HIROSHIMA MATH. J.
11 (1981), 425–441

On the behavior at infinity of non-negative superharmonic functions in a half space

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(Received January 19, 1981)

§ 1. Introduction

The aim of this paper is to generalize the results of Essén and Jackson [3].

Let $D=\{x=(x_1, x_2,..., x_p)\in R^p;\ x_1>0\}$, where $p\geqq 2$. Let u be a non-negative superharmonic function on D. Then it is known that u is uniquely decomposed as

$$u(x) = cx_1 + \int_D G(y, x)d\mu(y) + \int_{\partial D} x_1|y-x|^{-p}d\nu(y),$$

where c is a non-negative number, μ (resp. ν) is a (Radon) measure on D (resp. ∂D) (see Helms [4; p. 37 and p. 116]). Lelong-Ferrand [5; Théorème 1c] showed that

$$\lim_{|x|\to\infty, x\in D-E}(u(x)-cx_1)/x_1 = 0,$$

with a set E in D which is minimally thin at ∞. Recently Essén and Jackson [3; Theorem 4.6] proved that

$$\lim_{|x|\to\infty, x\in D-E}(u(x)-cx_1)/|x| = 0,$$

with a set E in D which is rarefied at ∞.

We introduce in § 2 the notion of a-minimal thinness ($0\leqq a\leqq 1$), which is identical to minimal thinness when $a=1$ and which is identical to rarefiedness when $a=0$. As our main result we shall prove in § 3 that

$$\lim_{|x|\to\infty, x\in D-E}(u(x)-cx_1)/(x_1^a|x|^{1-a}) = 0,$$

with a set E in D which is a-minimally thin at ∞. This result is best possible in the sense that if $E\subset D$ is unbounded and a-minimally thin at ∞ in D, then there exists a non-negative superharmonic function u on D such that

$$\lim_{|x|\to\infty, x\in E}(u(x)-cx_1)/(x_1^a|x|^{1-a}) = \infty.$$

We note that if $0\leqq a<a'\leqq 1$, then a set a-minimally thin at ∞ is a'-minimally thin at ∞; however the converse is not necessarily true as shown by an example in § 5. In § 4 the covering theorems, which were proved by Essén and

Jackson [3] for minimally thin sets and rarefied set, will be also obtained for a-minimally thin sets.

While this work has been in prepraration, Mizuta [6] extended our Theorem 3.2 to the case where the Green potential is of general order.

I would like to thank Professor Makoto Ohtsuka and Dr. Yoshihiro Mizuta for their help in preparing this paper.

§ 2. Preliminaries

We introduce the following notation.

(i) As in § 1, D denotes the half space $\{x=(x_1, x_2, \dots, x_p) \in R^p;\ x_1>0\}$, $p \geqq 2$.

(ii) Denote by $\hat{D}=D \cup \Delta$ the Martin compactification of D. Note that the Martin boundary Δ is the Alexandroff compactification of ∂D, i.e., $\Delta=\partial D \cup \{\infty\}$. Every point of Δ is a minimal boundary point. For these facts see e.g. [2; pp. 115–117].

(iii) If $x=(x_1, x_2, \dots, x_p)$, then $x'=(-x_1, x_2, \dots, x_p)$ denotes the reflection of x with respect to ∂D.

(iv) Let $G(x, y)=\phi(|x-y|)-\phi(|x-y'|)$ be the Green kernel for D, and let $G\mu(x)=\int G(y, x)d\mu(y)$ be the Green potential at x of a (Randon) measure μ, where $\phi(r)=r^{2-p}$ for $p \geqq 3$ and $=-\log r$ for $p=2$.

(v) Let $\tilde{K}(y, x)$ be the Martin kernel with reference point $e=(1, 0, \dots, 0)$, more precisely

$$\tilde{K}(y, x) = \begin{cases} 1 & \text{on } \{e\} \times \{e\} \\ G(y, x)/G(y, e) & \text{on } D \times D - \{e\} \times \{e\} \\ \lim_{z \to y, z \in D} G(z, x)/G(z, e) & \text{on } \Delta \times D. \end{cases}$$

We note that $\tilde{K}(y, x)$ is a minimal harmonic function of x for any $y \in \Delta$. It is known that $\tilde{K}(\infty, x)=x_1$ and that $\tilde{K}(y, x)=x_1|y-x|^{-p}|y-e|^p$ for any $y \in \partial D$ (see Brelot [2; p. 116]).

(vi) We introduce a Martin type kernel on $\hat{D} \times D$ as follows:

$$K(y, x) = \begin{cases} G(y, x)/y_1 & \text{on } D \times D \\ 2c_p x_1 |y-x|^{-p} & \text{on } \partial D \times D \\ x_1 & \text{on } \{\infty\} \times D. \end{cases}$$

Here $c_p=1$ when $p=2$, and $c_p=p-2$ when $p \geqq 3$. We note that $K(y, x)$ is continuous in the extended sense on $(D \cup \partial D) \times D$. Following Brelot [1; p. 31], we let $K^*(x, y)=K(y, x)$ be the associated kernel of K on $D \times \hat{D}$.

If μ is a measure on $\hat{D}$ (resp. D), we abbreviate $\int_{\hat{D}} K(y, x)d\mu(y)$ (resp. $\int_D K^*(y, x)d\mu(y)$) to $K\mu(x)$ (resp. $K^*\mu(x)$).

(vii) For a non-negative superharmonic function v on D, there exists a unique measure μ on D such that $v-G\mu$ is harmonic on D. This measure is called the measure associated with v on D (see Helms [4; p. 116]).

(viii) For a non-negative function u on D and $E\subset D$, let $\hat{R}_u^E$ be the regularized reduced function of u relative to E (Helms [4; p. 134]).

(ix) Let $s>1$ be fixed and define

$$I_n = \{x\in D;\ s^n \leqq |x| < s^{n+1}\}.$$

For $E\subset D$, we let $E(n)=E\cap I_n$ and $E(n)'=s^{-n}E_n=\{s^{-n}x;\ x\in E_n\}$.

(x) For a number a, $0\leqq a\leqq 1$, we define $f_a(x) = x_1^a$,

Definition 2.1. A set $E\subset D$ is called minimally thin at $y\in \Delta=\partial D\cup\{\infty\}$ if $\hat{R}^E_{\tilde{K}(y,\cdot)}\neq \tilde{K}(y,\cdot)$ (cf. Brelot [2; p. 122]).

