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Normal limits, half-spherical means and boundary measures of half-space Poisson integrals

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1. Introduction and results

Let D denote the Euclidean half-space $\boldsymbol{R}^n \times (0, +\infty)$, where $n \geq 1$, and let ∂D denote the Euclidean boundary of D. Arbitrary points of D and ∂D are denoted by $M=(X, x)$ and $P=(T, 0)$, respectively, where $X, T \in \boldsymbol{R}^n$ and $x \in (0, +\infty)$. We write $|M|$ for the Euclidean norm of M.

If μ is a signed measure on ∂D such that

$$\int_{\partial D} (1+|P|)^{-n-1} d|\mu|(P) < +\infty, \tag{1}$$

then the Poisson integral I_μ of μ is defined in D by the equation

$$I_\mu(M) = 2(s_{n+1})^{-1} \int_{\partial D} x|M-P|^{-n-1} d\mu(P),$$

where s_{n+1} is the surface-area of the unit sphere in $\boldsymbol{R}^{n+1}$. The condition (1) is necessary and sufficient for I_μ to be harmonic in D (see Flett [5], Theorem 6), and we say that a measure μ on ∂D is of class $\mathscr{F}$ if (1) is satisfied. If, further, μ is non-negative, we write $\mu \in \mathscr{F}^+$.

For each point P of ∂D and each positive number r, we write

$$\sigma(P, r) = \{M \in D \colon |M-P| = r\},$$
$$\tau(P, r) = \{Q \in \partial D \colon |Q-P| < r\},$$

and we denote surface-area measure on $\sigma(P, r)$ or ∂D by s.

If h is the difference of two non-negative harmonic functions in D, then the function $\mathscr{M}(h, P, \cdot)$, defined on $(0, +\infty)$ by

$$\mathscr{M}(h, P, r) = r^{-n-2} \int_{\sigma(P,r)} x h(M) ds(M),$$

is real-valued and continuous on $(0, +\infty)$ and is bounded on any interval of the form $[a, +\infty)$, where $a>0$. In the case where $h \geq 0$ in D, the mean $\mathscr{M}(h, P, r)$ is a decreasing function of r and a convex function of r^{-n-1} on $(0, +\infty)$. Papers dealing with this mean include those of Dinghas [2], [3], Kuran [9], [10] and the author [1].

Here we compare the behaviour of the three functions

$$I_\mu(T, y), \quad \mathscr{M}(I_\mu, P, r), \quad \mu(\tau(P, r))$$

as $y \to 0+$ and $r \to 0+$ for a measure μ of class $\mathscr{F}$. Analogous work on Gauss-Weierstrass integrals has been done by Watson [12]. Some of the work presented here is inspired by [12], but consideration of $\mathscr{M}(I_\mu, \cdot, \cdot)$ in this context seems to be new.

Before proceeding to the main results, we give the following lemma.

LEMMA 1. *If $\nu \in \mathscr{F}^+$ and $P=(T, 0) \in \partial D$, then the following are equivalent:*

(i) $\int_0^1 t^{-n-2}\nu(\tau(P, t))dt = +\infty$,

(ii) $\lim_{y\to 0+} y^{-1}I_\nu(T, y) = +\infty$,

(iii) $\lim_{r\to 0+} \mathscr{M}(I_\nu, P, r) = +\infty$.

THEOREM 1. *Suppose that $\mu \in \mathscr{F}$, $\nu \in \mathscr{F}^+$, that $P=(T, 0) \in \partial D$ and that one of the conditions* (i), (ii), (iii) *in Lemma* 1 *is satisfied. Then*

$$\liminf_{r\to 0+} \frac{\mu(\tau(P, r))}{\nu(\tau(P, r))} \le \liminf_{r\to 0+} \frac{\mathscr{M}(I_\mu, P, r)}{\mathscr{M}(I_\nu, P, r)} \le \liminf_{y\to 0+} \frac{I_\mu(T, y)}{I_\nu(T, y)}$$

$$\le \limsup_{y\to 0+} \frac{I_\mu(T, y)}{I_\nu(T, y)} \le \limsup_{r\to 0+} \frac{\mathscr{M}(I_\mu, P, r)}{\mathscr{M}(I_\nu, P, r)} \le \limsup_{r\to 0+} \frac{\mu(\tau(P, r))}{\nu(\tau(P, r))}.$$

