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Regular modules and V-modules

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Introduction and Notation. A ring R is called a (von Neumann) regular ring if for each a in R there exists an x in R such that $a=axa$. The notion of regularity has been extended to modules by D. Fieldhouse [6], R. Ware [20] and J. Zelmanowitz [21]. In this paper, following Zelmanowitz [21], we call a right R-module M *regular* if given any $m \in M$ there exists $f \in \mathrm{Hom}_R(M, R)$ with $mf(m)=m$. O. Villamayor has shown that every simple right R-module is injective if and only if every right ideal of R is an intersection of maximal right ideals. If a ring R satisfies these equivalent conditions, R is called a right V-ring. The notion of V-rings has been extended to modules by V. S. Ramamurthi [16] and H. Tominaga [19]. In this paper, following Tominaga [19], we call a module M_R a *V-module* if every R-submodule is an intersection of maximal R-submodules. Such a module M_R has also been called "co-semisimple" by K. R. Fuller [10]. The connections between the class of regular rings and the class of V-rings are studied by many authors (see the references of [7]).

In this paper, we shall consider the connections between the class of regular modules and the class of V-modules, and we shall study the relationship between these modules and their endomorphism rings. J. Fisher and R. Snider [9, Corollary 1.3] proved that a ring R is regular if and only if R is fully idempotent and every prime factor ring of R is regular. In § 2, we shall extend this result to modules (Theorem 2.3). In § 3, we consider V-modules and their endomorphism rings. We prove that a finitely generated projective module M_R is a V-module if and only if $\mathrm{End}_R(M)$ is a right V-ring and M_R is a self-generator. In § 4, we prove that a module M_R over a $P.I.$-ring R is regular if and only if it is a locally projective V-module (Theorem 4.4). R. Ware [20, Proposition 2.5] proved that if a projective module M_R over a commutative ring R is regular, then every simple homomorphic image of M_R is injective. The converse assertion was proved by V. S. Ramamurthi [16, Theorem 4] and Z. Maoulaoui [14, Proposition 1]. We shall prove this result for general regular modules over commutative rings. Finally, in § 5, we consider fixed subrings of automorphisms. We prove that if G is a finite group of automorphisms of a ring R such that $|G|^{-1} \in R$ and $J(R/I)=0$ for every G-invariant right ideal I of R, then the fixed subring R^G is a right V-ring.

Throughout this paper, R will denote an associative ring with identity and all modules considered are unitary right R-modules. Homomorphisms will be written on the side opposite to that of scalars. For any module M, M^* denotes $\mathrm{Hom}_R(M, R)$, and $S=S(M)$ denotes $\mathrm{End}_R(M)$. We denote by $Z(M)$ and $J(M)$ the singular submodule of M and the Jacobson radical of M, respectively. And we say that M is *semisimple* if $J(M)=0$. The annihilator ideal of M will be denoted by $\mathrm{Ann}_R(M)$: $\mathrm{Ann}_R(M)=\{r \in R \mid Mr=0\}$. The homomorphisms $(\, , \,)$: $M^* \otimes_S M \to R$ with $(f, m)=f(m)$ and $[\, , \,]$: $M \otimes_R M^* \to S$ with $[m, f]=mf$ are R-R-linear and S-S-linear, respectively. As is well known, (S, M^*, M, R) with these homomorphisms forms a Morita context. The images (M^*, M) and $[M, M^*]$ will be denoted by T and Δ, respectively. We denote by $U({}_S M_R)$ (resp. $U(M_R)$) the lattice of S-R-submodules (resp. R-submodules) of M, and by $U_T({}_R R)$ (resp. $U_T({}_R R_R)$) the lattice of all left ideals (resp. ideals) I of R with $TI=I$. Further, $U_\Delta(S_S)$ (resp. $U_\Delta({}_S S_S)$) denotes the lattice of all right ideals (resp. ideals) K of S with $K\Delta=K$. Given R-module M and A, we set $T_M(A)=\sum \{\mathrm{Im}(f) \mid f \in \mathrm{Hom}_R(M, A)\}$.

1. Preliminaries. Let R' be a ring (with or without identity). Following Tominaga [19], we say that a right R'-module $M \neq 0$ is *s-unital* if $u \in uR'$ for any $u \in M$. If $x_1, \dots, x_n$ are arbitrary elements of an s-unital module $M_{R'}$, then there exists $e \in R'$ such that $x_i e = x_i$ for all x_i ([19, Theorem 1]). Following B. Zimmerman-Huisgen [24], we say that a right R-module M is *locally projective* if M satisfies the following condition: For all diagrams

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle g} & & \\ F & \hookrightarrow & M & & \end{array}$$

with exact upper row and a finitely generated submodule F of M there is $g' \in \mathrm{Hom}_R(M, A)$ such that $g \mid F = fg' \mid F$. It is known that M_R is locally projective if and only if M is s-unital as a left Δ-module (see [24]). M_R is called a *self-generator* (resp. a *Σ-self-generator*) if $T_M(A)=A$ for all R-submodules A of M (resp. for all R-submodules A of M^n where n is any positive integer) (see [23]). Now, we begin with the following proposition.

PROPOSITION 1.1. *Let M be a right R-module. Then the following are equivalent*:

1) *M is s-unital as a right T-module.*

2) *The mapping $U_\Delta(S_S) \to U(M_R)$; $I \to IM$, is a lattice isomorphism.*

3) *M is a self-generator and $\Delta M = M$ (or equivalently $MT=M$).*

If M_R is locally projective, we may add:

4) *Every simple homomorphic image of any submodule of M_R is a homomorphic image of M_R.*

PROOF. 1)$\Rightarrow$2). It is clear that for any $L \in U(M_R)$, $L=[L, M^*]M$. Then we have $[L, M^*]=[L, M^*][M, M^*]=[L, M^*]\Delta$, and so $[L, M^*]\in U_\Delta(S_S)$. If $IM=KM$ for $I, K\in U_\Delta(S_S)$, we have $I=(IM, M^*)=(KM, M^*)=K$. Therefore, $U_\Delta(S_S)\to U(M_R)$; $I\to IM$, is a lattice isomorphism.

