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On asymptotic limits of nonoscillations in functional equations with retarded arguments

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1. Introduction

Our main aim in this paper is to study the asymptotic properties of the nonoscillatory solutions of the differential equation

$$L_n y(t) + a(t)h(y(g(t))) = f(t), \tag{1}$$

where $n \geq 2$ and L_n is a disconjugate differential operator defined by

$$L_n y(t) = p_n(t)(p_{n-1}(t)(\cdots(p_1(t)(p_0(t)y(t))')'\cdots)')'. \tag{2}$$

The following conditions are always assumed to hold:

(i) $p_i \in C([\alpha, \infty), (0, \infty))$, $0 \leq i \leq n$, and

$$\int_\alpha^\infty p_i^{-1}(t)dt = \infty, \quad 1 \leq i \leq n-1; \tag{3}$$

(ii) $a, f, g \in C([\alpha, \infty), R)$, a is of one sign, there exists a $t_0 > \alpha$ such that $0 < g(t) \leq t$ for $t \geq t_0$, and $g(t) \to \infty$ as $t \to \infty$;

(iii) $h \in C(R, R)$, h is nondecreasing, and sign $h(y) = $ sign y.

We introduce the notation:

$$L_0 y(t) = p_0(t)y(t), \quad L_i y(t) = p_i(t)(L_{i-1}y(t))', \quad 1 \leq i \leq n. \tag{4}$$

The domain $\mathscr{D}(L_n)$ of L_n is defined to be the set of all functions y: $[T_y, \infty) \to R$ such that $L_i y(t)$, $0 \leq i \leq n$, exist and are continuous on $[T_y, \infty)$. In what follows by a "solution" of equation (1) we mean a function $y \in \mathscr{D}(L_n)$ which is nontrivial in any neighborhood of infinity and satisfies (1) for all sufficiently large t. A solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise the solution is called nonoscillatory.

It is well known [5, 7] that in case $L_n y(t) = y^{(n)}(t)$ equation (1) has a nonoscillatory solution with a prescribed limit as $t \to \infty$ if

$$\int^\infty t^{n-1}|a(t)|dt < \infty \tag{5}$$

and

(6) $$\int^{\infty} t^{n-1}|f(t)|dt < \infty.$$

In this work we find conditions so that all nonoscillatory solutions of equation (1) approach limits as $t \to \infty$.

The literature on asymptotic nature of oscillatory and nonoscillatory solutions of functional equations is growing by the day. Our study in this paper is related to the works of Kitamura, Kusano and Naito [2], Kusano and Onose [3, 4], Philos and Staikos [6] and Singh [8]; but our results are different from and more complete than those obtained in the above papers. In fact, since the pioneering work of Hammett [1] the asymptotic study of nonoscillatory and oscillatory solutions of (functional) differential equations continues to offer new avenues to be explored.

2. Main results

Let $i_k \in \{1, 2,\dots, n-1\}$, $1 \leq k \leq n-1$, and t, $s \in [\alpha, \infty)$. We define $I_0 = 1$ and

(7) $$I_k(t, s; p_{i_k},\dots, p_{i_1}) = \int_s^t p_{i_k}^{-1}(r) I_{k-1}(r, s; p_{i_{k-1}},\dots, p_{i_1})dr.$$

It is easily verified that for $1 \leq k \leq n-1$

(8) $$I_k(t, s; p_{i_k},\dots, p_{i_1}) = \int_s^t p_{i_1}^{-1}(r) I_{k-1}(t, r; p_{i_k},\dots, p_{i_2})dr.$$

For the sake of brevity we employ the notation:

(9) $$J_i(t, s) = p_0^{-1}(t) I_i(t, s; p_1,\dots, p_i), \qquad J_i(t) = J_i(t, \alpha),$$

(10) $$K_i(t, s) = p_n^{-1}(t) I_i(t, s; p_{n-1},\dots, p_{n-i}), \quad K_i(t) = K_i(t, \alpha), \quad 0 \leq i \leq n-1.$$

LEMMA 1. *In addition to* (i)–(iii) *suppose* $a(t) > 0$ *and* $\int^{\infty} p_n^{-1}(t)|f(t)|dt < \infty$. *Let* $y(t)$ *be a nonoscillatory solution of equation* (1). *Then*