Remark 2.1. From the definition of $\tilde{K}$ and K, we have $K(y, x)=K(y, e)\cdot \tilde{K}(y, x)$ on $(D\cup\{\infty\})\times D$. Since $\tilde{K}$ and K are continuous in the extended sense on $(D\cup\partial D)\times D-\{e\}\times\{e\}$, the equality holds on $\hat{D}\times D$. We note that $K(y, x)$ plays the role of the Martin kernel in the following sense:

1. For any non-negative harmonic function h on D, there exists a unique measure ν on Δ such that

$$h(x) = K\nu(x).$$

From the Riesz decomposition theorem, for any non-negative superharmonic function u on D, there exists a unique measure μ on $\hat{D}$ such that

$$u(x) = K\mu(x)$$

2. K^* potential is lower semi-continuous on $D\cup\partial D$ (see Essén and Jackson [3; p. 240]).

Let $E\subset D$ be a bounded set. Then $\hat{R}^E_{f_1}$ is bounded and $\hat{R}^E_{f_1}\neq f_1$. Since $\tilde{K}(\infty, x)=f_1$ is a minimal harmonic function, the greatest harmonic minorant of $\hat{R}^E_{f_1}$ is zero. Therefore there exists a unique measure λ_E on D such that

$$\hat{R}^E_{f_1} = G\lambda_E \qquad \text{on} \quad D$$

or equivalently

$$y_1^{-1}\hat{R}^E_{f_1}(y) = K^*\lambda_E(y) \qquad \text{on} \quad D. \tag{2.1}$$

Definition 2.2 We call λ_E the fundamental distribution on E (cf. Lelong-Ferrand [5; p. 129] and Essén and Jackson [3; p. 237]).

Let $0 \leqq a \leqq 1$. If E is a bounded set, then $\hat{R}^E_{f_a}$ is a bounded superharmonic function on D. From Remark 2.1 we find a unique measure λ^a_E on $\hat{D} = D \cup \partial D \cup \{\infty\}$ such that $\hat{R}^E_{f_a} = K\lambda^a_E$. However $\hat{R}^E_{f_a}$ is bounded, so that $\lambda^a_E(\{\infty\}) = 0$, i.e. λ^a_E is a bounded (Radon) measure on $D \cup \partial D$.

DEFINITION 2.3. Let E be a bounded set. We define the a-mass of E by $\lambda^a(E) = \lambda^a_E(D \cup \partial D)$ for $0 \leqq a \leqq 1$, where λ^a_E is the measure on $D \cup \partial D$ such that $K\lambda^a_E = \hat{R}^E_{f_a}$.

LEMMA 2.1. *If E and Ω are bounded subsets of D and $E \subset \Omega$, then*

$$\lambda^a(E) = \int_{D \cup \partial D} K^*\lambda_E d\lambda^a_\Omega = \int_D f_a d\lambda_E = \int_D \hat{R}^E_{f_a} d\lambda_\Omega. \tag{2.2}$$

In particular

$$\lambda^1(E) = \int_D f_1 d\lambda_E = \int_D G\lambda_E d\lambda_E \quad \textit{and} \quad \lambda^0(E) = \lambda_E(D).$$

To prove Lemma 2.1 we need the following Lemmas 2.2 and 2.3, which were proved in Essén and Jackson [3; pp. 239–240].

LEMMA 2.2 *Let $E \subset D$. Let u, v be non-negative superharmonic functions in D, and μ^E_v be the measure associated with $\hat{R}^E_v$ in D. Then μ^E_v is concentrated on the set $\{x \in D;\ \hat{R}^E_u(x) = u(x)\}$.*

LEMMA 2.3. *Let $E \subset D$ be a bounded set. Let B_E be the set of points in ∂D at which E is not minimally thin. Then $K^*\lambda_E = 1$ on B_E and $\lambda^0_E(\partial D - B_E) = 0$.*

PROOF OF LEMMA 2.1. By Lemma 2.2, we see that λ_E is concentrated on the set

$$\{x \in D;\ \hat{R}^E_{f_a}(x) = x_1^a\} \subset \{x \in D;\ \hat{R}^\Omega_{f_a}(x) = x_1^a\}.$$

Hence

$$\int_D f_a d\lambda_E = \int_D \hat{R}^\Omega_{f_a} d\lambda_E = \int_D K\lambda^a_\Omega d\lambda_E = \int_{D \cup \partial D} K^*\lambda_E d\lambda^a_\Omega.$$

Since $y_1^{-1}\hat{R}^E_{f_1}(y) = K^*\lambda_E(y)$ on D, by Lemma 2.2 again,

$$\lambda^a_E(\{x \in D;\ \hat{R}^E_{f_1}(x) \neq x_1\}) = \lambda^a_E(\{x \in D;\ K^*\lambda_E(x) \neq 1\}) = 0.$$

If $a = 0$, then by Lemma 2.3

$$\lambda^0_E(\{x \in \partial D;\ K^*\lambda_E(x) \neq 1\}) = 0.$$

If $0 < a \leqq 1$, then the greatest harmonic minorant of $\hat{R}^E_{f_a}$ is zero, so that $\lambda^a_E(\partial D) = 0$. Thus λ^a_E is concentrated on the set

$$\{x \in D \cup \partial D;\ K^*\lambda_E(x) = 1\} \subset \{x \in D \cup \partial D;\ K^*\lambda_\Omega(x) = 1\}$$

in any case. Therefore

$$\lambda^a(E) = \int_{D \cup \partial D} d\lambda_E^a = \int_{D \cup \partial D} K^*\lambda_\Omega d\lambda_E^a = \int_D K\lambda_E^a d\lambda_\Omega = \int_D \hat{R}_{f_a}^E d\lambda_\Omega.$$

Taking $\Omega = E$, this value is also equal to

$$\int_D \hat{R}_{f_a}^E d\lambda_E = \int_D f_a d\lambda_E.$$

Hence we obtain (2.2).

REMARK 2.2. In Essén and Jackson [3], p. 237, the total mass of the fundamental distribution λ_E (resp. λ_E^0) is called the outer charge (resp. Green mass), and the Green energy of the fundamental distribution is called the outer power. Lemma 2.1 shows that the outer charge equals the Green mass, which we call the 0-mass, and that the outer power equals the 1-mass.

Now we shall show some properties of the a-mass.

LEMMA 2.4. *Let Ω be a bounded set in D.*

(i) *If $E \subset F \subset \Omega$, then $\lambda^a(E) \leqq \lambda^a(F)$.*

(ii) *If $E_j \subset \Omega$ and $E_j \uparrow E$, then $\lambda^a(E_j) \uparrow \lambda^a(E)$.*

(iii) *If $E_j \subset \Omega$, then $\lambda^a(\cup_{j=1}^\infty E_j) \leqq \sum_{j=1}^\infty \lambda^a(E_j)$.*

(iv) *If $E \subset \Omega$, then $\lambda^a(E) = \inf\{\lambda^a(O);\ O \supset E,\ O$ is open$\}$.*

PROOF. The properties (i), (ii) and (iii) readily follow from (2.2) and Brelot [2; p. 49]. As in the proof of (2.1) in [3; Lemma 2.1], we see that

$$\int_D \hat{R}_{f_a}^E d\lambda_\Omega = \lim_{n\to\infty} \int_D \hat{R}_{f_a}^{O(n)} d\lambda_\Omega$$

for some decreasing sequence $\{O(n)\}$ of open subsets of D containing E. By our Lemma 2.1, the left hand side is equal to $\lambda^a(E)$ and the right hand side is equal to $\lim_{n\to\infty} \lambda^a(O(n))$. Hence we obtain (iv).