2)⇒3) and 4). Trivial.

3)⇒1). Since M_R is a self-generator and $M=MT$, for each $m\in M$, we have $m\in mR=T_M(mR)=T_M(mR)T\subseteq mT$.

Next, we assume that M_R is a locally projective module.

4)⇒3). It suffices to show that M is a self-generator. Assume that there is an $N\in U(M_R)$ such that $T_M(N)\neq N$. Let $\bar{x}=x+T_M(N)$ be a non-zero element of $N/T_M(N)$ and let Y be a maximal submodule of $\bar{x}R$. By hypothesis, there is a non-zero R-homomorphism $h\colon M\to\bar{x}R/Y$. Now, we take an $m\in M$ such that $h(m)\neq 0$. Since M_R is locally projective, there is an $h'\in \mathrm{Hom}_R(M, xR+T_M(N))$ such that $h\mid mR=\pi h'\mid mR$, where π is the natural epimorphism: $xR+T_M(N)\to \bar{x}R/Y$. In particular, we have $0=\pi h'(m)=h(m)$. This is a contradiction.

A ring R is called fully right (resp. left) idempotent if $I^2=I$ for every right (resp. left) ideal I of R. And R is fully idempotent if $I^2=I$ for every ideal I of R. Let us call a module M_R *fully idempotent* if for every $m\in M$, $m\in S[m, M^*]mR$. Further M_R is called *fully right idempotent* (resp. *fully left idempotent*) if $m\in [m, M^*]mR$ (resp. $m\in S[m, M^*]m$) (cf. [13], [16]). A ring R is fully idempotent, fully right idempotent or fully left idempotent, according as R_R is .

PROPOSITION 1.2 (cf. [13, Theorem 7]). (1) *The following conditions are equivalent*:

1) *M_R is a fully right idempotent module.*

2) *For every R-submodule N of M, $N=[N, M^*]N$.*

3) *M_T is s-unital and $I^2=I$ for every $I\in U_\Delta(S_S)$.*

4) *M_T is s-unital and $N\cap IM=IN$ for every S-R-submodule N of M and every right ideal I of S.*

5) *M_T is s-unital and ${}_SM/N$ is flat for each S-R-submodule N of M.*

6) *M_T is s-unital and the functor* $\mathrm{Hom}_{R/\mathrm{Ann}_R(M/N)}(M/N, \text{-})$ *from the category* Mod-$R/\mathrm{Ann}_R(M/N)$ *to the category* Mod-S *preserves injective modules for each S-R-submodule N of M.*

(2) *The following conditions are equivalent*:

1) *M_R is a fully idempotent module.*

2) *For every S-R-submodule N of M, $N=[N, M^*]N$.*

3) *The mapping $U_T({}_RR_R)\to U({}_SM_R)$; $I\to MI$, is a lattice isomorphism and $I^2=I$ for each $I\in U_T({}_RR_R)$.*

4) *The mapping $U_T({}_RR_R)\to U({}_SM_R)$; $I\to MI$, is a lattice isomorphism and $N\cap MI=NI$ for every S-R-submodule N of M and for every ideal I of R.*

5) *The mapping $U_\Delta({}_SS_S)\to U({}_SM_R)$; $K\to KM$, is a lattice isomorphism and $K^2=K$ for every $K\in U_\Delta({}_SS_S)$.*

6) *The mapping* $U_\Delta({}_SS_S)\to U({}_SM_R)$; $K\to KM$, *is a lattice isomorphism and* $N\cap KM=KN$ *for every S-R-submodule N of M and for every ideal K of S.*

PROOF. (1). 1)⇔2) is clear.

2)⇒3). For each $m\in M$, we have that $m\in m(M^*, mR)\subseteq mT$. Let I be in $U_\Delta(S_S)$. Then $IM=[IM, M^*]IM=I\Delta IM=I^2M$, and therefore $I=[IM, M^*]=[I^2M, M^*]=I^2$.

3)⇒4). Let N be an S-R-submodule of M, and I a right ideal of S. It is clear that $N\cap IM\supseteq IN$. By Proposition 1.1, $N\cap IM=KM$ for some $K\in U_\Delta(S_S)$. Since $I\supseteq I\Delta\supseteq K\Delta=K$, it follows that $IN\supseteq KN\supseteq K(KM)=KM=N\cap IM$. Therefore, $IN=N\cap IM$.

4)⇔5). Since M_T is s-unital, ${}_SM$ is locally projective, and hence ${}_SM$ is flat (see [24]). Then, it is well known that ${}_SM/N$ is flat if and only if $N\cap IM=IN$ for each right ideal I of S.

5)⇔6). This follows from the well known fact that for a R'-R''-bimodule ${}_{R'}W_{R''}$, ${}_{R'}W$ is flat if and only if $\mathrm{Hom}_{R''}(W, \text{-})$: Mod-$R''\to$Mod-$R'$ preserves injective modules ([5, Proposition 6.28, p. 318]).

4)⇒3). If $I\in U_\Delta(S_S)$, then $IM=SIM\cap IM=ISIM=I^2M$. Therefore we obtain $I=I[M, M^*]=I^2[M, M^*]=I^2$.

3)⇒2). If N is an R-submodule of M, then by Proposition 1.1 there exists some $I\in U_\Delta(S_S)$ such that $N=IM$, and therefore $[N, M^*]N=[IM, M^*]IM=I\Delta IM=I^2M=IM=N$.

(2). 1)⇔2) is clear.

2)⇒3). For each $N\in U({}_SM_R)$, we have that $N=[N, M^*]N=M(M^*, N)$ and $(M^*, N)\in U_T({}_RR_R)$. If I is in $U_T({}_RR_R)$, then $MI=[MI, M^*]MI=MI^2$, and hence $I=(M^*, M)I=(M^*, MI^2)=I^2$.

