrak{p})=1$ for all $\mathfrak{p}\in\mathrm{Ass}_A(B/A)$. Since $Q(A)=Q(B)$, A satisfies (S_1). Let $Z=\{\mathfrak{q}\in\mathrm{Spec}(A)\mid\mathrm{ht}(\mathfrak{q})\geqq 2\}$, and let $Z'=\{Q\in\mathrm{Spec}(B)\mid\mathrm{ht}(Q)\geqq 2\}$. As we remarked above, it is sufficient to show that $T(Z, A)=A$. By (2.3), $\mathrm{Ass}_A(T(Z, A)\cap B/A)=\mathrm{Ass}_A(B/A)\cap Z=\phi$; hence $T(Z, A)\cap B=A$. On the other hand it follows from (2.9) that $\{Q\in\mathrm{Spec}(B)\mid Q\cap A\in Z\}\subseteq Z'$; hence by (1.9) $T(Z, B)\subseteq T(Z', B)$. Therefore $T(Z, A)\subseteq T(Z, B)\subseteq T(Z', B)=B$ because B satisfies (S_2). Thus we have $A=T(Z, A)\cap B=T(Z, A)$. This completes the proof.

§ 3. Finiteness of overrings

In this section, A will always denote a noetherian ring.

Theorem (3.1) *Let M be a finite A-module, and let N be an A-submodule of $Q_A(M)$ containing M. Then the following assertions are equivalent.*

(1) *N is a finite A-module.*

(2) $\mathrm{Ass}_A(N/M)$ *is a finite set, and* $T(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}) \cap N_\mathfrak{p}$ *is a finite $A_\mathfrak{p}$-module for every* $\mathfrak{p} \in \mathrm{Ass}_A(N/M)$.

Proof. (1)⇒(2) is clear. (2)⇒(1): Since $\mathrm{Ass}_A(N/M)$ is a finite set, we can choose a chain $\mathrm{Spec}(A) = Z_0 \supset Z_1 \supset \cdots \supset Z_n \supset \cdots$ of subsets of $\mathrm{Spec}(A)$ so that each Z_n is stable under specialization, $\bigcap_n Z_n = \phi$, $Z_n - Z_{n+1} \subseteq (Z_n)_{\mathrm{gen}}$ for every n, and $\mathrm{Ass}_A(N/M) \cap (Z_n - Z_{n+1})$ is empty or consists of one point for every n. Then there is an integer n_0 such that $\mathrm{Ass}_A(N/M) \cap Z_{n_0} = \phi$ because $\mathrm{Ass}_A(N/M)$ is a finite set and $\bigcap_n Z_n = \phi$; hence by (2.3), $T(Z_{n_0}, M) \cap N = M$. Assume now that $T(Z_{n+1}, M) \cap N$ is a finite A-module and $T(Z_{n+1}, M) \cap N \subset T(Z_n, M) \cap N$. We shall prove that $T(Z_n, M) \cap N$ is also finite. We put $L = T(Z_{n+1}, M) \cap N$ and $L' = T(Z_n, M) \cap N$. Note that $\mathrm{Ass}_A(N/M) \cap (Z_n - Z_{n+1}) \neq \phi$. In fact, if $\mathrm{Ass}_A(N/M) \cap (Z_n - Z_{n+1}) = \phi$, then by (1.7) (2), (1.10) (3) and (2.5), $L_\mathfrak{p} = M_\mathfrak{p} = L_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(A) - Z_{n+1}$; hence by (1.7) (2), $L' = L$ This is a contradiction. Therefore $\mathrm{Ass}_A(N/M) \cap (Z_n - Z_{n+1})$ consists of one point, say $\mathfrak{q}$. Since $L'_\mathfrak{q} = T(\mathfrak{q}A_\mathfrak{q}, M_\mathfrak{q}) \cap N_\mathfrak{q}$ is a finite $A_\mathfrak{q}$-module contained in $T(\mathfrak{q}A_\mathfrak{q}, M_\mathfrak{q})$, there exists an M-regular element t in $\mathfrak{q}$ such that $tL'_\mathfrak{q} \subseteq M_\mathfrak{q}$. Then also by (1.7) (2), (1.10) (3) and (2.5), $(tL')_\mathfrak{p} \subseteq L_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(A) - Z_{n+1}$; hence by (1.7) (2), $tL' \subseteq L$. Therefore L' is a finite A-module. Now, by induction on n, $T(Z_0, M) = N$ is a finite A-module. This completes the proof.

Corollary (3.2) (cf. [15, (2.6.2)]) *Let M be a finite A-module, and let Z be a subset of* $\mathrm{Spec}(A)$ *which is stable under specialization. Then the following assertions are equivalent.*

(1) *$T(Z, M)$ is a finite A-module.*

(2) $\Delta = \mathrm{Ass}_A(Q_A(M)/M) \cap Z$ *is a finite set, and* $T(\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p})$ *is a finite $A_\mathfrak{p}$-module for every* $\mathfrak{p} \in \Delta$.

Proof. We put $N = T(Z, M)$. By (2.3), $\mathrm{Ass}_A(N/M) = \mathrm{Ass}_A(Q_A(M)/M) \cap Z = \Delta$. Therefore the assertion follows from (3.1).

Corollary (3.3) (cf. [15, (3.1)]) *Let M and N be the same as in* (3.1), *and let $x(\neq 0)$ be an M-regular element. Then the following assertions are equivalent.*

(1) *N is a finite A-module.*

(2) (a) *N_x is a finite A_x-module, and*
(b) *$N_\mathfrak{p}$ is a finite $A_\mathfrak{p}$-module for every $\mathfrak{p} \in \mathrm{Ass}_A(M/xM)$.*

PROOF. (1)⇒(2) is clear. (2)⇒(1): Let $\mathfrak{p} \in \mathrm{Ass}_A(N/M)$. If $x \notin \mathfrak{p}$, then $\mathfrak{p}A_x \in \mathrm{Ass}_{A_x}(N_x/M_x)$, and if $x \in \mathfrak{p}$, then by (2.1) (3), $\mathfrak{p} \in \mathrm{Ass}_A(M/xM)$. Since $\mathrm{Ass}_{A_x}(N_x/M_x)$ and $\mathrm{Ass}_A(M/xM)$ are finite sets, $\mathrm{Ass}_A(N/M)$ is also a finite set. Moreover $N_\mathfrak{p}$ is a finite $A_\mathfrak{p}$-module for every $\mathfrak{p} \in \mathrm{Ass}_A(N/M)$. Therefore by (3.1), N is a finite A-module.