(11) $$\int^{\infty} p_n^{-1}(t) a(t) |h(y(g(t)))|dt < \infty.$$

PROOF. Without any loss of generality, suppose $T > t_0$ is large enough so that $y(g(t)) > 0$ for $t \geq T$. Dividing equation (1) by $p_n(t)$ and integrating from T to t, we have

(12) $$L_{n-1}y(t) - L_{n-1}y(T) + \int_T^t p_n^{-1}(r) a(r) h(y(g(r)))dr = \int_T^t p_n^{-1}(r) f(r)dr.$$

If (11) does not hold, then it follows from (12) that $L_{n-1}y(t) \to -\infty$ as $t \to \infty$.

This combined with condition (3) on $p_i(t)$ implies that $L_0 y(t) = p_0(t)y(t) \to -\infty$ as $t \to \infty$, a contradiction. The proof is complete.

LEMMA 2. *Suppose the conditions of Lemma* 1 *hold. Let* $y(t)$ *be a nonoscillatory solution of equation* (1). *Then*

$$y(t) = O(J_{n-1}(t)) \quad as \quad t \to \infty. \tag{13}$$

PROOF. Let $y(t)$ be a nonoscillatory solution of equation (1). From repeated integration of equation (1), we obtain that there exist constants $c_0, c_1, \ldots, c_{n-1}$ such that

$$p_0(t)y(t) = \sum_{i=0}^{n-1} c_i I_i(t, T; p_1, \ldots, p_i)$$
$$+ \int_T^t I_{n-1}(t, r; p_1, \ldots, p_{n-1}) p_n^{-1}(r) [f(r) - a(r)h(y(g(r)))] dr,$$

whence we see that

$$p_0(t)y(t)/I_{n-1}(t, \alpha; p_1, \ldots, p_{n-1})$$
$$\leq k + \int_T^t p_n^{-1}(s) [|f(r)| + a(r) | h(y(g(r)))|] dr$$

for $t \geq T$, where k is a positive constant. The conclusion now follows from Lemma 1.

We now state and prove one of the main results of this paper.

THEOREM 1. *Suppose* (i)–(iii) *hold. Further suppose that*

$$\int^\infty K_{n-1}(t) |f(t)| dt < \infty \tag{14}$$

and

$$\int^\infty K_{n-1}(t) |h(cJ_{n-1}(t)) a(t)| dt < \infty \tag{15}$$

for any constant $c \neq 0$. *If* $y(t)$ *is a nonoscillatory solution of equation* (1), *then* $p_0(t)y(t)$ *approaches a limit, finite or infinite, as* $t \to \infty$.

PROOF. Without any loss of generality we may suppose that $y(t)$ is eventually positive. Let T be such that $y(g(t)) > 0$ for $t \geq T$.

(a) *The case where* $a(t) > 0$. By Lemma 2 there exists a constant $c > 0$ such that

$$y(t) \leq cJ_{n-1}(t) \quad \text{for} \quad t \geq T. \tag{16}$$

If the conclusion is not true, then there exist two positive numbers β, δ such that

(17) $$\liminf_{t\to\infty} p_0(t)y(t) < \beta < \delta < \limsup_{t\to\infty} p_0(t)y(t).$$

Let $T_1 > T$ be so large that

(18) $$\int_{T_1}^{\infty} K_{n-1}(t)\,|f(t)|dt < \frac{\delta-\beta}{4}$$

and

(19) $$\int_{T_1}^{\infty} K_{n-1}(t)h(cJ_{n-1}(t))a(t)dt < \frac{\delta-\beta}{4}.$$

We observe that (17) implies that $L_i y(t)$, $1 \le i \le n-1$, are oscillatory and that there exist arbitrarily large numbers A and B such that $A<B$ and

$$p_0(A)y(A) < \beta < \delta < p_0(B)y(B).$$

Choose $A_0 < B_0 < A_1 < B_1$ so that $T_1 < A_0$,

(20) $$p_0(A_0)y(A_0) < \beta < \delta < p_0(B_0)y(B_0)$$

and

(21) $$p_0(A_1)y(A_1) < \beta < \delta < p_0(B_1)y(B_1).$$

Let $[s_1, s_2]$ be the smallest closed interval containing B_1 such that $p_0(s_1)y(s_1) = p_0(s_2)y(s_2) = \beta$ and