Much work has been done on angular limits and fine limits of ratios of harmonic (and even superharmonic) functions (see, for example, the papers of Brelot-Doob, Doob, and Naïm, cited in [8]). We remark that Theorem 1 fails if the upper and lower normal limits are replaced by angular or fine upper and lower limits. To see this it is enough to consider the case where ν is n-dimensional Lebesgue measure on ∂D (so that $I_\nu \equiv 1$) and μ is n-dimensional Lebesgue measure restricted to that part of ∂D which lies on one side of a hyperplane passing through P and meeting ∂D in an $(n-1)$-dimensional plane. In this case it is clear that $\mu(\tau(P, r))/\nu(\tau(P, r)) \to 1/2$ as $r \to 0+$, but I_μ/I_ν possesses neither an angular limit nor a fine limit at P.

By taking ν in Theorem 1 to be n-dimensional Lebesgue measure on ∂D, we obtain as a corollary the well-known result that if μ possesses a symmetric derivative l at P, then I_μ has a normal limit l at P. (See [8], Theorem 3.1, for a proof of the corresponding result for a ball.) By making other appropriate choices of ν, we obtain the following.

THEOREM 2. *Suppose that $a > 0$, and let f be a real-valued, continuous, increasing function on* $[0, a]$ *such that f is differentiable on* $(0, a)$ *and*

(2) $$\int_0^a t^{-n-2} f(t)dt = +\infty.$$

Define functions ω *and* ξ *on* $(0, a]$ *by*

$$\omega(r) = \begin{cases} (n+1)^{-1} r^{-n-1} f(0) & \text{if } f(0) \neq 0 \\ \int_r^a t^{-n-2} f(t)dt & \text{if } f(0) = 0 \end{cases}$$

and

$$\xi(y) = \begin{cases} 2(s_{n+1})^{-1} y^{-n} f(0) & \text{if } f(0) \neq 0 \\ (2n+2)(s_{n+1})^{-1} y \int_0^a t(y^2+t^2)^{-(n+3)/2} f(t)dt & \text{if } f(0) = 0. \end{cases}$$

If $\mu \in \mathscr{F}$ *and* $P=(T, 0) \in \partial D$, *then*

$$\liminf_{r\to 0+} \frac{\mu(\tau(P, r))}{f(r)} \leq \liminf_{r\to 0+} \frac{\mathscr{M}(I_\mu, P, r)}{\omega(r)} \leq \liminf_{y\to 0+} \frac{I_\mu(T, y)}{\xi(y)}$$

$$\leq \limsup_{y\to 0+} \frac{I_\mu(T, y)}{\xi(y)} \leq \limsup_{r\to 0+} \frac{\mathscr{M}(I_\mu, P, r)}{\omega(r)} \leq \limsup_{r\to 0+} \frac{\mu(\tau(P, r))}{f(r)}.$$

COROLLARY. *Suppose that* $\mu \in \mathscr{F}$ *and that* $P=(T, 0) \in \partial D$. *If* $0 \leq \alpha < n+1$, *then*

$$\liminf_{r\to 0+} r^{-\alpha} \mu(\tau(P, r)) \leq A_{\alpha,n} \liminf_{r\to 0+} r^{n+1-\alpha} \mathscr{M}(I_\mu, P, r)$$
$$\leq B_{\alpha,n} \liminf_{y\to 0+} y^{n-\alpha} I_\mu(T, y) \leq B_{\alpha,n} \limsup_{y\to 0+} y^{n-\alpha} I_\mu(T, y)$$
$$\leq A_{\alpha,n} \limsup_{r\to 0+} r^{n+1-\alpha} \mathscr{M}(I_\mu, P, r) \leq \limsup_{r\to 0+} r^{-\alpha} \mu(\tau(P, r)),$$

where

$$A_{\alpha,n} = n+1-\alpha$$

and

$$B_{\alpha,n} = s_{n+1}(n+1)^{-1}\{\mathrm{B}((\alpha+2)/2, (n+1-\alpha)/2)\}^{-1}.$$