COROLLARY (3.4) (cf. [8, Theoreme 2.1] and [15, (3.1)]) *Assume that A is a domain. Let x be a non-zero element of A. Then the following assertions are equivalent.*

(1) *For every $\mathfrak{p} \in \mathrm{Ass}_A(A/xA)$, the $xA_\mathfrak{p}$-adic completion of $A_\mathfrak{p}$ is reduced.*
(2) *For every $\mathfrak{p} \in \mathrm{Ass}_A(A/xA)$, $(A_\mathfrak{p})^g \cap \bar{A}_\mathfrak{p}$ is a finite $A_\mathfrak{p}$-module.*
(3) *$A_x \cap \bar{A}$ is a finite A-module.*

PROOF. (2)⇔(3): We put $N = A_x \cap \bar{A}$. Since $A_x = T(V(xA), A)$, if follows from (2.3) and (2.1) (3) that $\mathrm{Ass}_A(N/A) = \mathrm{Ass}_A(\bar{A}/A) \cap V(xA) \subseteq \mathrm{Ass}_A(A/xA)$. Therefore the assertion follows immediately from (3.1). (1)⇒(2): Let $\mathfrak{p} \in \mathrm{Ass}_A(A/xA)$, and let $(A_\mathfrak{p})^*$ be the $xA_\mathfrak{p}$-adic completion of $A_\mathfrak{p}$. Since $(A_\mathfrak{p})^*$ is faithfully flat over $A_\mathfrak{p}$, it follows from (1.4) that $((A_\mathfrak{p})^g \cap \bar{A}_\mathfrak{p}) \otimes_{A_\mathfrak{p}} (A_\mathfrak{p})^* \subseteq ((A_\mathfrak{p})^*)^g \cap \overline{(A_\mathfrak{p})^*}$. By (3.5) below, $((A_\mathfrak{p})^*)^g \cap \overline{(A_\mathfrak{p})^*}$ is finite over $(A_\mathfrak{p})^*$; hence $((A_\mathfrak{p})^g \cap \bar{A}_\mathfrak{p}) \otimes_{A_\mathfrak{p}} (A_\mathfrak{p})^*$ is also finite over $(A_\mathfrak{p})^*$. Therefore $(A_\mathfrak{p})^g \cap \bar{A}_\mathfrak{p}$ is finite over $A_\mathfrak{p}$. (3)⇒(1): (cf. [8, Theoreme 2.1]) Let $B = A_x \cap \bar{A}$. Then $B_x \cap \bar{A} = B$. Therefore by (2.3) and (2.6), $B_\mathfrak{q}$ is a DVR for every $\mathfrak{q} \in \mathrm{Ass}_B(B/xB)$. Let now $\mathfrak{p} \in \mathrm{Ass}_A(A/xA)$, and let $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ be the set of all associated prime ideals $\mathfrak{q}$ of xB such that $\mathfrak{q} \cap A \subseteq \mathfrak{p}$. Then $(A_\mathfrak{p})^* \subseteq (B_\mathfrak{p})^* \subseteq \widehat{B_{\mathfrak{q}_1}} \times \cdots \times \widehat{B_{\mathfrak{q}_n}} (= C)$, where $(A_\mathfrak{p})^*$ (resp. $(B_\mathfrak{p})^*$) is the $xA_\mathfrak{p}$-adic completion of $A_\mathfrak{p}$ (resp. $xB_\mathfrak{p}$-adic completion of $B_\mathfrak{p}$). Since C is reduced, $(A_\mathfrak{p})^*$ is reduced. This completes the proof.

LEMMA (3.5) *Assume that A is reduced, and is xA-adically complete for some regular element x in A. Then $A^g \cap \bar{A}$ is finite over A.*

PROOF. Let $B = A^g \cap \bar{A}$. Since $\bar{A}$ is a finite product of Krull domains and x is contained in the Jacobson radical of A, $\cap x^n B \subseteq \cap x^n \bar{A} = 0$. On the other hand A is xA-adically complete and, by (1.11), B/xB is finite over A/xA. Therefore B is finite over A (cf. [10, Lemma, p. 212]).

REMARK (3.6) (cf. [6, (5.10.17)]) Assume that A is a noetherian ring satisfying Serre's property (S_1). Let $Z = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{ht}(\mathfrak{p}) \geqq 2\}$, and put $A^{(1)} = T(Z, A)$. Corollary (3.2) gives a criterion for $A^{(1)}$ to be finite over A. Moreover if $A^{(1)}$ is finite over A, then $A^{(1)}$ satisfies (S_2). In fact, let $Z' = \{P \in \mathrm{Spec}(A^{(1)}) \mid \mathrm{ht}(P) \geqq 2.\}$. Since $Z' \subseteq \{P \in \mathrm{Spec}(A^{(1)}) \mid P \cap A \in Z\}$, we have $T(Z',$

$A^{(1)})\subseteq T(Z, A^{(1)})$; hence $A^{(1)}\subseteq T(Z', A^{(1)})\subseteq T(Z, A^{(1)})=T(Z, T(Z, A))=T(Z, A)$ $=A^{(1)}$. This shows that $A^{(1)}=T(Z', A^{(1)})$ and hence, by (1.5), $A^{(1)}$ satisfies (S_2).

We also have a criterion for an overring of A to be integral over A, which is a generalization of [15, (2.6.1)].