(22) $$\max\{p_0(t)y(t) : t \in [s_1, s_2]\} = p_0(s')y(s') > \delta.$$

Due to (20), (21) and (22) we have $T_1 < s_1 < s' < s_2$. Let $s_2 < t_1 < t_2 < \cdots < t_{n-1}$ be such that

(23) $$L_i y(t_i) = 0, \quad 1 \le i \le n-1.$$

On repeated integration from equation (1), we have in view of (22)

(24) $$(p_0(t)y(t))' = (-1)^{n-1} p_1^{-1}(t) \int_t^{t_1} p_2^{-1}(r_2) \int_{r_2}^{t_2} \cdots \int_{r_{n-2}}^{t_{n-2}} p_{n-1}^{-1}(r_{n-1}) \int_{r_{n-1}}^{t_{n-1}} \cdot$$
$$\cdot\, p_n^{-1}(r)\,[f(r) - a(r)h(y(g(r)))]dr dr_{n-1} dr_{n-2} \cdots dr_2.$$

Integrating (24) between s_1 and s', we have

$$\delta - \beta < \int_{s_1}^{s'} p_1^{-1}(r_1) \int_{r_1}^{t_1} p_2^{-1}(r_2) \int_{r_2}^{t_2} \cdots \int_{r_{n-2}}^{t_{n-2}} p_{n-1}^{-1}(r_{n-1}) \int_{r_{n-1}}^{t_{n-1}} \cdot$$
$$\cdot\, p_n^{-1}(r)\,[|f(r)| + a(r)h(y(g(r)))]dr dr_{n-1} dr_{n-2} \cdots dr_2 dr_1.$$

This in a manner of [9] gives

$$\delta-\beta<\int_{s_1}^{t_{n-1}} p_1^{-1}(r_1)\int_{r_1}^{t_{n-1}} p_2^{-1}(r_2)\int_{r_2}^{t_{n-1}}\cdots\int_{r_{n-2}}^{t_{n-1}} p_{n-1}^{-1}(r_{n-1})\int_{r_{n-1}}^{t_{n-1}}\cdot$$
$$\cdot\, p_n^{-1}(r)\,[|f(r)|+a(r)h(y(g(r)))]dr\,dr_{n-1}dr_{n-2}\cdots dr_2dr_1.$$

Using (8) and (10), the last integral can be rewritten as

$$\int_{s_1}^{t_{n-1}} I_{n-1}(r, s_1; p_{n-1},\dots, p_1)p_n^{-1}(r)\,[|f(r)|+a(r)h(y(g(r)))]dr$$
$$=\int_{s_1}^{t_{n-1}} K_{n-1}(r, s_1)\,[|f(r)|+a(r)h(y(g(r)))]dr.$$

Hence we obtain

$$\delta-\beta<\int_{s_1}^{\infty} K_{n-1}(t, s_1)\,[|f(t)|+a(t)h(y(g(t)))]dt. \tag{25}$$

From (16), (18), (19) and (25) it follows that

$$\delta-\beta<\int_{s_1}^{\infty} K_{n-1}(t)\,[|f(t)|+a(t)h(cJ_{n-1}(t))]dt$$
$$<\frac{\delta-\beta}{4}+\frac{\delta-\beta}{4}=\frac{\delta-\beta}{2}.$$

This contradiction completes the proof for the case where $a(t)>0$.

(b) *The case where* $a(t)<0$. Then either

$$\int_T^{\infty} p_n^{-1}(t)a(t)h(y(g(t)))dt=-\infty \tag{26}$$

or

$$\int_T^{\infty} p_n^{-1}(t)a(t)h(y(g(t)))dt>-\infty. \tag{27}$$

If (26) holds, then from (12) we see that $L_{n-1}y(t)\to\infty$ as $t\to\infty$. Clearly, this implies $L_0y(t)=p_0(t)y(t)\to\infty$ as $t\to\infty$. If (27) holds, then the argument of the proof of Lemma 2 shows that $y(t)=O(J_{n-1}(t))$ as $t\to\infty$. Once this growth estimate has been obtained, we can proceed exactly as in the proof for the case (a) to conclude that $p_0(t)y(t)$ approaches a limit, finite or infinite, as $t\to\infty$. This finishes the proof.