3)⇒4). Let N be an S-R-submodule of M, and I an ideal of R. Then $N\cap MI=ML$ for some $L\in U_T({}_RR_R)$. Since $I\supseteq TI\supseteq TL=L$, we see that $NI\supseteq NL\supseteq(ML)L=ML=N\cap MI$. Therefore, $NI=N\cap MI$.

4)⇒3). If $I\in U_T({}_RR_R)$, then $MI=MI\cap MI=MI^2$, and hence $I^2=I$.

3)⇒2). For each S-R-submodule N of M, we can find some $I\in U_T({}_RR_R)$, such that $N=MI$. Then $[N, M^*]N=[MI, M^*]MI=MI^2=MI=N$.

Similarly, interchanging R and S, we can prove 2)⇒5)⇒6)⇒5)⇒2).

Finally we state the following propositions without proofs. The proofs of them are similar to those of corresponding propositions in [13] and [21].

PROPOSITION 1.3. *If* M_R *is fully idempotent, then there hold the following*:

(1) $S=\mathrm{End}_R(M)$ *is a semiprime ring.*

(2) *The center of S is a regular ring.*

(3) *If* $S=S_1\oplus S_2\oplus\cdots\oplus S_n$ *with two-sided simple rings* S_i, *then* $M_i=S_iM$ *is S-R-simple and* $M=M_1\oplus\cdots\oplus M_n$.

PROPOSITION 1.4. (1) *$M_R=\oplus_{\alpha\in A}M_\alpha$ is fully right idempotent (resp. fully idempotent) if and only if each M_α is fully right idempotent (resp. fully idempotent).*

(2) *If R is fully right idempotent (resp. fully idempotent) then a projective module M_R is fully right idempotent (resp. fully idempotent).*

2. Regular modules. Following Zelmanowitz [21], we call a module M_R regular if given any $m\in M$ there exists $f\in \mathrm{Hom}_R(M, R)$ with $mf(m)=m$. Obviously, every regular module is locally projective. Moreover, we have the following

PROPOSITION 2.1. *The following conditions are equivalent:*

1) *M_R is a regular module.*

2) *M_R is locally projective and every homomorphic image of M_R is flat.*

3) *M_R is locally projective and for any submodule N of M_R and any left R-module L, the natural homomorphism $N\otimes_R L\to M\otimes_R L$ is a monomorphism.*

4) *M_R is locally projective and $MI\cap N=NI$ for every submodule N of M_R and every left ideal I of R.*

PROOF. By [6, Proposition 8.1] and [21, Theorem 2.3], 1)⇒2)⇒3)⇒4). We show that 4) implies 1). Let m be an element of M. Then $m\in mR\cap M(M^*, m)=m(M^*, m)$.

A module M_R is *prime* (resp. *semiprime*) if for every non-zero elements m, m_1 in M there holds $m(M^*, m_1)\neq 0$ (resp. $m(M^*, m)\neq 0$) (see [22]). It is well known that a ring R is fully idempotent if and only if every factor ring of R is semiprime. For locally projective modules, we have

PROPOSITION 2.2. *Let M_R be a locally projective module. Then the following conditions are equivalent:*

1) *M_R is a fully idempotent module.*

2) *For any S-R-submodule N of M, M/N is a semiprime $R/\mathrm{Ann}_R(M/N)$-module.*

3) *For any S-R-submodule N of M, $R/\mathrm{Ann}_R(M/N)$ is semiprime.*

PROOF. 1)⇒2). If M is fully idempotent, then the $R/\mathrm{Ann}_R(M/N)$-module M/N is also fully idempotent and so semiprime.

2)⇒1). If M_R is not fully idempotent, there is an $m\in M$ such that $m\notin S(m)M^*(m)R$. We set $\bar{M}_{\bar{R}}=M/N$, where $N=S(m)M^*(m)R$ and $\bar{R}=R/\mathrm{Ann}_R(\bar{M})$. Since $\bar{M}_{\bar{R}}$ is semiprime, there exists an $f\in(\bar{M}_{\bar{R}})^*$ such that $\bar{m}(f, \bar{m})\neq 0$. Then, by hypothesis, there is an $f^*\in M^*$ such that $\pi f^*|mR=fg|mR$ where π and g are natural epimorphisms $R\to\bar{R}$ and $M\to\bar{M}$, respectively. But $m(f^*, m)\in N$ implies $\bar{m}(f, \bar{m})=0$. This is a contradiction.

2)⇔3). Let N be a proper S-R-submodule of M and let $\bar{n}=n+N$ be a non-zero element of $\bar{M}=M/N$. Since M_R is locally projective, there are $m_1,\dots, m_k \in M, f_1,\dots,f_k \in M^*$ such that $\sum_{i=1}^{k} m_i f_i(n)=n$. Each f_i induces an element $\bar{f}_i$ in $\mathrm{Hom}_{\bar{R}}(\bar{M}, \bar{R})$, where $\bar{R}=R/\mathrm{Ann}_R(\bar{M})$. Then, $0\neq\bar{n}=\sum_{i=1}^{k}\bar{m}_i\bar{f}_i(\bar{n})\in[\bar{M}, \bar{M}^*](\bar{n})$. Now, 2)⇔3) is clear by [22, Proposition 1.1].

The following theorem is an extension of [9, Corollary 1.3] to modules.

THEOREM 2.3. *The following conditions are equivalent*:

1) *M_R is a regular module.*

2) *M_R is locally projective and fully idempotent, and for each prime ideal P of R, M/MP is a regular R/P-module.*

3) *M_R is locally projective and fully idempotent, and each prime factor module M/N_R $(N\subseteq_s M_R)$ is a regular $\bar{R}$-module, where $\bar{R}=R/\mathrm{Ann}_R(M/N)$.*

PROOF. A proof involves a slight modification of that of [9, Theorem 1.1].