PROPOSITION (3.7) *For an overring B of A, the following assertions are equivalent.*

(1) *B is integral over A.*

(2) *$(A_{\mathfrak{p}})^g \cap B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$ for every $\mathfrak{p}\in \mathrm{Ass}_A(B/A)$.*

PROOF. The implication (1)⇒(2) is obvious. (2)⇒(1): (cf. the proof of [15, (2.6.1)]) We may assume that B is generated by a single element x as an A-algebra, i.e., $B=A[x]$. We shall now construct a chain $A_0\subseteq A_1\subseteq\cdots(\subseteq B)$ of finite overrings of A as follows: We put $A_0=A$. Assume that $A_n (n\geqq 0)$ can be constructed already. If $x\in A_n$, then we put $A_{n+1}=A_n$. Now consider the case that $x \notin A_n$. Let $\mathfrak{p}$ be a minimal prime ideal of $A_n:_A x$. Then $x/1\in((A_n)_{\mathfrak{p}})^g$, and by (1.14), $((A_n)_{\mathfrak{p}})^g\cap B_{\mathfrak{p}}(=B_{\mathfrak{p}})$ is integral over $(A_{\mathfrak{p}})^g\cap B_{\mathfrak{p}}$. Therefore $B_{\mathfrak{p}}$ is integral (hence finite) over $A_{\mathfrak{p}}$. Choose a finite A-algebra A_{n+1} so that $A_n\subset A_{n+1}\subseteq B$ and $(A_{n+1})_{\mathfrak{q}}=B_{\mathfrak{q}}$ for all minimal prime ideals $\mathfrak{q}$ of $A_n:_A x$. Note that, by our construction, $A_n:_A x\subset A_{n+1}:_A x$ if $x\notin A_n$. Therefore if $x\notin A_n$ for all $n\geqq 0$, then the ascending chain $A:_A x\subset A_1:_A x\subset\cdots$ is not stable; this is a contradiction because A is noetherian. Thus we conclude that $x\in A_n$ for some n. This completes the proof.

§4. Finite (S_2)-overrings

In this section we shall study noetherian rings with finite (S_2)-overrings. To do this, we need a characterization of a noetherian local ring over which its global transform is essentially finite. Recall that an A-algebra B is essentially finite over A if there exist a finite A-subalgebra C of B and a multiplicative subset S of C such that $B=S^{-1}C$.

LEMMA (4.1) *Let B and R be A-algebras, and let C be a B-algebra. Then we have the following assertions.*

(1) *If B is essentially finite over A and C is essentially finite over B, then C is essentially finite over A.*

(2) *If C is essentially finite over A, then C is essentially finite over B.*

(3) *If B is essentially finite over A, then $B\otimes_A R$ is essentially finite over R.*

PROOF. The assertions (2) and (3) are obvious. (1): We may assume that C is finite over B. Choose an A-subalgebra B' of B and a multiplicative subset S of B' so that $B=S^{-1}B'$. Let $\{x_1,\dots, x_n\}$ be a set of generators of C over B.

We may assume that each x_i is integral over B'. We now put $C'=B'[x_1,\dots,x_n]$ $(\subseteq C)$. Then it is clear that $C=S^{-1}C'$.

LEMMA (4.2) *Let A be a noetherian local ring such that* $\dim A\geqq 2$ *and* $\operatorname{depth} A=1$. *Let C be a finite A-subalgebra of A^g. Then the following assertions are equivalent.*

(1) *$A^g=S^{-1}C$ for some multiplicative subset S of C.*

(2) $\operatorname{depth} C_Q\geqq 2$ *for every maximal ideal Q of C such that* $\operatorname{ht}(Q)\geqq 2$.

PROOF. (1)⇒(2): Choose a multiplicative subset S of C so that $A^g=S^{-1}C$. Let Q be a maximal ideal of C such that $\operatorname{ht}(Q)\geqq 2$. Then $(C_Q)^g=(C^g)_Q=(A^g)_Q=(S^{-1}C)_Q$. If $S\cap Q\neq\phi$, then $(C_Q)^g$ has no prime ideals over QC_Q; hence, by (1.11) (1), every regular element of C_Q is invertible in $(C_Q)^g$ and therefore, by (1.13), $\operatorname{ht}(Q)=1$. This is a contradication. Therefore $S\cap Q=\phi$ and hence $(C_Q)^g=C_Q$. This shows that $\operatorname{depth} C_Q\geqq 2$ (cf. (1.5)).

(2)⇒(1): (cf. the proof of (2.14)) Choose a regular element t of C so that every prime ideal, containing t, is a maximal ideal of height one and t is not contained in any maximal ideal whose height is graeter than one. Then it follows easily from (1.5) that $A^g=C_t$. This completes the proof.

LEMMA (4.3) *Let A be a residue ring of a regular local domain such that* $\dim A\geqq 2$ *and* $\operatorname{depth} A=1$. *Then the following assertions are equivalent.*

(1) *A^g is essentially finite over A.*

(2) $\dim A/\mathfrak{p}\geqq 2$ *for every embedded prime ideal $\mathfrak{p}$ of A.*

PROOF. (1)⇒(2): Choose a finite A-subalgebra C of A^g and a multiplicative subset S of C so that $A^g=S^{-1}C$. Let $\mathfrak{p}$ be an embedded prime ideal of A. Since C is an overring of A, there exists a unique embedded prime ideal P of C such that $P\cap A=\mathfrak{p}$. Let M be an arbitrary maximal ideal of C such that $M\supset P$. Since $\operatorname{ht}(M)\geqq 2$, it follows from (4.2) that $\operatorname{depth} C_M\geqq 2$. Therefore $\dim A/\mathfrak{p}=\dim C/P\geqq 2$ (cf. [10, Theorem 27, p. 100]).

(2)⇒(1): Since $A_\mathfrak{p}$ is essentially finite over A for every $\mathfrak{p}\in\operatorname{Spec}(A)$, by virtue of (1.17), we may assume that A has no minimal prime ideals $\mathfrak{p}$ such that $\dim A/\mathfrak{p}=1$. In this situation it follows from (1.16) that A^g is finite over A. This completes the proof.