LEMMA 2.5. *If E is a bounded set in D, then*

$$\lambda^a(kE) = k^{p+a-1}\lambda^a(E) \qquad \textit{for} \quad k > 0.$$

PROOF. Let Ω be a bounded set which includes E and kE. We note that $\hat{R}_{f_1}^{kE}(y) = k\hat{R}_{f_1}^E(k^{-1}y)$ and hence $K^*\lambda_{kE}(y) = K^*\lambda_E(k^{-1}y)$. By Remark 2.1, 2 we find that the latter equality holds on $D \cup \partial D$. Since $K(k^{-1}y, x) = k^{p-1}K(y, kx)$, we have from (2.2)

$$\lambda^a(kE) = \int_{D \cup \partial D} K^*\lambda_{kE}(y) d\lambda_\Omega^a(y) = \int_{D \cup \partial D} K^*\lambda_E(k^{-1}y) d\lambda_\Omega^a(y)$$

$$= \int_{D \cup \partial D} \left\{ \int_D K(k^{-1}y, x) d\lambda_E(x) \right\} d\lambda_\Omega^a(y)$$

$$= \int_D \left\{ \int_{D \cup \partial D} k^{p-1} K(y, kx) d\lambda_\Omega^a(y) \right\} d\lambda_E(x)$$

$$= k^{p-1} \int_D \hat{R}^\Omega_{f_a}(kx) d\lambda_E(x) = k^{p-1} \int_D k^a x_1^a d\lambda_E(x) = k^{p+a-1} \lambda^a(E).$$

DEFINITION 2.4. We say that $E \subset D$ is a-minimally thin at ∞ in D if

(2.3) $$\sum_{n=}^{\infty} \lambda^a(E(n)) s^{-n(p+a-1)} < \infty \qquad \text{for some} \quad s > 1.$$

By the aid of Lemma 2.5 we see that E is a-minimally thin at ∞ in D if and only if

(2.3′) $$\sum_{n=1}^{\infty} \lambda^a(E(n)') < \infty, \qquad \text{where} \quad E(n)' = s^{-n}E(n).$$

We shall abbreviate "a-minimally thin at ∞ in D" to "a-min. thin". The following lemma shows that the definition of a-min. thinness is independent of the choice of $s>1$. The proof is similar to that of Lemma 3.1 of Essén and Jackson [3], so we omit the proof.

LMEMA 2.6. *Let $\lambda^a(r)$ denote the a-mass of the set $E \cap \{x \in D; |x|<r\}$. Then E is a-min. thin if and only if*

(2.4) $$\int_1^\infty r^{-p-a} \lambda^a(r) dr < \infty.$$

REMARK 2.3. If $0 \leqq a < a' \leqq 1$, then (2.2) yields

$$\lambda^{a'}(E(n)) = \int_D f_{a'} d\lambda_{E(n)} \leqq s^{(n+1)(a'-a)} \int_D f_a d\lambda_{E(n)} = s^{(n+1)(a'-a)} \lambda^a(E(n))$$

since $s^{n+1} \geqq x_1$ for $x \in E(n)$. Therefore if E is a-min. thin then E is a'-min. thin. The converse is not true in general; but in case E is contained in a cone Γ in D with vertex at the origin and with axis parallel to the x_1 axis, there exists a positive constant C such that $C|x| \leqq x_1 \leqq |x|$ for $x \in \Gamma$ and $C^a s^{an} \lambda^0(E(n)) \leqq \lambda^a(E(n)) \leqq s^{(n+1)a} \lambda^0(E(n))$, so that E is a-min. thin if and only if E is 0-min. thin or equivalently 1-min. thin. Later we shall construct an example of a set which is a'-min. thin but not a-min. thin for $a < a'$ (see Proposition 5.1).

By an elementary caluclation we have the next lemma.

LEMMA 2.7. *If $p \geqq 3$, then there exist positive constants A and B such that*

$$A \leqq G(x, y)/(x_1 y_1 |x-y|^{2-p}|x-y'|^{-2}) \leqq B$$

for all $x, y \in D$. *If* $p=2$ *and* $\delta>0$, *then there exists positive constants* A *and* B_δ *such that*

$$Ax_1 y_1 |x-y|^{-2} \leqq G(x, y) \leqq B_\delta x_1 y_1 |x-y|^{-\delta}|x-y'|^{\delta-2} \qquad \textit{for all} \quad x, y \in D.$$

§ 3. The main results

We consider the function $g_\gamma(x)=x_1|x|^\gamma$, where γ is a real number. By an elementary calculation we find that $\Delta g_\gamma(x)=\gamma(p+\gamma)x_1|x|^{\gamma-2}$. Therefore g_γ is superharmonic in D if and only if $-p\leqq\gamma\leqq 0$. Since $\tilde{K}(\infty, x)=x_1$ is a minimal harmonic function on D, $\min(x_1, x_1|x|^\gamma)$ is a Green potential if $-p\leqq\gamma<0$. We set

$$Gv_\beta(x) = \min(x_1, x_1|x|^{1-p-\beta}) \qquad \text{for} \quad 1-p<\beta\leqq 1.$$

We note that

$$K^*v_\beta(x) = \begin{cases} 1 & \text{for} \quad |x|<1 \\ |x|^{1-p-\beta} & \text{for} \quad |x|\geqq 1. \end{cases} \tag{3.1}$$

Definition 3.1 (cf. [3; Definition 4.1]). For β, $1-p<\beta\leqq 1$, let $\mathcal{S}_\beta$ be the class of all non-negative superharmonic functions u on D for each of which there exists a measure μ on $D \cup \partial D$ such that

$$u(x) = \int_{D\cup\partial D} K(y, x)d\mu(y),$$

$$\int K^*v_\beta(y)d\mu(y) = \int K\mu(x)dv_\beta(x) = \int u(x)dv_\beta(x) < \infty. \tag{3.2}$$

Remark 3.1. If $u_1, u_2 \in \mathcal{S}_\beta$ and c is a positive constant, then u_1+u_2, $cu_1 \in \mathcal{S}_\beta$. If $1-p<\alpha<\beta\leqq 1$, then it is evident that $\mathcal{S}_\alpha \subset \mathcal{S}_\beta$. A non-negative superharmonic function $u(x)$ belongs to $\mathcal{S}_1$ if and only if for any $c>0$, $u(x)$ fails to dominate cx_1 on D (see [3; Remark 4.2]).