COROLLARY 1. *Under the conditions of Theorem* 1, *if equation* (1) *has a solution* $y(t)$ *such that*

$$\liminf_{t\to\infty} p_0(t)\,|y(t)|<\limsup_{t\to\infty} p_0(t)\,|y(t)|,$$

then $y(t)$ *is oscillatory.*

When specialized to the equation

$$y^{(n)}(t) + a(t)|y(g(t))|^{\gamma} \operatorname{sign} y(g(t)) = f(t), \tag{28}$$

the above results yield the following corollary.

COROLLARY 2. *Suppose that*

$$\int^{\infty} t^{n-1}|f(t)|dt < \infty \tag{29}$$

and

$$\int^{\infty} t^{(1+\gamma)(n-1)}|a(t)|dt < \infty. \tag{30}$$

Let $y(t)$ be a nonoscillatory solution of equation (28). *Then $y(t)$ tends to a limit, finite or infinite, as $t\to\infty$. Every solution $z(t)$ of equation* (28) *such that* $\liminf_{t\to\infty} |z(t)| < \limsup_{t\to\infty} |z(t)|$ *is oscillatory.*

EXAMPLE 1. All nonoscillatory solutions of the equation

$$y'''(t) + \frac{\sigma}{t^8} y^{5/3}(t-\pi) = \frac{\sigma(t-\pi)^{5/3}}{t^8}, \quad t>\pi,$$

where $\sigma=1$ or -1, approach limits as $t\to\infty$. In fact, $y(t)=t$ is one such solution. All conditions of Corollary 2 are easily verified.

EXAMPLE 2. The equation

$$y^{(iv)}(t) + \frac{\sigma y(t-\pi)}{t^6(2-\sin t)} = \frac{\sigma}{t^6} + \sin t, \quad t > \pi,$$

where $\sigma=1$ or -1, has a nonoscillatory solution $y(t)=2+\sin t$ which does not approach a limit as $t\to\infty$. Condition (30) of Corollary 2 holds, but condition (29) does not for this equation.

3. More on asymptotic limits

So far we have dealt with equation (1) in which the differential operator L_n obeys condition (3). Such an operator L_n is said to be *in canonical form.* Recently Trench [10] has shown that any differential operator of the form (2) can be put in canonical form in an essentially unique way. More precisely, if

$$L_n y(t) = p_n(t)(p_{n-1}(t)(\cdots(p_1(t)(p_0(t)y(t))')'\cdots)')'$$

and if (3) is not satisfied, then $L_n y(t)$ can be rewritten as

$$L_n y(t) = \tilde{p}_n(t)(\tilde{p}_{n-1}(t)(\cdots(\tilde{p}_1(t)(\tilde{p}_0(t)y(t))')'\cdots)')' \tag{31}$$

so that

$$\int_{\alpha}^{\infty} \tilde{p}_i^{-1}(t)dt = \infty, \quad 1 \leq i \leq n-1, \tag{32}$$

and the $\tilde{p}_i(t)$, $0 \leq i \leq n$, are determined up to positive multiplicative constants with product 1.

Actual computation leading to canonical form is not easy, so that it would be of practical interest to obtain an analogue of Theorem 1 for equation (1) with L_n not in canonical form without representing L_n in canonical form. The purpose of this section is to present an attempt in this direction.

By a *principal system* for L_n we mean a set of n solutions $X_1(t)$, $X_2(t)$,..., $X_n(t)$ of the equation $L_n x(t) = 0$ which are eventually positive and satisfy

$$\lim_{t\to\infty} \frac{X_i(t)}{X_j(t)} = 0 \quad \text{for} \quad 1 \leq i < j \leq n. \tag{33}$$

In case L_n is in canonical form the set of functions $\{J_0(t), J_1(t),..., J_{n-1}(t)\}$ defined by (9) is a principal system for L_n, and the set of functions $\{K_0(t), K_1(t), ..., K_{n-1}(t)\}$ defined by (10) is a principal system for the operator

$$M_n y(t) = p_0(t)(p_1(t)(\cdots(p_{n-1}(t)(p_n(t)y(t))')'\cdots)')', \tag{34}$$

which is also in canonical form. A basic property of principal systems is that if $\{X_1(t),..., X_n(t)\}$ and $\{\tilde{X}_1(t),..., \tilde{X}_n(t)\}$ are any two principal systems for L_n, then the limits

$$\lim_{t\to\infty} \frac{\tilde{X}_i(t)}{X_i(t)} > 0, \quad 1 \leq i \leq n, \tag{35}$$

exist and are finite. (See e.g. Trench [10].)