PROPOSITION (4.4) *Let $(A,\mathfrak{m})$ be a noetherian local ring such that* $\dim A\geqq 2$ *and* $\operatorname{depth} A=1$. *Let $(R,\mathfrak{n})$ be a faithfully flat A-algebra such that R is a residue ring of a regular local domain and $\mathfrak{n}=\sqrt{\mathfrak{m}R}$. Then the following assertions are equivalent.*

(1) *A^g is essentially finite over A.*

(2) *R^g is essentially finite over R* (i.e., $\dim R/\mathfrak{p}\geqq 2$ *for every embedded prime ideal $\mathfrak{p}$ of R*).

Proof. Note first that $R^g = A^g \otimes_A R$ (cf. (1.4)).

(1)⇒(2): The assertion follows from (4.1).

(2)⇒(1): Let $B = A^g \cap \bar{A}$. Then by (2.14), $A^g = B_t$ for some $t \in B$. Therefore $R^g = (B \otimes_A R)_t$. By our assumption, there exist a finite R-subalgebra C' of R^g and a multiplicative subset S of C' such that $R^g = S^{-1}C'$. Since C' is finite over R and $R^g = (B \otimes_A R)_t$, there exists a finite A-subalgebra C of B such that $C'_t \subseteq (C \otimes_A R)_t$. Therefore we may assume that $C' = C \otimes_A R$ for some finite A-subalgebra C of B. Since C' satisfies the condition (2) in (4.2), it follows from the faithful flatness of R over A that C also satisfies the condition (2) in (4.2); hence A^g is essentially finite over A.

We shall now study some properties of noetherian rings with finite (S_2)-overrings.

Lemma (4.5) *Let A be a noetherian ring satisfying* (S_1). *Consider the following conditions on A:*

(1) *There exists a finite* (S_2)*-overring of A.*

(2) $U = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}}$ *satisfies* $(S_2)\}$ *is a non-empty open subset of* $\mathrm{Spec}(A)$.

(3) $\Delta = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{ht}(\mathfrak{p}) \geqq 2$ *and* $\mathrm{depth}\, A_{\mathfrak{p}} = 1\}$ *is a finite set.*

Then (1)⇒(2)⇔(3).

Proof. (1)⇒(3): Let t be a regular element of A such that $tR \subseteq A$. Then A_t satisfies (S_2) and therefore $t \in \mathfrak{p}$ for every $\mathfrak{p} \in \Delta$; hence $\Delta \subseteq \mathrm{Ass}_A(A/tA)$. This shows that Δ is a finite set. (3)⇒(2): It is easy to see that $U = \mathrm{Spec}(A) - \bigcup_{\mathfrak{p} \in \Delta} V(\mathfrak{p})$; hence U is a non-empty open subset of $\mathrm{Spec}(A)$. (2)⇒(3): Since U is open and $\mathrm{Min}(A) \subseteq U$, there exists a regular element t of A such that $\mathrm{Spec}(A_t) \subseteq U$. On the other hand $\Delta \cap U = \phi$; hence $\Delta \subseteq V(tA)$. Therefore $\Delta \subseteq \mathrm{Ass}_A(A/tA)$. This shows that Δ is a finite set.

Lemma (4.6) *Let A be a noetherian local ring. Assume that A has a finite* (S_2)*-overring R. Then A^g is essentially finite over A.*

Proof. We put $B = A^g \cap R$. We may assume that $\dim A \geqq 2$ and depth $A = 1$. Now choose a regular element t of B so that every prime ideal, containing t, is a maximal ideal of height one and t is not contained in any maximal ideal whose height is greater than one. Then by (2.11) and (2.14), $B_t \subseteq A^g$ and R_t has no maximal ideals of height one. Therefore it follows from (1.5), (1.12) and (1.14) that $A^g = (B_t)^g \subseteq (R_t)^g = R_t$; hence A^g is finite over B_t. By (4.1) (1), A^g is essentially finite over A. This completes the proof.

The following therorem is our main result in this section.

THOEREM (4.7) *Let A be a noetherian ring satisfying Serre's property* (S_1), *and let Δ be the set of all prime ideals* $\mathfrak{p}$ *of A such that* $\mathrm{ht}(\mathfrak{p}) \geqq 2$ *and* depth $A_{\mathfrak{p}} = 1$. *Then the following assertions are equivalent.*

(1) *There exists a finite* (S_2)*-overring of A.*

(2) *Δ is a finite set, and for every $\mathfrak{p} \in \Delta$, $(A_{\mathfrak{p}})^g$ is essentially finite over $A_{\mathfrak{p}}$.*