Further (corresponding to the case $\alpha = n+1$*),*

$$\liminf_{r\to 0+} r^{-n-1} \mu(\tau(P, r)) \leq \liminf_{r\to 0+} (\log(1/r))^{-1} \mathscr{M}(I_\mu, P, r)$$
$$\leq 2^{-1} s_{n+1}(n+1)^{-1} \liminf_{y\to 0+} (y \log(1/y))^{-1} I_\mu(T, y)$$
$$\leq 2^{-1} s_{n+1}(n+1)^{-1} \limsup_{y\to 0+} (y \log(1/y))^{-1} I_\mu(T, y)$$
$$\leq \limsup_{r\to 0+} (\log(1/r))^{-1} \mathscr{M}(I_\mu, P, r) \leq \limsup_{r\to 0+} r^{-n-1} \mu(\tau(P, r)).$$

An obvious consequence of the Corollary is that if its hypotheses are satisfied and if either

$$\lim_{r\to 0+} r^{-\alpha}\mu(\tau(P, r))$$

or

$$\lim_{r\to 0+} r^{n+1-\alpha}\mathscr{M}(I_\mu, P, r)$$

exists for some α such that $0\le\alpha<n+1$, then

$$\lim_{y\to 0+} y^{n-\alpha}I_\mu(T, y)$$

exists. We give next some partial converses of these results. For this we require some new terminology. If g is a real-valued function on $(0, +\infty)$, we shall say that g is slowly increasing if for each positive number ε there exists a positive number δ such that

$$g(u) - g(v) > -\varepsilon$$

whenever $(1-\delta)v<u<v<\delta$. Thus the slowly increasing property is a condition on the behaviour of g near 0. In particular, it is easy to show that if the function $u\mapsto ug'(u)$ exists and is bounded above on some non-empty interval $(0, a)$, then g is slowly increasing.

THEOREM 3. *Suppose that* $\mu\in\mathscr{F}$, *that* $P=(T, 0)\in\partial D$ *and that* $0\le\alpha\le n+1$. *Suppose also that*

$$\lim_{y\to 0+} y^{n-\alpha}I_\mu(T, y) = l,$$

where l *is finite.*

(i) *If the function* $r\mapsto r^{n+1-\alpha}\mathscr{M}(I_\mu, P, r)$ *is slowly increasing and is bounded on* $(0, 1]$, *then*

$$\lim_{r\to 0+} r^{n+1-\alpha}\mathscr{M}(I_\mu, P, r) = s_{n+1}l\{(n+1)(n+3)\mathrm{B}((\alpha+2)/2, (n-\alpha+3)/2)\}^{-1}.$$

(ii) *If* $\alpha\ne n+1$ *and the function* $r\mapsto r^{-\alpha}\mu(\tau(P, r))$ *is slowly increasing and is bounded on* $(0, 1]$, *then*

$$\lim_{r\to 0+} r^{-\alpha}\mu(\tau(P, r)) = B_{\alpha,n}l.$$

Our final result limits the size of the set of points $P=(T, 0)$ in ∂D for which $|\mathscr{M}(I_\mu, P, r)|$ grows rapidly as $r\to 0+$ (and, *a fortiori*, limits the size of the set of points for which $|I_\mu(T, y)|$ grows rapidly as $y\to 0+$.) For each γ such that $0<\gamma\le n$, let m_γ denote the Hausdorff measure on ∂D constructed from the function $t\mapsto t^\gamma$. (See, for example, [7].) Then the Hausdorff dimension of a subset E of ∂D is

$$\inf\{\gamma\colon m_\gamma(E) = 0\}.$$

THEOREM 4. *Suppose that* $\mu\in\mathscr{F}$ *and that* $0\le\alpha\le n$. *Let*

$$S_\alpha = \{P \in \partial D\colon \limsup_{r\to 0+} r^{n+1-\alpha}|\mathscr{M}(I_\mu, P, r)| = +\infty\}$$

and

$$T_\alpha = \{P \in \partial D\colon \limsup_{r\to 0+} r^{n+1-\alpha}|\mathscr{M}(I_\mu, P, r)| > 0\}.$$

Then $m_\alpha(S_\alpha)=0$ *and the Hausdorff dimension of* T_α *is at most* α.