1)⇒2). Trivial.

2)⇒3). If $\bar{M}=M/N_R$ is prime for an S-R-submodule N, then $\bar{R}=R/\mathrm{Ann}_R(\bar{M})$ is a prime ring by [22, Proposition 1.1]. Hence M/N is a regular $\bar{R}$-module by 2).

3)⇒1). We have to show that for each $m\in M$ there exists an $f\in M^*$ such that $m=mf(m)$. Assume, to the contrary, that there exists an $m\in M$ such that $m=mx(m)$ has no solution in M^*. Then, by making use of the fact that M_R is locally projective and Zorn's lemma, we can choose an S-R-submodule N of M which is maximal with respect to the property that $\bar{m}=\bar{m}x(\bar{m})$ has no solution in $\mathrm{Hom}_{\bar{R}}(M/N, \bar{R})$ where $\bar{R}=R/\mathrm{Ann}_R(M/N)$, i.e. $m-mx(m)$ is not in N for every $x\in M^*$. By hypothesis, $\bar{M}=M/N_R$ is not prime. Therefore, there exist non-zero elements $\bar{m}_1$ and $\bar{m}_2$ in $\bar{M}$ such that $[\bar{m}_1, (\bar{M}_{\bar{R}})^*]\bar{m}_2=0$. Since $\bar{M}_{\bar{R}}$ is semi-prime by Proposition 2, it follows that $\overline{Sm_1R}\cap\overline{Sm_2R}=0$. By the choice of N and the fact that M_R is locally projective, there exist x and y in M^* with $m-m(x, m)\in Sm_1R+N$ and $m-m(y, m)\in Sm_2R+N$. Thus $m-m(x+y-x[m, y])m$ is in $(Sm_1R+N)\cap(Sm_2R+N)=N$. This contradicts the choice of N. Consequently M_R is regular and the proof is complete.

COROLLARY 2.4. *Let R be a ring all of whose prime factor rings are regular. Then every locally projective, fully idempotent module is regular.*

PROOF. Since every locally projective module over a regular ring is regular by [24, 2.3, 4)], our assertion is clear by Theorem 2.3.

3. V-modules. It was proved in [15] that for a ring R the following statements are equivalent:

1) Every simple right R-module is injective.

2) Every right R-module is semisimple.

3) Every right ideal of R is an intersection of maximal right ideals of R.

A ring R is called a right V-ring if R satisfies the above equivalent conditions. Following Tominaga [19], we call a module M_R a *V-module* if every R-submodule of M is an intersection of maximal R-submodules. Obviously, a ring R is a right V-ring if and only if the right R-module R_R is a V-module.

Let M be a right R-module. A right R-module N is defined to be *M-injective* in case for each monomorphism $f: K_R \to M_R$ and each homomorphism $g: K_R \to N_R$ there is an R-homomorphism $\bar{g}: M_R \to N_R$ such that $g = \bar{g}f$:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_R & \xrightarrow{f} & M_R \\ & & {\scriptstyle g}\downarrow & \swarrow{\scriptstyle \bar{g}} & \\ & & N_R & & \end{array}$$

The following proposition has been proved in [10, Proposition 3.1]. However we shall reprove it here because of the connection with the proof of Theorem 3.15.

PROPOSITION 3.1. *For a right R-module M the following conditions are equivalent*:

1) *M_R is a V-module.*

2) *Every simple right R-module is M-injective.*

3) *Every homomorphic image of M_R is cogenerated by a direct sum of simple modules.*

PROOF. 1)⇔3). Trivial.

1)⇒2). Let U be a simple right R-module and let f be a nonzero R-homomorphism from a submodule N of M to U. If $N' = \mathrm{Ker} f$, then there is a maximal submodule K of M_R such that $K \supseteq N'$ but $K \not\supseteq N$. Since N/N'_R is simple, it follows that $N \cap K = N'$. Then $M/K = (N+K)/K_R \simeq (N/N \cap K)_R = N/N'_R \simeq U_R$, and therefore f can be extended to an $\bar{f}$ in $\mathrm{Hom}_R(M, U)$. Hence U is M-injective.

2)⇒1). Let N be a proper submodule of M_R, and x a nonzero element of $\overline{M} = M/N$. Then by Zorn's lemma, there is a submodule Y of $\overline{M}_R$ which is maximal among the submodules X of $\overline{M}_R$ with $x \notin X$. Let D denote the intersection of all submodules Q of $\overline{M}_R$ with $Q \supsetneq Y$. Obviously x is in D, and D/Y_R is a simple module. Then by 2), D/Y is M-injective and so, $\overline{M}/Y$-injective by [2, Proposition 16.13, p. 188]. Therefore $\overline{M}/Y = D/Y \oplus K/Y$, where K is a submodule of $\overline{M}_R$. Since x does not belong to K, it follows that Y is a maximal submodule of $\overline{M}_R$. This implies that $\overline{M}_R$ is semisimple.

In case we restrict our attention to locally projective modules, we obtain the following

COROLLARY 3.2. *Let M_R be a locally projective module. Then the following are equivalent:*

1) *M_R is a V-module.*

2) *M_R is a self-generator and every simple homomorphic image of M_R is M-injective.*

3) *M_R is a self-generator and for any simple right R-module X, $\mathrm{Hom}_R(M, X)_S$ is injective.*

PROOF. 1)⇒2). Since every simple homomorphic image of any submodule of M_R is a homomorphic image of M_R (Proposition 3.1), M_R is a self-generator by Proposition 1.1.

2)⇒1). Obvious by Proposition 3.1.

2)⇔3). Since M_R is a Σ-self-generator by [23, Theorem 2.4], the equivalence of 2) and 3) is a consequence of [23, Corollary 1.5] and Proposition 3.1.

The following proposition, noted in [10], is immediate from Proposition 3.1 and [2, Proposition 16.13, p. 188].