Lemma 3.1. *If* $v\in\mathcal{S}_\beta$ *and* u *is a non-negative superharmonic function such that* $0\leqq u\leqq v$ *on* D, *then* $u\in\mathcal{S}_\beta$.

Proof. In view of Remark 3.1, we see that $u\in\mathcal{S}_1$. Hence we find a measure μ on $D\cup\partial D$ such that $u(x)=K\mu(x)$ and $\int u dv_\beta \leqq \int v dv_\beta < \infty$. This proves the lemma.

Lemma 3.2. *If* $u_n\in\mathcal{S}_\beta$ *and* $\sum\int u_n dv_\beta<\infty$, *then* $\sum u_n\in\mathcal{S}_\beta$.

PROOF. By Lebesgue's monotone convergence theorem $\int \sum u_n dv_\beta = \sum \int u_n dv_\beta < \infty$. Therefore it is sufficient to prove $\sum u_n \in \mathscr{S}_1$. Suppose that $\sum u_n(x) \geqq cx_1$ where $c \geqq 0$. Take an arbitrary integer m. Put $\sum_{n=1}^{m} u_n = K\xi$ and $\sum_{n>m} u_n = K\eta$. Then $(\xi+\eta)(\{\infty\}) \geqq c$. Since $u_n \in \mathscr{S}_1$, $\xi(\{\infty\})=0$, so that $\eta(\{\infty\}) \geqq c$. It follows that $\sum_{n>m} u_n(x) \geqq cx_1$. Hence

$$0 \leqq c\int x_1 dv_\beta(x) \leqq \sum_{n>m} \int u_n dv_\beta \to 0 \qquad \text{as} \quad m \to \infty.$$

Thus $c=0$ so that $\sum u_n \in \mathscr{S}_1$.

We investigate relations between the regularized reduced functions and the class $\mathscr{S}_\beta$. Let f be a non-negative function on D and $E \subset D$. Since $E = \cup_n E(n)$, where $E(n) = E \cap I_n$, it is obvious that $\hat{R}_f^E \leqq \sum \hat{R}_f^{E(n)}$, so that $\sum \hat{R}_f^{E(n)} \in \mathscr{S}_\beta$ implies $\hat{R}_f^E \in \mathscr{S}_\beta$. The converse does not hold generally. If we assume

$$\text{(3.3)} \qquad \liminf_{|x|\to\infty} f(x)/(x_1|x|^{\beta-1}) > 0,$$

then we can show the converse.

LEMMA 3.3 *Let $f(x)$ satisfy* (3.3). *If $\hat{R}_f^E \in \mathscr{S}_\beta$, then $\sum_{n=1}^{\infty} \hat{R}_f^{E(n)} \in \mathscr{S}_\beta$.*

PROOF. Since $\hat{R}_f^{E\cap\{y\in E;|y|\geqq 1\}} \leqq \hat{R}_f^E$ and $\{y \in E;\ |y| \geqq 1\}(n) = E(n)$ for $n \geqq 1$, if necessary, taking the intersection $E \cap \{y \in D \cup \partial D;\ |y| \geqq 1\}$, we may assume that E is included in the set $\{y \in D \cup \partial D;\ |y| \geqq 1\}$. Let $\hat{R}_f^E = K\mu$, where μ satisfies (3.2). Since the support of μ is included in the closure of E, $\mu(\{y \in D \cup \partial D;\ |y| < 1\}) = 0$. Noting (3.1), we have

$$\int |y|^{1-p-\beta} d\mu(y) < \infty.$$

We find a positive constant C and a natural number N such that $f(x) \geqq Cx_1|x|^{\beta-1}$ for $|x| > s^N$. Since $\sum_{n=1}^{N} \hat{R}_f^{E(n)} \leqq N\hat{R}_f^E \in \mathscr{S}_\beta$, it is sufficient to prove $\sum_{n>N} \hat{R}_f^{E(n)} \in \mathscr{S}_\beta$. We set $J_{n,k} = I_{n-k} \cup \cdots \cup I_n \cup \cdots \cup I_{n+k}$ for $k>0$. Let $n>N$ and $x \in E(n)$. If $|y| \leqq s^{n-k}$, then $s^k|y| \leqq |x|$ and $|x-y| \geqq (1-s^{-k})|x|$. From Lemma 2.7 we have

$$K(y, x) = G(y, x)/y_1 \leqq \text{Const.}\, x_1|x|^{\beta-1}(1-s^{-k})^{-p}(s^k|y|)^{1-p-\beta}.$$

Hence

$$\int_{|y|\leqq s^{n-k}} K(y, x) d\mu(y) \leqq \text{Const.}\, s^{k(1-p-\beta)} f(x).$$

Similarly we have

$$\int_{|y|\geqq s^{n+k+1}} K(y, x) d\mu(y) \leqq \text{Const.} f(x) \int_{|y|\geqq s^{k+N+1}} |y|^{1-p-\beta} d\mu(y).$$

Since $s^{k(1-p-\beta)}$ and $\int_{|y|\geqq s^{k+N+1}}|y|^{1-p-\beta}d\mu(y)$ tend to 0 as $k\to\infty$, choosing k large, we have

$$\int_{D\cup\partial D-J_{n,k}}K(y, x)d\mu(y)\leqq f(x)/2 \qquad \text{for all} \quad x\in E(n).$$

Since $\hat{R}_f^E\geqq f$ q.e. on $E(n)$, it follows that

$$\int_{J_{n,k}}K(y, x)d\mu(y)\geqq f(x)/2 \qquad \text{q. e. on} \quad E(n).$$

The definition of regularized reduced function yields

$$\int_{J_{n,k}}K(y, x)d\mu(y)\geqq \hat{R}_f^{E(n)}/2 \qquad \text{on} \quad D.$$

If we now sum both sides over n, then we have

$$(2k+1)\hat{R}_f^E=(2k+1)K\mu\geqq 2^{-1}\sum_{n>N}\hat{R}_f^{E(n)}.$$

By Lemma 3.1 we obtain $\sum_{n>N}\hat{R}_f^{E(n)}\in\mathscr{S}_\beta$. Thus the lemma follows.