2. Preparatory lemmas

Throughout this section, we suppose that $P=(T, 0)\in\partial D$, and given a measure μ on ∂D, we define a function μ^* on $[0, +\infty)$ by writing

$$\mu^*(0) = 0,\ \mu^*(r) = \mu(\tau(P, r)) \qquad (r > 0).$$

LEMMA 2. *If* $\mu\in\mathscr{F}$, *then*

(i) $r^{-n-1}\mu^*(r)\to 0 \qquad (r\to\infty)$,

(ii) $\int_1^\infty t^{-n-2}\mu^*(t)dt$ *exists and is finite*,

(iii) $\mathscr{M}(I_\mu, P, r)$ *is bounded and continuous on each interval* $[a, +\infty)$, *where* $a>0$, *and tends to* 0 *as* $r\to\infty$.

To prove (i), note that if $r>r_0>0$, then

$$|\mu^*(r) - \mu^*(r_0)|r^{-n-1} \le (|\mu|^*(r) - |\mu|^*(r_0))r^{-n-1}$$
$$\le \int_{r_0}^\infty t^{-n-1}d|\mu|^*(t) = \int_{\partial D\setminus\tau(P,r_0)} |P-Q|^{-n-1}d|\mu|(Q).$$

Since $\mu\in\mathscr{F}$, the last integral can be made arbitrarily small by taking r_0 to be sufficiently large, and (i) now follows easily.

If $r_2>r_1>0$, then

$$\left|\int_{r_1}^{r_2} t^{-n-2}\mu^*(t)dt\right| \le \int_{r_1}^{r_2} t^{-n-2}|\mu|^*(t)dt \tag{3}$$
$$= (n+1)^{-1}\left\{r_1^{-n-1}|\mu|^*(r_1) - r_2^{-n-1}|\mu|^*(r_2) + \int_{r_1}^{r_2} t^{-n-1}d|\mu|^*(t)\right\}.$$

Since

$$\int_{r_1}^{r_2} t^{-n-1}d|\mu|^*(t) = \int_{\tau(P,r_2)\setminus\tau(P,r_1)} |P-Q|^{-n-1}d|\mu|(Q),$$

we find, by using (i) and the fact that $\mu\in\mathscr{F}$, that the expression on the left-hand side of (3) can be made arbitrarily small by taking r_1 and r_2 to be sufficiently large. The result (ii) now follows from the general convergence principle for integrals.

By working with μ^+ and μ^- if necessary, we may suppose, in proving (iii), that $\mu \in \mathscr{F}^+$. The continuity of $\mathscr{M}(I_\mu, P, \cdot)$ is proved in [2], Theorem 1. It therefore suffices to note that, by [9], Theorem 4,

$$\mathscr{M}(I_\mu, P, r) \to 0 \qquad (r \to \infty),$$

since there is no positive constant c such that the function $M \mapsto cx$ minorizes I_μ in D.

LEMMA 3. *If $\mu \in \mathscr{F}$ and $r>0$, then*

$$\mathscr{M}(I_\mu, P, r) = \int_r^\infty t^{-n-2}\mu^*(t)dt.$$

LEMMA 4. *If $\mu \in \mathscr{F}$ and $y>0$, then*

$$I_\mu(T, y) = (2n+2)(s_{n+1})^{-1}y\int_0^\infty t(y^2+t^2)^{-(n+3)/2}\mu^*(t)dt$$

$$= 2(n+1)(n+3)(s_{n+1})^{-1}y^3\int_0^\infty t^{n+2}(y^2+t^2)^{-(n+5)/2}\mathscr{M}(I_\mu, P, t)dt.$$