PROPOSITION 3.3. (1) *Every submodule and every homomorphic image of a V-module are also V-modules.*

(2) *$\oplus_{\alpha\in A} M_\alpha$ is a V-module if and only if every M_α is a V-module.*

As immediate corollaries to Proposition 3.3, we have the following

COROLLARY 3.4. *Every module which is generated or finitely cogenerated by a V-module is also a V-module.*

COROLLARY 3.5. *Let R be a commutative ring, and M_R a finitely generated V-module. Then $R/\mathrm{Ann}_R(M)$ is a V-ring (and hence a regular ring).*

Let M_R be a module. Then it is clear that $J(M)=0$ if and only if M is cogenerated by the class of simple modules. Therefore, by Proposition 3.3, we have

PROPOSITION 3.6. *Let M_R be a V-module. Then for any submodule N of M_R, $\mathrm{Ann}_R(N)$ and $\mathrm{Ann}_R(M/N)$ are intersections of maximal right ideals of R.*

Every right V-ring is fully right idempotent ([15, Corollary 2.2]). However, V-modules need not be fully right idempotent. For example, any simple right R-module which is not isomorphic to any right ideal of R is not a fully right idempotent module but a V-module. However, we shall show that every locally projective V-module is fully right idempotent. In advance of proving this we shall give some definitions: Let M_R and N_R be two right R-modules. Then N_R

is said to be *p-M-injective* if every R-homomorphism of any cyclic submodule of M_R into N_R can be extended to an R-homomorphism of M_R into N_R. If every simple right R-module is p-M-injective, M_R is called a *p-V-module.* Needless to say every V-module is a p-V-module.

PROPOSITION 3.7. *If M_R is a locally projective, p-V-module, then M_R is fully right idempotent. In particular, every locally projective V-module is fully right idempotent.*

PROOF. Assume, to the contrary, that there exists an $m\in M$ such that $m\notin[m, M^*]mR$. Then, by Zorn's lemma, there is a submodule Y of M_R which is maximal among the submodules X of M_R such that $[m, M^*]mR\subseteq X\subsetneq mR$. We consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & mR & \stackrel{i}{\longrightarrow} & M_R \\ & & \downarrow{\scriptstyle p} & & \\ & & mR/Y & & \end{array}$$

where i is the inclusion map and p is the natural epimorphism. Since the simple right R-module mR/Y is p-M-injective by hypothesis, there is $q\colon M\to mR/Y$ such that $p=qi$. We consider also the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & mR & \stackrel{i}{\longrightarrow} & M & & \\ & & & & \downarrow{\scriptstyle q} & & \\ & & R & \stackrel{h}{\longrightarrow} & mR/Y & \longrightarrow & 0 \end{array}$$

where h is the natural epimorphism. Since M_R is locally projective, there is $j\colon M\to R$ such that $qi=hj\,|\,mR$. Then we have $m+Y=qi(m)=hj(m)=h(1)j(m)$. Since $h(1)j(m)\subseteq[m, M^*]mR+Y=Y$, it follows that $m\in Y$. This is a contradiction.

By the above proof, we can easily see the following

PROPOSITION 3.8. *The following are equivalent:*
1) *M_R is regular.*
2) *M_R is locally projective and every right R-module is p-M-injective.*
3) *M_R is locally projective and for each $m\in M$, mR is p-M-injective.*

For an ideal I of R, an R-module M is called *I-accessible* in case $MI=M$.

PROPOSITION 3.9. *Assume that M_R is quasi-projective or T-accessible. If M_R is a self-generator and S is a right V-ring, then M_R is a V-module.*

PROOF. By [23, Theorem 2.4], M_R is a Σ-self-generator. If X_R is simple, then $\mathrm{Hom}_R(M, X)_S$ is simple or zero by [2, Exercise 18, p. 191] and by [23, Theorem 4.5], and so, by [23, Corollary 1.5], X_R is M-injective. Therefore M_R is a V-module by Proposition 3.1.

The next corresponds to a theorem of R. Ware concerning regular modules (see [20, Theorem 3.9]).

COROLLARY 3.10. *Let R be a commutative ring, and M_R a locally projective module. If S is a right V-ring, then M_R is a V-module.*

PROOF. Since M_R is locally projective over a commutative ring R, M is s-unital as a T-module by [24, 2.3, 3)], and hence M_R is a self-generator by Proposition 1.1. Therefore by Proposition 3.9, M_R is a V-module.

Now we consider the endomorphism ring of a finitely generated, projective V-module.

THEOREM 3.11. *Let M_R be a finitely generated, projective module. Then the following are equivalent:*

1) *M_R is a V-module.*

2) *M_R is a self-generator (or equivalently M_T is s-unital) and S is a right V-ring.*

PROOF. Recall first that every locally projective V-module is a self-generator (Corollary 3.2). Since M_R is finitely generated projective, we see that $\Delta = S$. Assume that M_R is a self-generator. Then, by Proposition 1.1, the lattice $U(S_S)$ is isomorphic to the lattice $U(M_R)$. Therefore S is a right V-ring if and only if M_R is a V-module.

COROLLARY 3.12 (cf. [15, Theorem 2.5]). *If M_R is a finitely generated, projective module over a right V-ring R, then the endomorphism ring S is a right V-ring.*

By Proposition 1.2, we can easily see the following

PROPOSITION 3.13. *Let M_R be a finitely generated projective module. If M_R is fully (right) idempotent, then S is a fully (right) idempotent ring.*

COROLLARY 3.14. *If a finite dimensional, non-singular, projective module M_R is fully right idempotent, then it is a direct sum of finitely many S-R-simple modules. In particular, a noetherian, projective, fully right idempotent module is a direct sum of finitely many S-R-simple modules.*

PROOF. By [22, Theorem 3.5], S is a semiprime right Goldie ring. On the other hand, S is fully right idempotent by Proposition 3.13. Hence S is a direct sum of finitely many simple rings by [15, Lemma 3.1]. Now, our assertion is clear by (3) of Proposition 1.3 and [22, Proposition 3.1].