LEMMA 3.4. *Let $f(x)$ satisfy* (3.3). *Let $W_n=\cup_{k\geqq n}I_k$. If $\{A_j\}$ is a sequence of sets such that $\hat{R}_f^{A_j}\in\mathscr{S}_\beta$, then there exists an increasing sequence $\{n(j)\}$ of natural numbers such that $\sum_j\hat{R}_f^{A(j,n(j))}\in\mathscr{S}_\beta$, where $A(j, n)=A_j\cap W_n$.*

PROOF. Since $\hat{R}_f^{A(j,n)}\leqq\sum_{k\geqq n}\hat{R}_f^{A(j,k)}$, we have $\int\hat{R}_f^{A(j,n)}dv_\beta\to 0$ as $n\to\infty$ by Lemma 3.3. Take an increasing sequence $\{n(j)\}$ such that $\int\hat{R}_f^{A(j,n(j))}dv_\beta<2^{-j}$. Then $\sum_j\int\hat{R}_f^{A(j,n(j))}dv_\beta<1$. By Lemma 3.2 we have $\sum_j\hat{R}_f^{A(j,n(j))}\in\mathscr{S}_\beta$.

THEOREM 3.1 (cf. [3; Theorems 4.4 and 4.5]). *Let $f(x)=x_1^a|x|^{\beta-a}$ with $1-p<\beta\leqq 1$. Then the following three statements are equivalent*:

(i) *E is a-min. thin.*

(ii) $\sum_{n=1}^\infty\hat{R}_f^{E(n)}\in\mathscr{S}_\beta$.

(iii) $\hat{R}_f^E\in\mathscr{S}_\beta$.

PROOF. In this proof, constants of comparison will be positive. Obviously $f(x)=x_1^a|x|^{\beta-a}$ satisfies (3.3). Hence Lemma 3.3 yields that (ii) and (iii) are equivalent. We note

$$\text{Const. }\hat{R}_{f_a}^{E(n)}\leqq s^{-n(\beta-a)}\hat{R}_f^{E(n)}\leqq\text{Const. }\hat{R}_{f_a}^{E(n)},$$

$$\text{Const. }\lambda^a(E(n))\leqq s^{-n(1-p-\beta)}\int|y|^{1-p-\beta}d\lambda_{E(n)}^a(y)\leqq\text{Const. }\lambda^a(E(n)),$$

$$\int\hat{R}_{f_a}^{E(n)}dv_\beta=\int K\lambda_{E(n)}^a dv_\beta=\int K^*v^\beta d\lambda_{E(n)}^a=\int|y|^{1-p-\beta}d\lambda_{E(n)}^a(y).$$

Hence

$$\text{Const. } s^{n(1-a-p)}\lambda^a(E(n)) \leqq \int \hat{R}_f^{E(n)} dv_\beta \leqq \text{Const. } s^{n(1-a-p)}\lambda^a(E(n)),$$

which implies that (i) and (ii) are equivalent.

REMARK 3.2. In the case $a=\beta=1$ Theorem 3.1 shows that E is 1-min. thin if and only if $\hat{R}_{f_1}^E \in \mathscr{S}_1$, which implies that $\hat{R}_{f_1}^E$ is a Green potential. It follows that E is 1-min. thin if and only if E is minimally thin at ∞.

Now we discuss the behavior at ∞ of functions in $\mathscr{S}_\beta$, which is characterized by the use of a-min. thin sets. From Lemma 3.4 we have the following theorem.

THEOREM 3.2 (cf. [3; Theorem 4.7]). *Let* $f(x)=x_1^q|x|^{\beta-a}$. *If* $u \in \mathscr{S}_\beta$, *then there exists a set* $E \subset D$ *which is a-min. thin such that*

$$\lim_{|x|\to\infty, x\in D-E} u(x)/f(x) = 0.$$

Conversely, if E is unbounded and a-min. thin, then there exists $u \in \mathscr{S}_\beta$ *such that*

$$\lim_{|x|\to\infty, x\in E} u(x)/f(x) = \infty.$$

PROOF. We set $A_j=\{x\in D;\ u(x)/f(x) \geqq j^{-1}\}$ for each positive integer j. We note that by Theorem 3.1, A_j is a-min. thin since $\hat{R}_f^{A_j} \leqq ju \in \mathscr{S}_\beta$. Applying Lemma 3.4, we find $\{n(j)\}$ such that $\sum_j \hat{R}_f^{A(j;n(j))} \in \mathscr{S}_\beta$. Set $\bigcup_{j=1}^\infty A(j, n(j))=E$. Since $\hat{R}_f^E \leqq \sum_j \hat{R}_f^{A(j;n(j))}$, $\hat{R}_f^E \in \mathscr{S}_\beta$ and hence E is a-min. thin by Theorem 3.1. If $x\notin E$, then $x \notin A(j, n(j))$ for every j. It follows that if $|x| \geqq s^{n(j)}$, then $x \notin A_j$. This implies that $u(x)/f(x)<j^{-1}$. Thus we have $u(x)/f(x)\to 0$ as $|x|\to\infty$, $x\in D-E$.

For the converse we take an open set $O \supset E$ such that O is a-min. thin. This is possible since λ^a is continuous to the right (Lemma 2.4 (iv)). By Theorem 3.1 we have

$$\sum_{n=1}^\infty \hat{R}_f^{O(n)} \in \mathscr{S}_\beta, \qquad \text{where} \quad O(n) = O \cap I_n,$$

which implies

$$\sum_{n=1}^\infty \int \hat{R}_f^{O(n)} dv_\beta < \infty.$$

We find an increasing sequence $\{c_n\}$ of positive numbers such that $c_n \uparrow \infty$ and

$$\sum_{n=1}^\infty c_n \int \hat{R}_f^{O(n)} dv_\beta < \infty.$$

Set $u=\sum_{n=1}^\infty c_n \hat{R}_f^{O(n)}$. By the aid of Lemma 3.2 we have $u \in \mathscr{S}_\beta$. Since $O(n)$ is included in the interior of $O(n-1) \cup O(n)$,

$$\hat{R}_f^{O(n-1)}(x) + \hat{R}_f^{O(n)}(x) \geq \hat{R}_f^{O(n-1)\cup O(n)}(x) \geqq f(x)$$

for $x \in O(n)$. Hence, if $x \in E(n) \subset O(n)$, then

$$u(x) \geqq c_{n-1}\hat{R}_f^{O(n-1)}(x) + c_n\hat{R}_f^{O(n)}(x) \geqq c_{n-1}f(x).$$

Therefore

$$\lim_{|x|\to\infty, x\in E} u(x)/f(x) = \infty.$$

§4. Covering theorems

In this section we consider only the case $p \geqq 3$. Now we estimate the a-mass of a ball. We use the following notation: $B=B(t, r, R)$ denotes the open ball of radius r and with center $\xi=(t, x_2, \dots, x_p)$, where $t>0$ and $R=|\xi|$. Let $\mathscr{H}$ be the collection of all sets of the form $B \cap D$ with $0<r\leqq tp^{1/2}$. One notes that if $F \subset D \cup \partial D$ is a closed cube with sides parallel to the coordinate axes, then the ball B whose center is the center of F and whose diameter is that of F satisfies $B \cap D \in \mathscr{H}$.