To prove Lemma 3, we note first that by an easily justified change of order of integration

$$\mathscr{M}(I_\mu, P, r) = (s_{n+1})^{-1}r^{-n-2}\int_{\partial D}\int_S x^2|M-Q|^{-n-1}ds(M)d\mu(Q),$$

where S denotes the sphere of centre P and radius r. Next, by noticing that the (ball) Poisson integral of the function $M \mapsto x^2$ on S is the function

$$M \mapsto x^2 - (n+1)^{-1}\{|P-M|^2 - r^2\},$$

we find that the inner integral in the above equation is $s_{n+1}(n+1)^{-1}r$ when $Q \in \tau(P, r)$. Similarly, by using the Poisson integral formula for the complement of a closed ball (see [8], Lemma 9.1), we find that when $Q \in \partial D \backslash \bar{\tau}(P, r)$ the inner integral is $s_{n+1}(n+1)^{-1}r^{n+2}|P-Q|^{-n-1}$. By the monotone convergence theorem, this continues to hold when $Q \in \partial D \backslash \tau(P, r)$. It now follows that

$$\mathscr{M}(I_\mu, P, r) = (n+1)^{-1}\{r^{-n-1}\mu^*(r) + \int_r^\infty t^{-n-1}d\mu^*(t)\}$$

$$= (n+1)^{-1}\lim_{t\to\infty}(t^{-n-1}\mu^*(t)) + \int_r^\infty t^{-n-2}\mu^*(t)dt,$$

and the result follows by Lemma 2(i).

In proving Lemma 4, we may again suppose that $\mu \in \mathscr{F}^+$. To prove the first equation in Lemma 4, we have

$$2^{-1}s_{n+1}I_\mu(T, y) = y\int_{\partial D}(y^2+|P-Q|^2)^{-(n+1)/2}d\mu(Q) = y\int_0^\infty (y^2+t^2)^{-(n+1)/2}d\mu^*(t)$$

$$= y\lim_{r\to\infty}(y^2+r^2)^{-(n+1)/2}\mu^*(r) + (n+1)y\int_0^\infty t(y^2+t^2)^{-(n+3)/2}\mu^*(t)dt.$$

Since, by Lemma 2(i), the limit here is 0, the result follows.

Since μ^* is increasing on $[0, +\infty)$, it follows from Lemma 3 that

$$\mathscr{M}'(I_\mu, P, r) = -r^{-n-2}\mu^*(r)$$

for all but countably many non-negative values of r. Hence by the first equation in Lemma 4,

$$s_{n+1}(2n+2)^{-1}I_\mu(T, y) = -y\int_0^\infty t^{n+3}(y^2+t^2)^{-(n+3)/2}\mathscr{M}'(I_\mu, P, t)dt$$

$$= -y[t^{n+3}(y^2+t^2)^{-(n+3)/2}\mathscr{M}(I_\mu, P, t)]_o^\infty$$

$$+ (n+3)y^3\int_0^\infty t^{n+2}(y^2+t^2)^{-(n+5)/2}\mathscr{M}(I_\mu, P, t)dt.$$

Since $\mathscr{M}(I_\mu, P, t)\to 0$ as $t\to\infty$ and $t^{n+3}\mathscr{M}(I_\mu, P, t)\to 0$ as $t\to 0+$, by Lemmas 2(iii) and 3, the second equation in Lemma 4 follows.

3. Proof of Lemma 1 and Theorem 1

The equivalence of (i) and (iii) in Lemma 1 follows from Lemmas 3 and 2(ii). By Lemma 4, for each positive y

$$y^{-1}I_\nu(T, y) = (2n+2)(s_{n+1})^{-1}\int_0^\infty t(y^2+t^2)^{-(n+3)/2}\nu^*(t)dt,$$

so that, by the monotone convergence theorem,

$$\lim_{y\to 0+} y^{-1}I_\nu(T, y) = (2n+2)(s_{n+1})^{-1}\int_0^\infty t^{-n-2}\nu^*(t)dt.$$

Hence, in view of Lemma 2(ii), we have the equivalence of (i) and (ii).

In proving Theorem 1, we need only consider the inequalities between the upper limits, for if these are proved, the inequalities between the lower limits will follow by working with $-\mu$ instead of μ.