Rings all of whose singular simple modules are injective are studied in [1] and [17]. For a right R-module M, we obtain the following

THEOREM 3.15. *The following are equivalent:*
1) *Every singular simple right R-module is M-injective.*
2) $Z(M) \cap J(M) = 0$ *and* $J(M/N) = 0$ *for any essential submodule N of* M_R.
3) *Every singular simple submodule of* M_R *is a direct summand of* M_R *and* $J(M/N) = 0$ *for any essential submodule N of* M_R.

PROOF. 1)⇒2). If N is an essential submodule of M_R then, by making use of the same argument as in the proof of 2)⇒1) of Proposition 3.1, we can prove that $J(M/N) = 0$. Now suppose that $Z(M) \cap J(M)$ contains a nonzero element m. Then by Zorn's lemma, there is a submodule Y of M_R which is maximal among the submodules X of M_R with $m \notin X$. Since $\bar{m}R = (mR + Y)/Y$ is a singular simple module, by hypothesis we have $M/Y = \bar{m}R \oplus Y'/Y$ for some submodule Y' of M_R. Since $m \notin Y'$, $Y' = Y$, and hence Y is a maximal submodule of M_R. This contradicts the choice of m.

2)⇒3). Let X be a singular simple submodule of M_R. Since $Z(M) \cap J(M) = 0$, there is a maximal submodule Y of M_R such that $X \cap Y = 0$. Then there holds that $M = X \oplus Y$.

3)⇒1). Let X_R be a singular simple module, and N an essential submodule of M_R with a nonzero R-homomorphism $f: N \to X$. If $K = \mathrm{Ker} f$ is not essential in M, then K is a direct summand of N_R, and so $N = K \oplus I$ for some submodule I of M_R. Since I ($\simeq X$) is a singular simple submodule of M_R, by hypothesis we see that $M = I \oplus L$ for some submodule L_R. Then f can be extended to an R-homomorphism of M to X. If $K = \mathrm{Ker} f$ is essential in M, we can also extend f to an R-homomorphism of M to X (see the proof of 1)⇒2) of Proposition 3.1).

A ring R is called in [17] a generalized V-ring or, for short, a GV-ring if every singular simple right R-module is injective. We call a module M_R a *GV-module* if one of the equivalent conditions in Theorem 3.15 is satisfied. Again by [2, Proposition 16.13, p. 188], we obtain the following

PROPOSITION 3.16. (1) *Every submodule and every homomorphic image of a GV-module are also GV-modules.*
(2) $\oplus_{\alpha \in A} M_\alpha$ *is a GV-module if and only if every* M_α *is a GV-module.*

Since a module M_R is a GV-module if and only if every simple right R-module is either projective or M-injective (see Theorem 3.15), the proof of [17, Proposition 3.4] enables us to obtain the following

PROPOSITION 3.17. *Let R be a ring in which every primitive idempotent is* *central. Then M_R is a V-module if and only if it is a GV-module.*

4. Regular modules versus V-modules. We shall begin this section with the following theorem which corresponds to [7, Theorem 14].

THEOREM 4.1. *Let M_R be a fully right idempotent module. If M/MP_R is a V-module for each primitive ideal P of R, then M_R is a V-module.*

PROOF. Let X_R be a simple module, and N_R a submodule of M_R. Let f be a nonzero element of $\mathrm{Hom}_R(N, X)$. Then $P = \mathrm{Ann}_R(X)$ is a right primitive ideal of R. By Proposition 1.1, $N = AM$ for some $A \in U_\Delta(S_S)$ and $MP = BM$ for some ideal B of S. Noting that $AM \cap BM = ABM = AMP$ (Proposition 1.2 (1)), one will easily see that the map f' defined by $a + b \mapsto f(a)$ $(a \in AM,\ b \in BM)$ is an extension of f in $\mathrm{Hom}_R(AM + BM, X)$. Since R/P is a right primitive ring and $M/\mathrm{Ker} f'$ can be regarded as an R/P-module, we can prove that X is M-injective (see the proof of 1)⇒2) of Proposition 3.1).

We say that R is a *P.I.-ring* if R satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible. Since every primitive factor ring of a $P.I.$-ring R is simple artinian by Kaplansky [12], we obtain the following

COROLLARY 4.2. *Let R be a P.I.-ring. If M_R is fully right idempotent, then M_R is a V-module.*

Now we intend to extend the results in [3] to modules. First, we require the following lemma.

LEMMA 4.3. *Let M_R be a locally projective module. If S is regular and M is s-unital as a right T-module, then M_R is regular.*

PROOF. By Proposition 1.1, for any $m \in M$, there is I in $U_\Delta(S_S)$ with $mR = IM$. Then $m = \sum a_i m_i$ with some $a_i \in I$ and $m_i \in M$. If we set $I' = \sum a_i S$, it is easy to see that $mR = I'M$. Since S is regular, the right ideal I' is generated by an idempotent e. Then mR $(= eM)$ is a direct summand of M_R and is projective. Thus we conclude that M_R is regular by [21, Theorem 2.2].

THEOREM 4.4. *Let R be a P.I.-ring, and M a right R-module. Then the following conditions are equivalent*:

1) *M_R is a regular module.*
2) *M_R is a locally projective V-module.*
3) *M_R is locally projective and fully right idempotent.*

PROOF. 1)⇒2). By Corollary 4.2.

2)⇒3). By Proposition 3.7.