PROOF. The implication (1)⇒(2) follows from (4.5) and (4.6). (2)⇒(1): For a finite overring B of A, we put $\Delta(B) = \{Q \in \mathrm{Spec}(B) \mid \mathrm{ht}(Q) \geqq 2$ and depth $B_Q = 1\}$, $\Delta^*(B) = \{Q \in \Delta(B) \mid (B_Q)^g$ is not finite over $B_Q\}$, $n(B) = \inf\{\mathrm{ht}(Q \cap A) \mid Q \in \Delta(B)\}$, and $n^*(B) = \sup\{\mathrm{ht}(Q \cap A) \mid Q \in \Delta^*(B)\}$. We shall show that there exists a finite overring R of A such that $\Delta^*(R) = \phi$. If this is done, then by (3.2) and (3.6), R has a finite (S_2)-overring; hence so does A. If $\Delta^*(A) = \phi$, there is nothing to prove. Therefore we may assume that $\Delta^*(A) \neq \phi$. Since $\Delta(A)$ is a finite set and $(A_{\mathfrak{p}})^g$ is essentially finite over $A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \Delta(A)$, there exists a finite A-subalgebra C of $Q(A)$ such that, for every $\mathfrak{p} \in \Delta(A)$, $(A_{\mathfrak{p}})^g \cap C_{\mathfrak{p}}$ satisfies the conditions in (4.2). We now put $B = T(Z, A) \cap C$ where $Z = \bigcup_{\mathfrak{p} \in \Delta} V(\mathfrak{p})$, and we shall show that B satisfies the condition (2) above. Choose a regular element t of A so that $tB \subseteq A$. Since $B_t = A_t$, $\{Q \in \Delta(B) \mid t \in Q\}$ is a finite set. On the other hand $\{Q \in \Delta(B) \mid t \in Q\}$ is also a finite set because it is a subset of $\mathrm{Ass}_B(B/tB)$. Therefore $\Delta(B)$ is a finite set. Let Q be a regular prime ideal of B such that $\mathrm{ht}(Q) \geqq 2$. We put $\mathfrak{q} = Q \cap A$. Since $B_{\mathfrak{q}}$ is finite over $A_{\mathfrak{q}}$, it follows from (1.14) that $(B_{\mathfrak{q}})^g$ is finite over $(A_{\mathfrak{q}})^g$. Suppose that $\mathfrak{q} \notin \Delta^*(A)$. Then $(A_{\mathfrak{q}})^g$ is finite over $A_{\mathfrak{q}}$. Therefore $(B_{\mathfrak{q}})^g$ is finite over $B_{\mathfrak{q}}$; hence $(B_Q)^g = ((B_{\mathfrak{q}})^g)_Q$ is finite over $B_Q = (B_{\mathfrak{q}})_Q$. In particular, $Q \notin \Delta^*(B)$ whenever $\mathfrak{q} \notin \Delta^*(A)$. Suppose next that $\mathfrak{q} \in \Delta^*(A)$. Then $(A_{\mathfrak{q}})^g$ is essentially finite over $A_{\mathfrak{q}}$; hence by (4.1), $(B_{\mathfrak{q}})^g$ is essentially finite over $B_{\mathfrak{q}}$. Therefore $(B_Q)^g$ is essentially finite over B_Q. Thus B satisfies the condition (2) above. Moreover we have $Q \notin \Delta(B)$ whenever $\mathfrak{q} \in \Delta(A) \cap Z_{\mathrm{gen}}$ (= the set of minimal elements of $\Delta(A)$). In fact if $\mathfrak{q} \in \Delta(A) \cap Z_{\mathrm{gen}}$, then $B_{\mathfrak{q}} = (A_{\mathfrak{q}})^g \cap C_{\mathfrak{q}}$ and therefore, by the choice of C, depth $B_Q \geqq 2$ (cf. (4.2)); hence $Q \notin \Delta(B)$. This shows that $n^*(B) > n(B)$.

Therefore, inductively, we can construct a sequence $B_0 = A \subseteq B_1 = B \subseteq \cdots \subseteq B_n \subseteq \cdots$ of finite overrings of A with the following properties:

(a) *Each B_n satisfies the condition* (2), *and*

(b) $n^*(B_0) \geqq n^*(B_1) \geqq \cdots \geqq n^*(B_n) \geqq n(B_n) > \cdots > n(B_1) > n(B_0)$ if $\Delta^*(B_i) \neq \phi$ *for* $i = 1, \ldots, n$.

We must then have that $\Delta^*(B_j) = \phi$ for some j. This completes the proof.

REMARK (4.8) Recently, in [2] M. Brodmann proved the following assertion: A noetherian local domain A whose formal fibres satisfy the first Serre property (S_1) adimits a finite (S_2)-overring if and only if the set $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}}$ is (S_2)$\}$ is open in Spec (A).

If formal fibres of A satisfy (S_1), then by (4.4), $(A_{\mathfrak{p}})^g$ is essentially finite over

$A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$. Therefore his result follows from (4.7) and (4.5).

§ 5. Seminormality and global transforms

Throughout this section, A will denote a noetherian ring. Let B be a finite overring of A. Recall that A is seminormal in B if and only if, for an element b of B, $b \in A$ whenever $b^2, b^3 \in A$. This characterization of seminormality is sufficient to follow what we discuss in this section. In particular if A is seminormal in B, then $A:_A B$ is a radical ideal of B.

We shall first study a finite overring B of A such that $\operatorname{Ass}_A(B/A)$ consists of one point $\mathfrak{p}$. Since $\operatorname{Ass}_A(B/A) = \{\mathfrak{p}\}$, $A:_A N$ is a $\mathfrak{p}$-primary ideal of A for every A-submodule N of B such that $N \nsubseteq A$. Let $\{P_1, \ldots, P_r\}$ be the set of all prime ideals of B which lie over $\mathfrak{p}$. We define an A-subalgebra C_0 of B by the following pull back diagram:

$$(*) \qquad \begin{array}{ccc} C_0 & \longrightarrow & B \\ \downarrow & & \downarrow \\ k(\mathfrak{p}) & \longrightarrow & \prod_i k(P_i) \end{array}$$

where $k(\mathfrak{p}) \to \prod_i k(P_i)$ and $B \to \prod_i k(P_i)$ are the natural ring homomorphisms, and inductively we define a chain $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq \cdots$ of A-subalgebras of B by the following pull back diagrams:

$$(**) \qquad \begin{array}{ccc} C_{n+1} & \longrightarrow & C_n \\ \downarrow & & \downarrow \\ k(\mathfrak{p}) & \longrightarrow & C_n \otimes_A k(\mathfrak{p}) \end{array}$$

$n = 0, 1, \ldots,$ where $k(\mathfrak{p}) \to C_n \otimes_A k(\mathfrak{p})$ and $C_n \to C_n \otimes_A k(\mathfrak{p})$ are the natural ring homomorphisms. C_0 is what we call the ring obtained from B by glueing over $\mathfrak{p}$. By definition, $\mathfrak{p}C_n$(resp. $P_1 \cap \cdots \cap P_r$) is also an ideal of C_{n+1}(resp. C_0). We put $P = P_1 \cap \cdots \cap P_r$. Then P is the only one prime ideal of C_0 lying over $\mathfrak{p}$, and $k(\mathfrak{p}) = k(P)$; in particular $P \cap C_n$ is the only one prime ideal of C_n lying over $\mathfrak{p}$.