We start with the final inequality. Let the value of the last upper limit be λ. The inequality is trivial if $\lambda = +\infty$. We therefore suppose that $\lambda < +\infty$ and let Λ be such that $\lambda < \Lambda < +\infty$. Then there exists a positive number r_0 such that $u^*(r) < \Lambda\nu^*(r)$ for all r such that $0 < r < r_0$. By Lemmas 3, 2(ii) and 1, if $0 < r < r_0$, then

$$\mathscr{M}(I_\mu, P, r) = \int_r^\infty t^{-n-2}\mu^*(t)dt = \int_r^{r_0} t^{-n-2}\mu^*(t)dt + O(1) \qquad (r \to 0+)$$

$$< \Lambda \int_r^{r_0} t^{-n-2}\nu^*(t)dt + O(1) = (\Lambda + o(1)) \int_r^\infty t^{-n-2}\nu^*(t)dt$$

$$= (\Lambda + o(1))\mathscr{M}(I_\nu, P, r),$$

whence the required inequality follows.

In proving the penultimate inequality in Theorem 1, we may suppose that the value of the penultimate upper limit, k say, is not $+\infty$. Let K be such that $k < K < +\infty$. Then there is a positive number r_1 such that

$$\mathscr{M}(I_\mu, P, t) < K\mathscr{M}(I_\nu, P, t)$$

whenever $0 < t < r_1$. Hence by Lemmas 4, 2(iii) and 1,

$$2^{-1}s_{n+1}(n+1)^{-1}(n+3)^{-1}I_\mu(T, y) = y^3\int_0^{r_1} t^{n+2}(y^2 + t^2)^{-(n+5)/2}\mathscr{M}(I_\mu, P, t)dt$$

$$+ O(y^3) \qquad (y \to 0+)$$

$$< Ky^3 \int_0^{r_1} t^{n+2}(y^2 + t^2)^{-(n+5)/2}\mathscr{M}(I_\nu, P, t)dt + O(y^3)$$

$$= Ky^3\int_0^\infty t^{n+2}(y^2 + t^2)^{-(n+5)/2}\mathscr{M}(I_\nu, P, t)dt + O(y^3)$$

$$= K2^{-1}s_{n+1}(n+1)^{-1}(n+3)^{-1}I_\nu(T, y) + O(y^3)$$

$$= (K + o(1))2^{-1}s_{n+1}(n+1)^{-1}(n+3)^{-1}I_\nu(T, y),$$

and the required inequality follows.

4. Proof of Theorem 2

To deal with the case where $f(0) = 0$, define a measure ν on ∂D by writing

$$\nu(E) = (s_n)^{-1} \int_{E \cap \tau(P,a)} f'(|P-Q|)\,|P-Q|^{1-n}ds(Q),$$

for each Borel subset E of ∂D. Then

$$\nu^*(r) = f(r) \ (0 \le r \le a), \quad \nu^*(r) = f(a) \ (r \ge a),$$

so that condition (i) in Lemma 1 holds. Hence, by Lemmas 4 and 1,

$$I_\nu(T, y) = (2n+2)(s_{n+1})^{-1}y\int_0^a t(y^2 + t^2)^{-(n+3)/2}f(t)dt + O(y) \qquad (y \to 0+)$$

$$= \xi(y) + o(I_\nu(T, y)),$$

so that

$$I_\nu(T, y) = (1 + o(1))\xi(y).$$

Also, by Lemma 3,

$$\mathscr{M}(I_\nu, P, r) = \int_r^a t^{-n-2} f(t)dt + O(1) = (1 + o(1))\omega(r) \qquad (r \to 0+).$$

The case where $f(0)=0$ of Theorem 2 now follows from Theorem 1.

To deal with the case where $f(0)\neq 0$, let ν be $f(0)\delta_p$, where δ_p is the Dirac measure concentrated at P. Then $\nu^*(r)=f(0)$ for all positive r. Also,

$$I_\nu(M) = 2(s_{n+1})^{-1} f(0) x |M-P|^{-n-1} \qquad (M \in D),$$

so that, in particular,

$$I_\nu(T, y) = 2(s_{n+1})^{-1} f(0) y^{-n} = \xi(y),$$

and by Lemma 3,

$$\mathscr{M}(I_\nu, P, r) = f(0)\int_r^\infty t^{-n-2} dt = \omega(r).$$

Again we can apply Theorem 1.