3)⇒1). If M_R is prime, then $R/\mathrm{Ann}_R(M)$ is a prime ring by [22, Proposition 1.1]. Hence, according to Theorem 2.3, it is sufficient to show that a faithful, prime and fully right idempotent module M over a prime $P.I.$-ring R is regular. Let C be the center of R. First we shall show that M is C-torsion-free. Suppose there exists a nonzero $m' \in M$ and a nonzero $c' \in C$ such that $m'c'=0$. Since M_R is faithful, there is a nonzero $m'' \in M$ such that $m''c' \neq 0$. Then, we have $m'(M^*, m''c') = m'c'(M^*, m'') = 0$. This contradicts the primeness of M_R. Since M_R is fully right idempotent, for each $m \in M$ and each nonzero $c \in C$, there are $f_1, \ldots, f_n \in M^*$ and $r_1, \ldots, r_n \in R$ such that $mc = \sum_{i=1}^n mcf_i(mc)r_i = (\sum_{i=1}^n mf_i(m)r_i)c^2$. Hence we can define $mc^{-1} = \sum_{i=1}^n mf_i(m)r_i$, and then M has a Q-module structure, where Q is the ring of central quotients of R. By [18, Corollary 1], Q is a simple artinian ring. Since M_Q is completely reducible, by [21, Theorem 2.8] we may assume that M is an irreducible Q-module. Since $\mathrm{End}_R(M) \simeq \mathrm{End}_Q(M)$ is a division ring by Shur's lemma, M_R is a regular module by Lemma 4.3.

A module M_R is said to be *semi-artinian* if every nonzero homomorphic image of M_R has the nonzero socle. The next is an extension of [7, Theorem 17] to modules.

PROPOSITION 4.5. *Let M_R be a finitely generated, projective, semi-artinian module. Then the following conditions are equivalent*:

1) *M_R is a regular module.*
2) *M_R is a fully right idempotent module.*

PROOF. 1)⇒2). Trivial.

2)⇒1). By Proposition 1.1, the lattice $U(S_S)$ is isomorphic to the lattice $U(M_R)$. Therefore S_S is also semi-artinian. Since S is fully right idempotent (Proposition 3.13), S is regular by [7, Theorem 17], and hence M_R is regular by Lemma 4.3.

As an immediate consequence of Propositions 3.7 and 4.5, we obtain

COROLLARY 4.6. *Let M_R be a finitely generated, projective, semi-artinian module. If M_R is a V-module, then M_R is regular.*

A ring R is said to be *normal* if every idempotent is central. For example, reduced rings and right and left duo rings are normal.

LEMMA 4.7. *Let R be normal. If M_R is a regular module, then every simple homomorphic image of M_R is injective. In particular, M_R is a V-module.*

PROOF. If M_R is regular, then for every $m \in M$, mR is projective and is a direct summand of M_R by [21, Theorem 2.2]. Therefore we may assume that

M_R is cyclic (and projective). Since R is normal, $M_R \simeq eR_R$ for some central idempotent $e \in R$. Since the ring eR is regular and normal, it is a strongly regular ring, and hence a right V-ring by [4, Theorem]. The second assertion is clear by Corollary 3.2.

For a locally projective module M over a commutative ring R, we have

THEOREM 4.8. *Let R be a commutative ring. Then the following conditions are equivalent*:

1) *M_R is regular.*
2) *M_R is a locally projective V-module.*
3) *M_R is a locally projective GV-module.*
4) *M_R is fully right idempotent.*
5) *M_R is locally projective and every simple homomorphic image of M_R is injective.*
6) *M_R is locally projective and every simple homomorphic image of M_R is M-injective.*

PROOF. 1)⇒2). By Corollary 4.2.

2)⇒4). By Proposition 3.7.

2)⇔3). This is included in Proposition 3.17.

4)⇒1). Since M_R is fully right idempotent, for any $m \in M$ we have that $m \in [m, M^*]mR$. Since R is commutative, the right multiplication of any element of R is in S. Therefore $m \in [m, M^*]Sm = [m, M^*]m$. Consequently, M_R is regular.

1)⇒5). By Lemma 4.7.

5)⇒6). Trivial.

6)⇒2). Since M_R is locally projective over a commutative ring R, M is s-unital as a T-module by [24, 2.3, 3)], and hence M is a self-generator by Proposition 1.1. Therefore M_R is a V-module by Corollary 3.2.

REMARK. For a projective module M_R, Ware [20, Proposition 2.5] has proved that 1)⇒5), Ramamurthi [16, Theorem 4] has proved that 5)⇒4)⇒1), and Maoulaoui [14, Proposition 1] has also proved that 5)⇒1).

In case R is a $P.I.$-ring, the implication 1)⇒5) in Theorem 4.8 does not remain valid (in spite of the assertion in Maoulaoui [14, Proposition 2]).

EXAMPLE. Let K be a field. If we set $R=\begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ and $I=\begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$, then R is a $P.I.$-ring and I is a minimal right ideal and is a direct summand of R_R. Therefore I_R is a regular module ([20, Proposition 2.1]). However, I_R is not injective, because the homomorphism $f: \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}_R \to I_R$ defined by $f\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$ can not be extended to a homomorphism of R_R into I_R.

Ware [20, Theorem 3.8] proved that if M is a projective module over a commutative ring R and S is a regular ring, then M_R is regular. We shall generalize this result to locally projective modules (see also [21, Theorem 3.8]).

THEOREM 4.9. *Let R be a commutative ring. If M is a locally projective R-module and S is a regular ring, then M_R is a regular module.*

PROOF. By [24, 2.3, 3)], M is s-unital as a right T-module. Then by Lemma 4.3, M_R is regular.

We conclude this section with the following

PROPOSITION 4.10. *Let M_R be a projective V-module. If M_R is quasi-injective, then M_R is regular.*

PROOF. By [20, Proposition 1.1 (2)], $J(S)\subseteq \mathrm{Hom}_R(M, J(M_R))$. Then $J(M_R)=0$ implies $J(S)=0$. Since M_R is quasi-injective, it is well known that S $(=S/J(S))$ is von Neumann regular ([2, Exercise 28, p. 217]). Hence M_R is regular by Lemma 4.3.