Lemma (5.1) *With the same notation and assumptions as above, we have the following assertions.*

(1) *C_0 is seminormal in B.*
(2) *A is not seminormal in C_n if $C_n \neq A$.*
(3) *$A = C_0$ if and only if A is seminormal in B.*
(4) *A is seminormal in B if and only if so is A in* $\operatorname{End}_A(\mathfrak{p}) \cap B$.
(5) *$A = C_n$ for some n.*

Proof. The assertion (1) is clear. (2): Let n be a non-negative integer such that $C_n \neq A$. To prove that A is not seminormal in C_n, it is sufficient to show that $A_\mathfrak{p}$ is not seminormal in $(C_n)_\mathfrak{p}$. Note that $A_\mathfrak{p} \neq (C_n)_\mathfrak{p}$ because $\mathrm{Ass}_A(C_n/A) = \{\mathfrak{p}\}$. Since the above pull back diagrams (*) and (**) commute with the localization with respect to $A-\mathfrak{p}$, we may assume that A is local and $\mathfrak{p}$ is the maximal ideal of A. Let Q be the prime ideal of C_n lying over $\mathfrak{p}$. Suppose contrarily that A is seminormal in C_n. Since $A:_A C_n$ is a radical ideal of C_n, and also is $\mathfrak{p}$-primary, $Q = A:_A C_n = \mathfrak{p}$; hence by definition $C_n = A + Q = A + \mathfrak{p} = A$. This is a contradiction. Therefore A is not seminormal in C_n. The assertion (3) is now obvious by (1) and (2). (5): Since $A:_A C_n$ is a $\mathfrak{p}$-primary ideal or A, to prove that $A:_A C_n = A$ for some n, we may assume that A is local and $\mathfrak{p}$ is the maximal ideal of A. By definition $C_{n+1} = A + \mathfrak{p} C_n$. Therefore $A:_A \mathfrak{p} C_n = A:_A C_{n+1}$. Suppose now that $A:_A C_n \neq A$ for all n. Then each $A:_A C_n$ is a $\mathfrak{p}$-primary ideal different from $\mathfrak{p}$; hence $A:_A C_{n+1} = A:_A \mathfrak{p} C_n = (A:_A C_n):_A \mathfrak{p} \subset A:_A C_n$. Therefore we have an ascending chain $A:_A C_0 \subset A:_A C_1 \subset \cdots \subset A:_A C_n \subset \cdots$ of $\mathfrak{p}$-primary ideals. This is a contradiction; hence $A:_A C_n = C_n$ for some n. (4): Suppose that A is not seminormal in B. Then $A = C_{n+1} \subset C_n$ for some n. Since $C_n \subseteq \mathrm{End}_A(\mathfrak{p}) \cap B$, it follows from (2) that A is not seminormal in $\mathrm{End}_A(\mathfrak{p}) \cap B$. The converse is trivial. Thus the lemma is proved.

Let R be an overring of A. We say that A is seminormal in R if A is seminormal in every finite A-subalgebra of R.

THEOREM (5.2) *Let R be an overring of A. Then the following assertions are equivalent.*

(1) *A is seminormal in R.*
(2) *$A_\mathfrak{p}$ is seminormal in $(A_\mathfrak{p})^g \cap R_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}_A(R/A)$.*
(3) *$A_\mathfrak{p}$ is seminormal in $\mathrm{End}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}) \cap R_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}_A(R/A)$.*

PROOF. We may assume that R is finite over A. The implication (1)⇒(3) is obvious, and (3)⇒(2) follows from (5.1) because $\mathrm{Ass}_{A_\mathfrak{p}}((A_\mathfrak{p})^g \cap R_\mathfrak{p}/A_\mathfrak{p}) = \{\mathfrak{p}A_\mathfrak{p}\}$ (cf. (2.3)). (2)⇒(1): Suppose that A is not seminormal in R. Then there is an element b of $R-B$ such that $b^2, b^3 \in A$. Let $\mathfrak{p}$ be a minimal prime ideal of $A:_A b$. By definition, $\mathfrak{p} \in \mathrm{Ass}_A(R/A)$. Since $b/1 \in (A_\mathfrak{p})^g \cap R_\mathfrak{p}$ and $A_\mathfrak{p}$ is seminormal in $(A_\mathfrak{p})^g \cap R_\mathfrak{p}$, we have $b/1 \in A_\mathfrak{p}$. This is a contradiction. Therefore A is seminormal in R.

REMARK (5.3) The above theorem shows that A is seminormal in R if and only if $A_\mathfrak{p}$ is seminormal in $R_\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}_A(R/A)$ (cf. [5] and [7]).

REMARK (5.4). Let B be a finite overring of A. Choose a chain $Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ of subsets of $\mathrm{Spec}(A)$, which are stable under specialization, so that $\cap Z_n = \phi$ and, for each n, $\mathrm{Ass}_A(B/A) \cap (Z_n - Z_{n+1})$ consists of one point or is empty. We may assume that $\mathrm{Ass}_A(B/A) \cap (Z_i - Z_{i+1}) = \{\mathfrak{p}_{i+1}\}$ for $i = 0, \ldots, m$

and $\mathrm{Ass}_A(B/A)\cap Z_{m+1}=\phi$. We now put $B_i=T(Z_i, A)\cap B$ for $i=0,\dots, m$. Since $(B_i)_{\mathfrak{p}_i}=A_{\mathfrak{p}_i}$, there exists a unique prime ideal P_i of B_i such that $P_i\cap A=\mathfrak{p}_i$. Therefore we have a chain $A=B_{m+1}\subset B_m\subset\cdots\subset B_1\subset B_0=B$ of finite overrings of A such that $\mathrm{Ass}_{B_i}(B_{i-1}/B_i)=\{P_i\}$ and $k(P_i)=k(\mathfrak{p}_i)$ $(i=1,\dots, m+1)$. Thus the following lemma gives us another proof of [19, Th. 2.4].