LEMMA 4.1. *Let $H=B \cap D \in \mathscr{H}$, where $B=B(t, r, R)$. Then we have*

$$\text{Const. } t^{1+a}r^{p-2} \leqq \lambda^a(H) \leqq \text{Const. } t^{1+a}r^{p-2}, \tag{4.1}$$

where the constants of comparison are positive.

PROOF. Constants of comparison will be positive in this proof. We start with the case $t>r$; in this case $H=B$. From the minimum principle we have $\hat{R}_{f_1}^{\partial B}=f_1$ on B, which implies that $\hat{R}_{f_1}^{\partial B}=\hat{R}_{f_1}^{B}$ on D. By the aid of Lemma 2.1 we have $\lambda^a(\partial B)=\lambda^a(B)$. Let ξ be the center of B and $y \in \partial B$. Then, noting that supp $\lambda_{\partial B}^a \subset \partial B$ and $t=\xi_1 \leqq |\xi-y'| \leqq$ Const. $\xi_1=$ Const. t, we have from Lemma 2.7

$$\text{Const. } K\lambda_{\partial B}^a(\xi) \leqq t^{-1}\int_{\partial B}|\xi-y|^{2-p}d\lambda_{\partial B}^a(y) \leqq \text{Const. } K\lambda_{\partial B}^a(\xi).$$

Hence,

$$\text{Const. } \hat{R}_{f_a}^{\partial B}(\xi) \leqq t^{-1}r^{2-p}\lambda^a(\partial B) \leqq \text{Const. } \hat{R}_{f_a}^{\partial B}(\xi) \leqq \text{Const. } t^a.$$

Since $\hat{R}_{f_a}^{\partial B}$ is harmonic on B and continuous on $\bar{B}$, it follows that

$$\hat{R}_{f_a}^{\partial B}(\xi) = \sigma_p^{-1}r^{1-p}\int_{\partial B}\hat{R}_{f_a}^{\partial B}(y)d\sigma(y) = \sigma_p^{-1}r^{1-p}\int_{\partial B}y_1^a d\sigma(y) \geqq \text{Const. } t^a,$$

where σ is the surface measure on ∂B and σ_p is the surface area of a unit ball in R^p. Thus we have

$$\text{Const. } t^{a+1}r^{p-2} \leqq \lambda^a(\partial B) = \lambda^a(B) \leqq \text{Const. } t^{a+1}r^{p-2}.$$

If $t \leqq r$, then we have $t \leqq r \leqq tp^{1/2}$. The ball B' whose center lies at the center of B and whose radius equals $r/p^{1/2}$ is contained in B. Let ζ be the projection of the center of B on ∂D. Then $\{x \in D; |x-\zeta|<2r\}$ contains B. By the above argument and Lemmas 2.4 and 2.5, we have

$$\text{Const. } r^{p+a-1} \leqq \lambda^a(B') \leqq \lambda^a(H) \leqq \lambda^a(\{x \in D; |x-\zeta| < 2r\}) \leqq \text{Const. } r^{p+a-1}.$$

Thus the proof of the lemma is complete.

THEOREM 4.1 (cf. [3; Theorem 4.1]). *Set $H_n = B(t_n, r_n, R_n) \cap D$, and suppose $H_n \subset I_n$. Then $E = \cup_n H_n$ is a-min. thin if and only if*

$$\sum_{n=1}^{\infty} (t_n/R_n)^{1+a}(r_n/R_n)^{p-2} < \infty. \tag{4.2}$$

PROOF. Since $H_n \subset I_n$, we have $s^n \leqq R_n \leqq s^{n+1}$. From (4.1) we see that there exist positive constants C and C' such that $Ct_n^{1+a}r_n^{p-2} \leqq \lambda^a(H_n) \leqq C't_n^{1+a}r_n^{p-2}$. Hence Theorem 4.1 follows from Definition 2.4.

DEFINITION 4.1 (cf. [3; Definition 5.1]). Let h: $[0, \infty) \to [0, \infty)$ be a non-decreasing continuous function such that $h(0)=0$. If $H = B(t, r, R) \cap D \in \mathscr{H}$, then we define a premeasure (see Rogers [7; p. 9]) by

$$\tau_h^a(H) = t^{1+a}h(r), \qquad H \in \mathscr{H}.$$

DEFINITION 4.2 (cf. [3; Definition 5.2]). We define

$$L_h^a(E) = \inf_{\{H_\alpha\}} \sum_\alpha \tau_h^a(H_\alpha),$$

where the infimum is taken over all countable coverings $\{H_\alpha\} \subset \mathscr{H}$ of E.

REMARK 4.1. If $h(r) = r^{p-2}$, then there exist positive constants which depend only on p such that Const. $\lambda^a(H) \leqq \tau_h^a(H) \leqq$ Const. $\lambda^a(H)$ (Lemma 4.1). By using the monotone and countably subadditive properties of the a-mass λ^a (Lemma 2.4), we have Const. $\lambda^a(E) \leqq L_h^a(E)$, where the constant is positive.

The same argument as in Essén and Jackson [3; pp. 255–260] shows the following lemma.

LEMMA 4.2. *If E is contained in $\{x \in D; |x| < b\}$, and if h is defined as in Definition 4.1 and satisfies*

$$\int_0^\infty \phi(r)dh(r) < \infty,$$

then there exists a constant C which depends only on p and b such that

$$L_h^a(E) \leqq C\lambda^a(E).$$

THEOREM 4.2 (cf. [3; Theorem 5.1]). *Let $E \subset D$ be a-min. thin. If a function h: $[0, \infty) \to [0, \infty)$ is non-decreasing, continuous and satisfies $h(0)=0$ and*

$$\int_0^\infty \phi(r)dh(r) < \infty,$$

then there exists a covering $\{H_n\}$ in $\mathscr{H}$ of E such that $H_n = B_n \cap D$, $B_n = B(t_n, r_n, R_n)$ and

$$\sum_n (t_n/R_n)^{1+a} h(r_n/R_n) < \infty.$$

PROOF. We recall the definition of a-min. thinnness. From (2.3′) we have $\sum_n \lambda^a(E(n)') < \infty$. Since $E(n)' \subset \{x \in D; |x| < s\}$, applying Lemma 4.2, we have $\sum_n L_h^a(E(n)') \leqq \sum_n C\lambda^a(E(n)') < \infty$. We can cover $E(n)'$ by a sequence $\{H'_{n,j}\}$ in $\mathscr{H}$ such that

