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Minimal conditions for weak subideals of Lie algebras

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1.

Minimal conditions for subideals of Lie algebras were investigated in [1] and [3]. Among other things the following result was shown:

$$\bigcap_{n=1}^{\infty} \text{Min-}\lhd^{n} \mathfrak{X} = \text{Min-si}\, \mathfrak{X}$$

for any I-closed class $\mathfrak{X}$ of Lie algebras. On the other hand, we introduced the notion of weak subideals in [4], which generalizes the notion of subideals. Thus in this paper we shall establish the similar result concerning the minimal conditions for weak subideals of Lie algebras.

2.

Throughout the paper we employ the notations and terminology in [1] and [4], and all Lie algebras are over a field of arbitrary characteristic.

We denote by Min-wsi (resp. Min-$\leq^{n}$) the class of Lie algebras satisfying the minimal condition for weak subideals (resp. n-step weak subideals). For a class $\mathfrak{X}$ of Lie algebras we denote by Min-wsi $\mathfrak{X}$ (resp. Min-$\leq^{n}\mathfrak{X}$) the class of Lie algebras which satisfy the minimal condition for weak subideals (resp. n-step weak subideals) belonging to $\mathfrak{X}$. Similarly, we define Min-wasc $\mathfrak{X}$ and Min-$\leq^{\alpha}\mathfrak{X}$ where α is an ordinal.

We call a class $\mathfrak{X}$ of Lie algebras WSI-closed if H wsi $L \in \mathfrak{X}$ implies $H \in \mathfrak{X}$. Hence $\mathfrak{X}$ is WSI-closed if it is S-closed.

Now we shall state the following three lemmas, which can be shown easily.

LEMMA 1 ([4]). *Let L be a Lie algebra and let m, n be any integers ≥ 0. Then*:

(a) *If $H \leq^{m} K \leq^{n} L$, then $H \leq^{m+n} L$.*

(b) *If $H \leq^{m} L$ and $K \leq L$, then $H \cap K \leq^{m} K$.*

(c) *Let f be a homomorphism from L onto a Lie algebra $\bar{L}$. If $H \leq^{m} L$, then $f(H) \leq^{m} \bar{L}$.*

LEMMA 2. Min-wsi *is* E-*closed.*

LEMMA 3 ([2]). *If H* wsi *L, then $H^{\omega} = \bigcap_{n=1}^{\infty} H^{n} \lhd L$.*

3.

We shall first show the following

THEOREM 1. Min-wsi $= \bigcap_{n=1}^{\infty}$ Min-$\leq^n$.

PROOF. It is clear that Min-wsi $\leq \bigcap_{n=1}^{\infty}$ Min-$\leq^n$. Assume that Min-wsi $\neq \bigcap_{n=1}^{\infty}$ Min-$\leq^n$ and take a Lie algebra L such that

$$L \notin \text{Min-wsi} \quad \text{and} \quad L \in \bigcap_{n=1}^{\infty} \text{Min-}\leq^n.$$

Then there exists a minimal M with respect to $M \lhd L$ and $M \notin$ Min-wsi. This can be shown by using the fact that $L \in$ Min-$\lhd$.

We now assert that if N wsi M and $N \neq M$ then $N \in$ Min-wsi. In fact, let $N \leq^n M$. Then for any integer $i > 0$

$$N^i \lhd N \leq^n M \lhd L,$$

whence $N^i \leq^{n+2} L$ by Lemma 1. Since $L \in$ Min-$\leq^{n+2}$, there exists an integer $c > 0$ such that

$$N^c = N^{c+1} = \cdots = N^{\omega}.$$

By Lemma 3 we have $N^{\omega} \lhd L$. Hence by the minimality of M

$$N^{\omega} \in \text{Min-wsi}.$$

Since $L \in$ Min-$\leq^{n+3}$, $N^i \in$ Min-$\lhd$ and therefore

$$N^i/N^{i+1} \in \text{Min-}\lhd.$$

Since N^i/N^{i+1} is abelian, it follows that N^i/N^{i+1} is finite-dimensional for $1 \leq i \leq c-1$. Now we use Lemma 2 to see that $N \in$ Min-wsi, as was asserted.

Since $M \notin$ Min-wsi, there exists a strictly descending chain $(H_i)_{i \geq 1}$ of weak subideals of M. By Lemma 1, $(H_i)_{i \geq 2}$ is a strictly descending chain of weak subideals of H_2. This contradicts the fact that $H_2 \in$ Min-wsi, which follows from the assertion in the preceding paragraph since H_2 wsi M and $H_2 \neq M$.

We shall next show the following

THEOREM 2. *For any* WSI-*closed class* $\mathfrak{X}$ *of Lie algebras,*

$$\text{Min-wsi}\,\mathfrak{X} = \bigcap_{n=1}^{\infty} \text{Min-}\leq^n \mathfrak{X}.$$

PROOF. Assume that $L \in \bigcap_{n=1}^{\infty}$ Min-$\leq^n \mathfrak{X}$. Let H be an arbitrary weak subideal of L belonging to $\mathfrak{X}$. Then we assert that $H \in$ Min-$\leq^m$ for any integer

$m>0$. In fact, let $H \leq^n L$ and let $(K_i)_{i \geq 1}$ be a descending chain of m-step weak subideals of H. Then $K_i \leq^{m+n} L$ by Lemma 1, and $K_i \in \mathfrak{X}$ by the wsi-closedness of $\mathfrak{X}$. Since $L \in \text{Min-}\leq^{m+n} \mathfrak{X}$, there exists an integer j such that $K_j = K_{j+1} = \cdots$. Hence $H \in \text{Min-}\leq^m$, as was asserted. Now by Theorem 1 we have

$$H \in \bigcap_{m=1}^{\infty} \text{Min-}\leq^m = \text{Min-wsi}.$$

Especially $H \in \text{Min-wsi}\, \mathfrak{X}$. It follows that

$$L \in \text{Min-wsi}\, \mathfrak{X}.$$

Hence $\bigcap_{n=1}^{\infty} \text{Min-}\leq^n \mathfrak{X} \leq \text{Min-wsi}\, \mathfrak{X}$. The converse is evident.

REMARK. *For any class $\mathfrak{X}$ of Lie algebras,*

$$\text{Min-wasc}\, \mathfrak{X} = \bigcap_{\alpha>0} \text{Min-}\leq^\alpha \mathfrak{X}.$$

In fact, assume that $L \in \bigcap_{\alpha>0} \text{Min-}\leq^\alpha \mathfrak{X}$. Take an ordinal α of cardinality $> \dim L$. Then if H wasc L, $H \leq^\alpha L$. Since $L \in \text{Min-}\leq^\alpha \mathfrak{X}$, we have $L \in \text{Min-wasc}\, \mathfrak{X}$.

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