Lemma (5.5) (cf. (5.1) (5)) *Let B be a finite overring of A. Assume that $\mathrm{Ass}_A(B/A)=\{\mathfrak{p}\}$. Consider the following chain $B=B_0\supseteq B_1\supseteq\cdots$ of A-subalgebras of B defined by the following pull back diagrams:*

$$\begin{array}{ccc} B_{n+1} & \longrightarrow & B_n \\ \downarrow & & \downarrow \\ k(\mathfrak{p}) & \longrightarrow & B_n\otimes_A k(\mathfrak{p}) \end{array}$$

$n=0, 1, 2,\dots$ *where $k(\mathfrak{p})\to B_n\otimes_A k(\mathfrak{p})$ and $B_n\to B_n\otimes_A k(\mathfrak{p})$ are the natural ring homomorphisms. Then $B_n=A$ for some n.*

Proof is similar to that of (5.1) (5).

Lemma (5.6) *With the same notations and assumptions as in* (2.4), *if A is seminormal in R, then B is seminoraml in B'.*

Proof. Let x be an element of R such that $x^2, x^3\in B$. We put $I=(A:_A x^2)\cap(A:_A x^3)$. Since $(Ix)^2, (Ix)^3\subseteq A$, it follows from our assumption that $Ix\subseteq A$; hence $x\in T(Z, A)\cap R=B$ because $V(I)\subseteq Z$. This shows that B is seminormal in R. Therefore B is seminormal in B'.

We now consider a finite overring B of A in which A is seminormal. We define, inductively, a chain $B=B_0\supseteq B_1\supseteq B_2\supseteq\cdots$ of A-subalgebras of B as follows: Assume that B_n is already defined. If $B_n=A$, then we put $B_{n+1}=A$. Now cosider the case that $B_n\neq A$. Let $\mathfrak{p}_{n+1}$ be a minimal prime ideal of $A:_A B_n$. We then put $B_{n+1}=$the ring obtained from B_n by glueing over $\mathfrak{p}_{n+1}$.

Proposition (5.8) ([20, Theorem 2.1] and [7, Theorem 1.13]) *With the same notations and assumptions as above, we have the following assertions.*

(1) *$B_n=A$ for some n.*

(2) $\mathrm{Ass}_A(B/A)=\{\mathfrak{p}_1,\dots, \mathfrak{p}_n\}$ *if $B_{n-1}\supset B_n=A$.*

The assertion (1) in (5.8) is the structure theorem of seminormal rings due to C. Traverso [20], and the assertion (2) has been shown by J. V. Leahy and M. A. Vitulli [7]. However we give here another proofs of these results using Z-transforms.

Proof of (5.8). We put $Z_0=\mathrm{Spec}(A)$. Since $\mathfrak{p}_1$ is a minimal element of

$\mathrm{Ass}_A(B/A)$, we can choose a subset Z_1 of $\mathrm{Spec}(A)$ which is stable under specialization so that $\mathrm{Ass}_A(B/A)\cap(Z_0-Z_1)=\{\mathfrak{p}_1\}$. We put $B_1'=T(Z,A)\cap B$, and we show that $B_1'=B_1$. By (2.4) and (5.6), B_1' is seminormal in B and $\mathrm{Ass}_{B_1'}(B/B_1')=\{P_1'\}$ where P_1' is a unique prime ideal of B_1' lying over $\mathfrak{p}_1$ (note that $(B_1')_{\mathfrak{p}_1}=A_{\mathfrak{p}_1}$). Since $k(\mathfrak{p}_1)=k(P_1')$, it follows from (5.1) (3) that $B_1'=B_1$. By induction, it is now easy to see that there exists a chain $Z_0\supset Z_1\supset Z_2\supset\cdots$ of subsets of $\mathrm{Spec}(A)$ such that each Z_n is stable under specialization, $\mathrm{Ass}_A(B_n/A)\cap(Z_n-Z_{n+1})=\{\mathfrak{p}_{n+1}\}$ whenever $B_n\neq A$, and $B_n=T(Z_n,A)\cap B$ for $n=1,2,\dots$. In particular $\mathfrak{p}_i\neq\mathfrak{p}_j$ if $i\neq j$, and $\{\mathfrak{p}_1,\mathfrak{p}_2,\dots\}\subseteq\mathrm{Ass}_A(B/A)$. Since $\mathrm{Ass}_A(B/A)$ is a finite set, $B_n=A$. Then $\mathrm{Ass}_A(B/A)=\cup(\mathrm{Ass}_A(B/A)\cap(Z_i-Z_{i+1}))=\cup(\mathrm{Ass}_A(B_i/A)\cup(Z_i-Z_{i+1}))=\{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$. This completes the proof.

Appendix

In this appendix, as an application of (2.12), we shall give an alternative proof of the theorem of Mori-Nagata:

Let R be the derived normal domain of a noetherian domain A. Then we have the following assertions.

(1) *For each $\mathfrak{p}\in\mathrm{Spec}(A)$, there are only a finite number of prime ideals P of R such that $P\cap A=\mathfrak{p}$, and if P is such a prime ideal of R, then $[k(P):k(\mathfrak{p})]$ is finite.*

(2) *R is a Krull domain, i.e.,*

(a) *R_P is a* DVR *for every height one prime ideal P of R,*

(b) *for every $x(\neq 0)$ in R, there are only a finite number of height one prime ideals of R which contain x, and*

(c) *$R=\cap R_P$ where P runs through all height one prime ideals of R.*

Proof. (b): We may assume that $x\in A$. Then it follows from (2.1) (3) and (2.9) that every height one prime ideal of R containing x lies over some prime ideal in $\mathrm{Ass}_A(A/xA)$. Since $\mathrm{Ass}_A(A/xA)$ is a finite set, the assertion is clear by (2.12). Similarly we can prove the assertion (a).