$$\sum_j (t'_{n,j})^{1+a} h(r'_{n,j}) < L_h^a(E(n)') + 2^{-n},$$

where $H'_{n,j} = B'_{n,j} \cap D$, $B'_{n,j} = B(t'_{n,j}, r'_{n,j}, R'_{n,j})$, and $R'_{n,j} \geqq 1$. We set $t_{n,j} = s^n t'_{n,j}$, $r_{n,j} = s^n r'_{n,j}$, $R_{n,j} = s^n R'_{n,j}$ and $H_{n,j} = B(t_{n,j}, r_{n,j}, R_{n,j}) \cap D$. We observe $\bigcup_{n,j} H_{n,j} \supset \bigcup_n E(n) = E$, $r'_{n,j} \geqq r_{n,j}/R_{n,j}$ and $t'_{n,j} \geqq t_{n,j}/R_{n,j}$. Since h is non-decreasing,

$$\sum_{n,j} (t_{n,j}/R_{n,j})^{1+a} h(r_{n,j}/R_{n,j}) < \infty.$$

Our proof is now complete.

If $h(r) = r^\beta$ for $0 \leqq r \leqq 1$ and $=1$ for $r > 1$, then $\int_0^\infty \phi(r)dh(r) < \infty$ for each $\beta > p-2$. Hence we have the following corollary.

COROLLARY. *Let $E \subset D$ be a-min. thin. For each $\beta > p-2$, E can be covered by a sequence $\{H_n\}$ in $\mathscr{H}$ such that $H_n = B(t_n, r_n, R_n) \cap D$ and $\sum_n (t/R_n)^{1+a} (r_n/R_n)^\beta < \infty$.*

§5. Examples

In this section we use the following notation.

(i) Let π be the projection from D onto ∂D, namely,

$$\pi(x) = \pi((x_1, x_2, \dots, x_p)) = (0, x_2, \dots, x_p).$$

(ii) For $t > 0$ and a bounded set $A \subset \partial D$ we define $A(t)$ by

$$A(t) = \{x = (x_1, x_2, \dots, x_p) \in D; \pi(x) \in A, 0 < x_1 < t\}.$$

We investigate the behavior of $\lambda^a(A(t))$ when $0 < t < 1$.

LEMMA 5.1. *Let $p \geqq 2$. If A is a bounded subset of ∂D, then there exists a constant C_1 which depends only on p and A such that*

$$\lambda^a(A(t)) \leqq C_1 t^a.$$

PROOF. From Lemma 2.1 we have

$$\lambda^a(A(t)) = \int \hat{R}_{f_a}^{A(t)} d\lambda_{A(1)} \leqq t^a \lambda^0(A(1)).$$

Since $A(1)$ is a bounded set, $\lambda^0(A(1)) < \infty$. Hence the lemma follows.

LEMMA 5.2. *If $p \geqq 3$ and A is a bounded subset of ∂D which has an interior point in ∂D, then there exists a positive constant C_2 which depends only on p and A such that*

$$C_2 t^a \leqq \lambda^a(A(t)).$$

PROOF. Let $A'(t) = \{x = (x_1, x_2, \dots, x_p) \in D;\ \pi(x) \in A,\ x_1 = t/2\}$ and μ_t be the restriction of the $(p-1)$-dimensional Lebesgue measure on $A'(t)$. Let $x, y \in A'(t)$. Put $r = |x-y| = |\pi(x) - \pi(y)|$. Since $|y - x'|^2 \geqq r^2 + t^2$, we have from Lemma 2.7

$$K^*\mu_t(x) \leqq \text{Const.} \int_{A'(t)} [tr^{2-p}/(t^2+r^2)] d\mu_t \leqq \text{Const.} \int_0^\infty [t/(t^2+r^2)] dr = \text{Const.}$$

Therefore we have

$$\lambda^a(A(t)) \geqq \lambda^a(A'(t)) = \int d\lambda^a_{A'(t)} \geqq \text{Const.} \int K^*\mu_t d\lambda^a_{A'(t)}$$

$$= \text{Const.} \int K\lambda^a_{A'(t)} d\mu_t = \text{Const.} \int \hat{R}_{f_a}^{A'(t)} d\mu_t,$$

where the constants are positive. Since μ_t has finite energy, μ_t vanishes on any set of capacity zero. Hence

$$\int \hat{R}_{f_a}^{A'(t)} d\mu_t = \int R_{f_a}^{A'(t)} d\mu_t = (t/2)^a \int d\mu_t \geqq 2^{-1}|A| t^a,$$

where $|A|$ is the $(p-1)$-dimensional Lebesgue measure of A. Since A has an interior point in ∂D, $0 < |A| < \infty$ and the lemma follows.

LEMMA 5.3. *Take any $\varepsilon > 0$. If A is as in Lemma 5.2 and $p = 2$, then there exists a positive constant C_3 which depends only on A and ε such that*

$$C_3 t^{a+\varepsilon} \leqq \lambda^a(A(t)).$$

PROOF. Without loss of generality we may assume that $0 < \varepsilon < 1$. Applying Lemma 2.7 with $\delta = 1 - \varepsilon$, we have

$$K^*(y, x) \leqq \text{Const.}\ y_1 |y-x|^{\varepsilon-1} |y-x'|^{-1-\varepsilon}.$$

Letting $A'(t)$ and μ_t be as in the proof of Lemma 5.2, we have

$$K^*\mu_t(x) \leqq \text{Const.}\, t^{-\varepsilon} \quad \text{on} \quad A'(t),$$

$$\lambda^a(A(t)) \geqq \lambda^a(A'(t)) \geqq C_3 t^{a+\varepsilon},$$

where C_3 is a positive constant which depends only on A and ε.

Lemma 5.3 is not sharp, but if $a=0$, then we have the following lemma.