THEOREM 2. *Let $\{X_1(t),..., X_n(t)\}$ be a principal system for L_n and let $\{Y_1(t),..., Y_n(t)\}$ be a principal system for M_n defined by* (34). *Let $y(t)$ be a nonoscillatory solution of equation* (1). *Then $y(t)/X_1(t)$ approaches a limit, finite or infinite, as $t\to\infty$ if*

$$\int^{\infty} Y_n(t)\,|f(t)|dt < \infty \tag{36}$$

and

$$\int^{\infty} Y_n(t)\,|h(cX_n(t))a(t)|dt < \infty \tag{37}$$

for any constant $c \neq 0$.

PROOF. We represent L_n in canonical form, that is, in the form (31) satisfying

(32). Let $\{\tilde{X}_1(t),\dots, \tilde{X}_n(t)\}$ and $\{\tilde{Y}_1(t),\dots, \tilde{Y}_n(t)\}$ stand for the sets of functions $\{\tilde{J}_0(t),\dots, \tilde{J}_{n-1}(t)\}$ and $\{\tilde{K}_0(t),\dots, \tilde{K}_{n-1}(t)\}$, respectively, where $\tilde{J}_i(t)$ and $\tilde{K}_i(t)$, $0\le i\le n$, are constructed from $\tilde{p}_i(t)$, $0\le i\le n$, according to the rules (9) and (10), respectively.

Theorem 1 says that the function $\tilde{p}_0(t)y(t)(=y(t)/\tilde{J}_0(t)=y(t)/\tilde{X}_1(t))$ tends to a limit as $t\to\infty$ if

$$\int^{\infty} \tilde{Y}_n(t)\,|f(t)|dt < \infty \tag{38}$$

and

$$\int^{\infty} \tilde{Y}_n(t)\,|h(c\tilde{X}_n(t))a(t)|dt < \infty \tag{39}$$

for any constant $c\neq 0$. Since (35) implies that (38) and (39) are equivalent to (36) and (37), respectively, the conclusion readily follows.

Let us now consider equation (1) in which L_n satisfies the condition

$$\int_a^{\infty} p_i^{-1}(t)dt < \infty, \quad 1\le i\le n-1. \tag{40}$$

Using (7), define for $0\le i\le n$

$$j_i(t) = p_0^{-1}(t)I_i(\infty, t;\, p_1,\dots, p_i), \tag{41}$$

$$k_i(t) = p_n^{-1}(t)I_i(\infty, t;\, p_{n-1},\dots, p_{n-i}). \tag{42}$$

It is easily verified that $\{j_{n-1}(t),\dots, j_0(t)\}$ and $\{k_{n-1}(t),\dots, k_0(t)\}$ form principal systems for L_n and M_n, respectively. This fact leads to the following corollary to Theorem 2.

COROLLARY 3. *Let $y(t)$ be a nonoscillatory solution of equation* (1) *with L_n satisfying* (40). *Suppose that*

$$\int^{\infty} p_n^{-1}(t)\,|f(t)|dt < \infty$$

and

$$\int^{\infty} p_n^{-1}(t)\,|h(cp_0^{-1}(t))a(t)|dt < \infty$$

for any constant $c\neq 0$. Then $y(t)/j_{n-1}(t)$ approaches a limit, finite or infinite, as $t\to\infty$.

EXAMPLE 3. Consider the equation

$$(t^3y''(t))'' + \frac{\sigma}{t^2}\,y^3(t^{1/6}) = \frac{12+\sigma}{t^3}, \quad t>1, \tag{43}$$

where $\sigma=1$ or -1. Put $L_4y(t)=(t^3y''(t))''$. Then the operator M_4 associated with L_4 (see (34)) coincides with L_4, and integration of $L_4x(t)=0$ shows that

$$\{1/t,\ 1,\ \log t,\ t\}$$

is a principal system for $L_4=M_4$. All the hypotheses of Theorem 2 are satisfied, and so for every nonoscillatory solution $y(t)$ of equation (43), $ty(t)$ tends to a limit as $t\to\infty$. One such solution is $y(t)=1/t^2$.

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