(c): To prove the assertion, it is fufficient to show that if Q is a minimal prime ideal of $xR:_R y$ for some x and y in R with $y\notin xR_Q$, then $\mathrm{ht}(Q)=1$. Suppose contrarily that $\mathrm{ht}(Q)\geqq 2$. We may assume that A is a local ring with the maximal ideal $\mathfrak{q}=Q\cap A$, and $x,y\in A$. Moreover replacing A by $(A^g\cap\bar{A})_{Q\cap(A^g\cap\bar{A})}$, we may assume that there exist no height one maximal ideals of R (cf. (2.11)). Since Q is a minimal prime ideal of $xR:_R y$, $\mathfrak{q}^n ys\subseteq xR$ for some positive integer n and $s\in R-Q$. Choose a finite overring B of A so that $s\in B$ and $\mathfrak{q}^n ys\subseteq xB$. Note that, by virtue of (2.12), $B^g\subseteq R$. Since $\mathfrak{q}^n B^g ys\subseteq xB^g$ and $\dim B^g/\mathfrak{q}^n B^g=0$, $ys\in xB^g\subseteq xR$; hence $y\in xR_Q$. This is a contradiction. This completes the proof of (2).

(1): We may assume that A is a local domain with the maximal ideal $\mathfrak{p}$, and it is sufficient to show that (1) is true for $\mathfrak{p}$. We shall use induction on $n=\dim A$. If $n=1$, then the assertion follows from the theorem of Krull-Akizuki (or (2.12)). Now assume that $n \geqq 2$. We shall first consider the case that depth $A \geqq 2$. Let x, y be a regular sequence of length two in $\mathfrak{p}$, and let X be an indeterminate. Then $\mathfrak{q}=(xX-y)A(X)$ is a prime ideal of $A(X)$ and $A(X)_{\mathfrak{q}}$ is a DVR; hence there exists a unique prime ideal Q of $\bar{A}(X)$ such that $Q \cap A(X)=\mathfrak{q}$. By virtue of (2.1) (3) and (2.9), Q is the only one minimal prime ideal of $(xX-y)\cdot\bar{A}(X)$; hence every maximal ideal of $\bar{A}(X)$ contains Q. Therefore the natural map $\mathrm{Spec}(\bar{A}(X)/Q) \to \mathrm{Spec}(\bar{A})$ induces a bijection $\mathrm{Max}(\bar{A}(X)/Q) \simeq \mathrm{Max}(\bar{A})$. (Note that every maximal ideal of $\bar{A}(X)$ is of the form $M\bar{A}(X)$, $M \in \mathrm{Max}(\bar{A})$ (cf. [11, Chap. I, p. 18]). Since $\dim A(X)/\mathfrak{q} \leqq n-1$ and $\bar{A}(X)/Q \subseteq \overline{A(X)/\mathfrak{q}}$, the induction hypothesis implies that $\mathrm{Max}(\bar{A}(X)/Q)$ $(=\mathrm{Max}(\bar{A}))$ is a finite set and, for every $M \in \mathrm{Max}(\bar{A})$, $k(M\bar{A}(X)/Q)(=k(M)(X))$ is a finite algebraic extension of $k(\mathfrak{p}A(X)/\mathfrak{q})(=k(\mathfrak{p})(X))$; hence $[k(M): k(\mathfrak{p})]$ is finite. Now consider the case that depth $A=1$. By (2.12), it is sufficient to prove that $B=A^g \cap \bar{A}$ satisfies (1) for its maximal ideals. Let M be a maximal ideal of B. If $\mathrm{ht}(M) \geqq 2$, then by (2.12) depth $B_M \geqq 2$; hence this case is established already. If $\mathrm{ht}(M)=1$, then by (2.12) B_M is a DVR; hence the assertion is obvious. This completes the proof.

References

[1] M. Brodmann, Ueber die minimale Dimension der assoziierten Primideale der Komplettion eines lokalen Integritätsbereiches, Comment. Math. Helv. **50** (1975), 219–232.

[2] ———, Finiteness of ideal transforms, J. Algebra (to appear).

[3] N. Bourbaki, Algèbre Commutative, Chap. 5, 6, Hermann, Paris, 1964.

[4] D. Ferrand-M. Raynaud, Fibres formelles d'un anneaux local noethérien, Ann. Sci. Ecole Norm. Sup. (4) **3** (1970), 295–311.

[5] S. Greco-C. Traverso, On seminormal schemes, Serie di Pubbl. Ist. Mat. Pol. Torino, Serie I, Preprint N. 10, 1978.

[6] A. Grothendieck, Eléments de géométrie algébrique IV (Second partie), Publ. Math. IHES. **24**, Paris, 1965.

[7] J. V. Leahy-M. A. Vitulli, Seminormal rings and weakly normal varieties, (preprint).

[8] J. Marot, Topologies d'anneau principal, C. R. Acad. Sci. Paris Sér. A **288** (1979), 713–715.

[9] J. Matijevic, Maximal ideal transforms of noetherian rings, Proc. Amer. Math. Soc. **54** (1976), 49–52.

[10] H. Matsumura, Commutative Algebra, Benjamin, 1970.

[11] M. Nagata, Local Rings, Interscience, 1962.

[12] ———, A treatise on the 14-th problem of Hilbert, Mem. Coll. Sci. Univ. Kyoto, Ser. A Math. **30** (1956–57), 57–70.

[13] ———, A theorem of finite generation of a ring, Nagoya Math. J. **27** (1966), 193–205.

[14] J. Nishimura, On ideal transforms of noetherian rings, I, J. Math. Kyoto Univ. **19** (1979), 41–46.
[15] ———, On ideal transforms of noetherian rings, II, J. Math. Kyoto Univ. **20** (1980), 149–154.
[16] D. G. Northcott, Finite Free Resolutions, Cambridge tracts in mathematics; no. 71, 1976.
[17] J. Querré, Sur un théorème de Mori-Nagata, C. R. Acad. Sci. Sér. A Paris **285** (1977), 323–324.
[18] L. J. Ratliff, On prime divisors of the integral closure of a principal ideal, J. reine Angew. Math. **255** (1972), 210–220.
[19] G. Tamone, Su una generalizzazione della nozione di incollamento, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **65** (1978), 107–112.
[20] C. Traverso, Seminormality and Picard group, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **24** (1970), 585–595.
[21] H. Yanagihara, On glueing of prime ideals, Hiroshima Math. J. (to appear).
[22] K. Yoshida, On the birational integral extensions and the prime ideals of depth one, (in preparation).

Department of Mathematics,
Faculty of Science,
Hiroshima University