If we take $f(r)=r^\alpha$ for each $r \in [0, 1]$, where $0 \le \alpha \le n+1$, then the conditions of Theorem 2 are fulfilled and for $r \in (0, 1]$

$$\omega(r) = \begin{cases} (n+1)^{-1} r^{-n-1} & (\alpha = 0) \\ (n+1-\alpha)^{-1}(r^{\alpha-n-1} - 1) & (0 < \alpha < n+1) \\ -\log r & (\alpha = n+1). \end{cases}$$

Also, if $y \in (0, 1]$, then in the case where $\alpha=0$

$$\xi(y) = 2(s_{n+1})^{-1} y^{-n},$$

and in the case where $0<\alpha\le n+1$,

$$\begin{aligned} 2^{-1}(n+1)^{-1} s_{n+1}\xi(y) &= y\int_0^1 t^{\alpha+1}(y^2 + t^2)^{-(n+3)/2} dt \\ &= y^{\alpha-n-1}\int_0^1 (t/y)^{\alpha+1}(1 + t^2/y^2)^{-(n+3)/2} dt \\ &= 2^{-1} y^{\alpha-n}\int_0^{1/y^2} u^{\alpha/2}(1+u)^{-(n+3)/2} du, \end{aligned}$$

so that in the case where $\alpha=n+1$

$$\xi(y) = 2(n+1)(s_{n+1})^{-1} y(-\log y + O(1)) \qquad (y \to 0+)$$

and in the case where $0<\alpha<n+1$

$$\xi(y) = (n+1)(s_{n+1})^{-1}y^{\alpha-n}\left\{\int_0^\infty u^{\alpha/2}(1+u)^{-(n+3)/2}du + o(1)\right\} \qquad (y\to 0+)$$

$$= \{(n+1)(s_{n+1})^{-1}\mathrm{B}((\alpha+2)/2, (n-\alpha+1)/2) + o(1)\}y^{\alpha-n}.$$

Collecting together these results and applying Theorem 2, we obtain the Corollary.

5. Proof of Theorem 3

The proof depends on the following form of a Tauberian theorem of Wiener.

THEOREM A. *Let ϕ and ψ be real-valued functions on* $(0, +\infty)$ *such that*

(i) $\int_0^\infty |\phi(t)|dt < +\infty$,

(ii) $\int_0^\infty \phi(t)t^{-iu}dt \neq 0$ *for each real number* u,

(iii) ψ *is bounded on* $(0, +\infty)$ *and slowly increasing,*

(iv) *there is a real number k such that*

$$u^{-1}\int_0^\infty \phi(t/u)\psi(t)dt \longrightarrow k\int_0^\infty \phi(t)dt \qquad (u\to 0+).$$

Then $\psi(u)\to k$ as $u\to 0+$.

This result is given by Hardy [6], Theorems 233 and 235. Note that in Theorem 233 Hardy gives details of the corresponding result in the case where $u\to\infty$. To pass to the case where $u\to 0+$ (Theorem 235), it is necessary to observe that a function ψ on $(0, +\infty)$ is slowly increasing if and only if the function $t\mapsto\psi(t^{-1})$ is slowly decreasing in the sense of [6], § 6.2.

To prove Theorem 3(i), we note that, by Lemmas 4 and 2(iii),

$$(4) \quad y^{n-\alpha}I_\mu(T, y) = 2(n+1)(n+3)(s_{n+1})^{-1}y^{n+3-\alpha}\int_0^1 t^{n+2}(y^2+t^2)^{-(n+5)/2}$$

$$\times \mathscr{M}(I_\mu, P, t)dt + o(1) \qquad (y\to 0+)$$

$$= 2(n+1)(n+3)(s_{n+1})^{-1}y^{-1}\int_0^\infty \phi_1(t/y)\psi_1(t)dt + o(1),$$

where

$$\phi_1(t) = t^{\alpha+1}(1+t^2)^{-(n+5)/2} \qquad (t \geq 0)$$

and

$$\psi_1(t) = \begin{cases} t^{n+1-\alpha}\mathscr{M}(I_\mu, P, t) & (0 < t \leq 1) \\ 0 & (t > 1). \end{cases}$$