LEMMA 5.4. *If A is as in Lemma 5.2, then there exists a positive constant C_4 which depends only on p and A such that*

$$C_4 \leqq \lambda^0(A(t)) \qquad \textit{for any} \quad t,\ 0<t<1.$$

PROOF. There exist $r_0>0$ and $\zeta \in \partial D$ such that $U=\{x \in \partial D;\ |x-\zeta|<r_0\} \subset A$. Let $V=\{x \in \partial D;\ |x-\zeta|<r_0/2\}$ and μ be the restriction of the $(p-1)$-dimensional Lebesgue measure on V. We find that $K\mu(x)$ is harmonic in D and continuous on $(D \cup \partial D)-\{x \in \partial D;\ |x-\zeta|=r_0/2\}$ and that

$$K\mu = 1 \quad \text{on} \ V, \qquad K\mu = 0 \quad \text{on} \ \partial D - U,$$

$$0 \leqq K\mu \leqq 1 \quad \text{on} \ D \cup \partial D, \qquad \lim_{|x|\to\infty} K\mu(x) = 0.$$

Since $\hat{R}_1^{U(t)}=1$ on $U(t)$, we have

$$\liminf_{z\to x, z\in D} (\hat{R}_1^{U(t)} - K\mu(z)) \geqq 0$$

for all $x \in \partial D \cup \{\infty\}$. By the maximum principle we have

$$\hat{R}_1^{U(t)} \geqq K\mu \qquad \text{on} \quad D.$$

Since $K^*\lambda_{U(t)}=1$ on V, it follows from Lemma 2.1 that

$$\lambda^0(A(t)) \geqq \lambda^0(U(t)) = \int_D \hat{R}_1^{U(t)} d\lambda_{U(t)} \geqq \int_D K\mu d\lambda_{U(t)}$$

$$= \int_{D\cup\partial D} K^*\lambda_{U(t)} d\mu = \int_{\partial D} K^*\lambda_{U(t)} d\mu = \int_{\partial D} d\mu = |V| > 0.$$

Thus the lemma follows.

We have already shown in Remark 2.3 that if E is contained in a cone Γ with vertex at the origin and with axis parllel to the x_1 axis, then the a-min. thinness of E is equivalent to the minimal thinness. On the other hand $\lambda^a((D-\Gamma)(n)')= \lambda^a((D-\Gamma)(1))>0$ for every positive integer n, and hence $D-\Gamma$ is not a-min. thin. Now we give examples of a-min. thin sets conatained in $D-\Gamma$.

DEFINITION 5.1. Let $h(r)$ be a positive monotone function on $[0, \infty)$. We define D_h by

$$D_h = \{x = (x_1, x_2,\dots, x_p) \in D;\ 0 < x_1 < h(|\pi(x)|\}.$$

REMARK 5.1. From Lemma 5.4 it follows that D_h is not 0-min. thin. In particular a strip domain $\{x \in D;\ 0<x_1<b\}$, $b>0$, is not 0-min. thin.

Hereafter let h be non-decreasing and satisfy

$$\limsup_{r\to\infty} h(r)/r < \infty. \tag{5.1}$$

Take θ, $0<\theta<\pi/2$, and r_0, $r_0>0$, so that $h(r)<r\tan\theta$ for $r\geqq r_0$, i.e. $D_h \cap \{x \in D;\ |x|<r_0\} \subset D_{r\tan\theta}$. Choose s, $s>1$, such that that $s\cos\theta>1$. We set

$$E = D_h,\ F = \{x \in \partial D;\ 1 < |x| < s\cos\theta\},$$
$$G = \{x \in \partial D;\ \cos\theta \leqq |x| \leqq s\}.$$

From the assumption we have $F \subset \pi(E(n)') \subset G$ if n is large. Set $m_n = h(s^n)/s^n$ and $M_n = h(s^{n+1})/s^n$. Then we obtain

$$F(m_n) \subset E(n)' \subset G(M_n) \qquad \text{if} \quad n \text{ is large},$$

where $F(m_n) = \{x \in D;\ \pi(x) \in F,\ 0<x_1<m_n\}$ and $G(M_n) = \{x \in D;\ \pi(x) \in G,\ 0<x_1<M_n\}$. Since the a-mass is monotone, $\lambda^a(F(m_n)) \leqq \lambda^a(E(n)') \leqq \lambda^a(G(M_n))$. By Lemma 5.1 we have $\lambda^a(G(M_n)) \leqq C_1 M_n^a$, so that

$$\lambda^a(E(n)') \leqq C_1 M_n^a \qquad \text{for} \quad p \geqq 2 \tag{5.2}$$

if n is large. Similarly there exist positive constants C_2 and C_3 such that

$$C_2 m_n^a \leqq \lambda^a(E(n)') \qquad \text{for} \quad p \geqq 3, \tag{5.3}$$

$$C_3 m_n^{a+\varepsilon} \leqq \lambda^a(E(n)') \qquad \text{for} \quad p = 2 \quad \text{and} \quad \varepsilon > 0, \tag{5.4}$$

if n is large.

PROPOSITION 5.1. *Let $h(r)$ be a non-negative non-decreasing function on* $[0, \infty)$ *and satisfy* (5.1). *Then D_h is a-min. thin if*

$$\int_1^\infty [(h(r)/r)^a/r]dr < \infty.$$

In case $p \leqq 3$, D_h is not a-min. thin if

$$\int_1^\infty [(h(r)/r)^a/r]dr = \infty.$$

In case $p=2$, D_h is not a-min. thin if

$$\int_1^\infty [(h(r)/r)^{a+\varepsilon}/r]dr = \infty \qquad \textit{for some} \quad \varepsilon > 0.$$

PROOF. Since $h(r)$ is non-decreasing, we have

$$M_n = h(s^{n+1})/s^n \geqq m_n = h(s^n)/s^n = M_{n-1}/s$$

and

$$(h(s^n)/s^{n+1})^a(s^{n+1}-s^n)/s^{n+1} \leqq \int_{s^n}^{s^{n+1}} [(h(r)/r)^a/r]dr \leqq (h(s^{n+1})/s^n)^a(s^{n+1}-s^n)/s^n.$$

Therefore

$$\text{Const.}\ M_{n-1}^a \leqq \int_{s^n}^{s^{n+1}} [(h(r)/r)^a/r]dr \leqq \text{Const.}\ M_n^a \leqq \text{Const.}\ m_{n+1}^a,$$

where the constants of comparison are positive and depend only on a and s. Similarly

$$\int_{s^n}^{s^{n+1}} [(h(r)/r)^{a+\varepsilon}/r]dr \leqq \text{Const.}\ m_{n+1}^{a+\varepsilon}.$$

Using (5.2), (5.3) and (5.4), we obtain the proposition.

REMARK 5.2. If $0<a\leqq 1$ and $h(r)=b$, $b>0$, then h satisfies (5.5). Hence, a strip domain $\{x \in D;\ 0<x_1<b\}$ is a-min. thin for any a, $0<a\leqq 1$. If $h_\alpha(r)=(e/\alpha)^\alpha$ for $0\leqq r<e^\alpha$ and $=r/(\log r)^\alpha$ for $r\geqq e^\alpha$, then $\int_1^\infty [(h_\alpha(r)/r)^a/r]dr<\infty$ if and only if $a\alpha>1$. Therefore in case $p\geqq 3$ D_{h_α} is a-min. thin if and only if $a\alpha>1$, and in case $p=2$ D_{h_α} is a-min. thin if $a\alpha>1$ and D_{h_α} is not a-min. thin if $a\alpha<1$.

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