5. Fixed subrings. Let G be a finite group which acts on R (by means of a homomorphism into the automorphism group of R). For $r\in R$ and $g\in G$ we will let r^g denote the image of r under g. The *skew group ring* $R*G$ is defined to be $\oplus_{g\in G}\, gR$ with multiplication given as follows: If r, $s\in R$ and g, $h\in G$, then $(gr)(hs)=ghr^h s$. Throughout this section, U will represent a skew group ring of R with G.

We say that U is *R-projective* if N is a U-submodule of a right U-module M such that N, when viewed as an R-module, is an R direct summand of M, then N is a U direct summand. If the order of G is invertible in R then by the proof of [8, Theorem 1.3] we can easily see that U is R-projective.

THEOREM 5.1. *Assume that $|G|$ is invertible in R. Then the following conditions are equivalent:*

1) *M_U is a V-module.*
2) *For any U-submodule N of M_U, $J(M/N_R)=0$.*

PROOF. 1)$\Rightarrow$2). Let X be a maximal U-submodule of M. Since the simple U-module M/X_U is finitely generated over R, there is a maximal R-submodule Y of M such that $Y\supseteq X$. Then Yg is a maximal R-submodule for every $g\in G$ and there holds that $\bigcap_{g\in G} Yg=X$. Therefore $J(M/N_R)\subseteq J(M/N_U)=0$ for every U-submodule N of M.

2)$\Rightarrow$1). Let X be a U-submodule of M and let x be an element of M such that $x\notin X$. Then by Zorn's lemma, there is a U-submodule Y of M which is maximal among the U-submodules B of M with $x \notin B$. Since $J(M/Y_R)=0$, there is a maximal R-submodule L such that $x \notin L \supseteq Y$. Since $\bigcap_{g\in G} Lg = Y$, we can regard M/Y as an R-submodule of the completely reducible module $\oplus_{g\in G} M/Lg$. Let D denote the intersection of all U-submodules P of M with $P \supsetneqq Y$. Then $x \in D$, and D/Y is a simple U-module. Since U is R-projective, D/Y is a direct summand of M/Y_U. So we can write $D/Y \oplus E/Y = M/Y$ with some U-submodule E of M. Since x does not belong to E, it follows that $E = Y$, and therefore Y is a maximal U-submodule of M.

COROLLARY 5.2. *Assume that $|G|$ is invertible in R. If R is a right V-ring, then U is also a right V-ring.*

Now, we shall consider the fixed subring of automorphisms. In what follows G will be a finite group of automorphisms of R. Then R is a right U-module, where the multiplication of $u = \sum_{g\in G} g t_g \in U$ and $r \in R$ is given by $\sum_{g\in G} r^g t_g$. If the order of G is invertible in R, $e = |G|^{-1} \sum_{g\in G} g$ is an idempotent of U and $R_U \simeq eU_U$ by [8, Corollary 1.4]. A right ideal I of R is said to be G-invariant if $I^g \subseteq I$ for all $g \in G$.

THEOREM 5.3. *Assume that $|G|$ is invertible in R. Then the following are equivalent*:

1) *For any G-invariant right ideal I of R, $J(R/I)=0$.*

2) *The fixed subring R^G is a right V-ring and R is s-unital as a right ReR-module.*

PROOF. 1)⇒2). By 1) and Theorem 5.1 R_U is a V-module. Then, by Theorem 3.11, $\mathrm{End}_U(R)$ is a right V-ring and R is s-unital as a right ReR-module, because R_U is a cyclic, projective U-module and the trace ideal of R_U is $UeU = ReR$. Since $\mathrm{End}_U(R)$ is isomorphic to R^G by [8, Lemma 1.2], R^G is a right V-ring.

2)⇒1). Reversing the above process, we can easily see that 2) implies 1).

COROLLARY 5.4. *Assume that R is a fully right idempotent ring without $|G|$-torsion. If R^G is a right V-ring, then $J(R/I)=0$ for every G-invariant right ideal I of R.*

PROOF. Since R is fully right idempotent, there are r_i, s_i in R such that $|G| = \sum_i |G| r_i |G| s_i$. Since R has no $|G|$-torsion, we have $|G|^{-1}$ in R. By [11, Theorem 1], U is also fully right idempotent. Then by (2) of Proposition 1.4, R_U is fully right idempotent. Therefore, by Theorem 5.3, we see that $J(R/I)=0$ for every G-invariant ideal I of R.

By the above proof and Lemma 1.1, we have

PROPOSITION 5.5. *Assume that R is a fully right idempotent ring without $|G|$-torsion. Then the lattice of right ideals of R^G is isomorphic to the lattice of G-invariant right ideal of R by the homomorphism*: $I \to IR$.

Corresponding to Theorem 5.3, we obtain the following

THEOREM 5.6. *If R has no $|G|$-torsion, then the following are equivalent*:

1) *Every finitely generated G-invariant right ideal of R is a direct summand of R_R.*

2) *Every cyclic G-invariant right ideal of R is a direct summand of R_R.*

3) *R^G is regular and R is s-unital as a right ReR-module.*

PROOF. 1)⇒2). Trivial.

2)⇒3). By 2), $|G|R_R$ is a direct summand of R_R. Since R has no $|G|$-torsion, we have $|G|R=R$, and hence $|G|$ is invertible in R. Since U is R-projective and R_U is projective, 2) is equivalent to that R_U is regular ([21, Theorem 2.2]). Therefore, R^G ($\simeq \mathrm{End}_U(R)$) is regular by [20, Theorem 3.6], and R is s-unital as a right ReR-module by Proposition 1.2.

3)⇒1). Since R^G is a regular ring without $|G|$-torsion, $|G|$ is invertible in R. By Theorem 4.3, R_U is regular, and hence 1) holds by [21, Theorem 2.2].

COROLLARY 5.7. *Assume that R is a fully right idempotent ring without $|G|$-torsion. If R^G is regular, then every finitely generated G-invariant right ideal is a direct summand of R_R.*

PROOF. As was seen in the proof of Corollary 5.4, R_U is fully right idempotent. Therefore by Theorem 5.6, the proof is complete.

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