The functions ϕ_1 and ψ_1 satisfy the hypotheses (i) and (iii) of Theorem A. Also, by (4), the hypothesis (iv) is satisfied with

$$k = s_{n+1} l\{2(n+1)(n+3)\int_0^\infty \phi_1(t)dt\}^{-1}$$
$$= s_{n+1} l\{(n+1)(n+3)\mathrm{B}((\alpha+2)/2, (n-\alpha+3)/2)\}^{-1}.$$

The hypothesis (ii) is also satisfied, since

$$\int_0^\infty \phi_1(t)t^{-iu}dt = \int_{-\infty}^\infty e^v\phi_1(e^v)e^{-iuv}dv$$
$$= 2^{-1}\mathrm{B}((n+3-\alpha+iu)/2, (2+\alpha-iu)/2) \neq 0.$$

(See, for example, [4], p. 120, formula (21).) We can now apply Theorem A to obtain $\psi_1(r)\to k$ as $r\to 0+$, which is the required result.

To prove Theorem 3(ii), we note first that by Lemmas 4 and 2(ii),

$$y^{n-\alpha}I_\mu(T, y) = (2n+2)(s_{n+1})^{-1}y^{n+1-\alpha}\int_0^1 t(y^2+t^2)^{-(n+3)/2}\mu^*(t)dt + o(1)$$
$$(5) \qquad = (2n+2)(s_{n+1})^{-1}y^{-1}\int_0^\infty \phi_2(t/y)\psi_2(t)dt + o(1),$$

where

$$\phi_2(t) = t^{\alpha+1}(1+t^2)^{-(n+3)/2} \qquad (t \geq 0)$$

and

$$\psi_2(t) = \begin{cases} t^{-\alpha}\mu^*(t) & (0 < t \leq 1) \\ 0 & (t > 1). \end{cases}$$

The functions ϕ_2 and ψ_2 satisfy the hypotheses (i) and (iii) of Theorem A. Also, by (5), the hypothesis (iv) is satisfied with

$$k = s_{n+1}l\{(2n+2)\int_0^\infty \phi_2(t)dt\}^{-1} = B_{\alpha,n}l.$$

The hypothesis (ii) is also satisfied, since

$$\int_0^\infty \phi_2(t)t^{-iu}dt = \int_{-\infty}^\infty e^v\phi_2(e^v)e^{-iuv}dv$$
$$= 2^{-1}\mathrm{B}((n+1-\alpha+iu)/2, (2+\alpha-iu)/2) \neq 0$$

([4], *loc. cit.*). We can now apply Theorem A to obtain $\psi_2(r)\to k$ as $r\to 0+$, which is the required result.

6. Proof of Theorem 4

This proof is borrowed from [12]. We can assume that S_α and T_α are bounded. By the corollary of Theorem 2,

$$\limsup_{r\to 0+} r^{n+1-\alpha}|\mathscr{M}(I_\mu, P, r)| \leq \limsup_{r\to 0+} r^{n+1-\alpha}\mathscr{M}(I_{|\mu|}, P, r)$$
$$\leq (A_{\alpha,n})^{-1} \limsup_{r\to 0+} r^{-\alpha}|\mu|(\tau(P, r)).$$

Further, if $J(P, r)$ denotes the closed n-dimensional cube in ∂D with centre P, edge length $2r$ and faces orthogonal to the co-ordinate axes, then

$$\limsup_{r\to 0+} r^{-\alpha}|\mu|(\tau(P, r)) \leq (2\sqrt{n})^\alpha \limsup_{r\to 0+} (2r\sqrt{n})^{-\alpha}|\mu|(J(P, r))$$
$$\leq (2\sqrt{n})^\alpha \lim_{\delta\to 0+} [\sup_J \{(\operatorname{diam}(J))^{-\alpha}|\mu|(J) \colon P\in J, \operatorname{diam}(J) < \delta\}]$$

where J denotes any non-trivial, n-dimensional interval in ∂D. Hence if Z denotes the set where this last limit is infinite, we have $S_\alpha \subseteq Z$. By [11], Lemma 4, $m_\alpha(Z) = 0$, so that $m_\alpha(S_\alpha) = 0$.

The result for T_α is trivial when $\alpha = n$, and when $0 \leq \alpha < n$ it follows from the fact that $T_\alpha \subseteq S_\beta$ for each β such that $\alpha < \beta \leq n$.

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