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A characterization of uniform convexity and applications to accretive operators

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1. Introduction

It is well known that geometric properties of a Banach space X correspond to analytic properties of the (normalized) duality mapping $J: X \to 2^{X^*}$. For instance X is strictly convex if and only if J is strictly monotone; X^* is uniformly convex if and only if J is singlevalued and uniformly continuous on bounded subsets of X. On the other hand, geometric properties of Banach spaces enter the theory of accretive operators, since accretiveness is defined via the duality mapping, and additional properties of J induce better behaviour of accretive operators.

In this paper we shall prove that a Banach space X is uniformly convex if and only if its duality mapping is in some sense uniformly strictly monotone. This result will then be applied to accretive operator theory. It is known that an accretive operator A is locally bounded at interior points of its domain $D(A)$, provided X^* is uniformly convex. We shall show that this statement is also true in case X is uniformly convex. Furthermore, we shall extend a criterion for maximal monotonicity of the sum of two maximal monotone operators on Hilbert spaces to accretive operators in case X and X^* are uniformly convex.

2. A characterization of uniform convexity

Let X be a real Banach space with norm $|\cdot|$; then X^* is the normal dual of X and (x, x^*) denotes the value of x^* at $x \in X$. The open ball in X (resp. X^*) with center x (resp. x^*) and radius $r > 0$ is denoted by $B_r(x)$ (resp. $B_r^*(x^*)$). Also, their closures are denoted by $\bar{B}_r(x)$ and $\bar{B}_r^*(x^*)$, respectively.

Recall that X is said to be *uniformly convex* if to each $\varepsilon \in (0, 2]$ there exists $\delta(\varepsilon) > 0$ such that

$$|x+y| \leq 2(1-\delta) \quad \text{whenever} \quad |x| = |y| = 1 \quad \text{and} \quad |x-y| \geq \varepsilon.$$

The (normalized) *duality mapping* $J: X \to 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^*: (x, x^*) = |x|^2, |x| = |x^*|\}.$$

It is well known that J is an everywhere defined monotone operator which is weakly* upper semicontinuous and, by the Theorem of Bishop and Phelps [4], has dense range in X^*. Moreover, by James' characterization of reflexivity [4], J is surjective if and only if X is reflexive. J is singlevalued and uniformly continuous on bounded subsets of X if and only if X^* is uniformly convex. X is strictly convex if and only if J is strictly monotone, i.e. $(x-y, x^*-y^*)>0$ for all $x, y \in X$, $x^* \in Jx$, $y^* \in Jy$ with $x \neq y$. These facts are proved for instance in the book of Cioranescu [3]; cp. also [10].

We shall use the

DEFINITION. A function ω: $[0, \infty) \to [0, \infty)$ belongs to *class* $\mathscr{C}$ if ω is nondecreasing, $\omega(\rho)>0$ for $\rho>0$ and $\omega(0)=0$.

Now we are ready to give our characterization of uniformly convex spaces.

THEOREM 1. *A Banach space X is uniformly convex if and only if to each $R>0$ there is a function ω_R of class $\mathscr{C}$ such that*

$$(x-y,\, x^*-y^*) \geq \omega_R(|x-y|)\,|x-y| \qquad \textit{for all} \quad x,\, y \in \bar{B}_R(0), \quad x^* \in Jx, \quad y^* \in Jy. \tag{1}$$

PROOF. Let X be uniformly convex and $R>0$. Let $\omega(0)=0$,

$$\omega(\rho) = \inf\{|x-y|^{-1}\cdot(x-y,\, x^*-y^*): \quad x,\, y \in \bar{B}_R(0),\ |x-y| \geq \rho,\ x^* \in Jx,\ y^* \in Jy\}$$

for $\rho \in (0, 2R]$, and $\omega(\rho)=\omega(2R)$ for $\rho \geq 2R$. Obviously, ω is nondecreasing and so it remains to prove $\omega(\rho)>0$ for $\rho>0$. Assume on the contrary that $\omega(\rho)=0$ for some $0<\rho\leq 2R$. Then there are sequences (x_n), $(y_n) \subset \bar{B}_R(0)$, $x_n^* \in Jx_n$, $y_n^* \in Jy_n$ such that $|x_n-y_n| \geq \rho$ and $(|x_n|-|y_n|)^2 \leq (x_n-y_n,\, x_n^*-y_n^*) \to 0$. Hence we may assume $\lim_{n\to\infty}|x_n| = \lim_{n\to\infty}|y_n| = a>0$. Then

$$\liminf_{n\to\infty}\big|\,|x_n|^{-1}x_n - |y_n|^{-1}y_n\big| = a^{-1}\liminf_{n\to\infty}|x_n-y_n| \geq \rho/a,$$

and therefore, by the uniform convexity of X,

$$\limsup_{n\to\infty}|x_n+y_n| = a\limsup_{n\to\infty}\big|\,|x_n|^{-1}x_n + |y_n|^{-1}y_n\big| \leq 2a(1-\delta)$$

for some $\delta>0$. On the other hand,

$$|x_n|^2 + |y_n|^2 - (x_n,\, y_n^*) - (y_n,\, x_n^*) = (x_n-y_n,\, x_n^*-y_n^*) \to 0 \quad \text{as} \quad n\to\infty,$$

hence $\lim_{n\to\infty}(x_n+y_n,\, x_n^*) = 2a^2$, which leads to the contradiction

$$2a^2 = \lim_{n\to\infty}(x_n+y_n,\, x_n^*) \leq a\limsup_{n\to\infty}|x_n+y_n| \leq 2a^2(1-\delta).$$

To prove sufficiency, we first show that the range of J is closed. Let $x_n^* \in Jx_n$

and $x_n^* \to x^*$. Then (x_n) is bounded, say $|x_n| \leq R$, and by hypotheses we find $\omega_R \in \mathscr{C}$ such that

$$\omega_R(|x_n - x_m|)\,|x_n - x_m| \leq (x_n - x_m,\, x_n^* - x_m^*) \leq |x_n - x_m| \cdot |x_n^* - x_m^*|.$$

Hence (x_n) is a Cauchy-sequence and therefore $x_n \to x$ for some $x \in X$, $x^* \in Jx$, and so J has closed range. Thus J is surjective since $R(J)$ is also dense in X^*, which in turn implies that X is reflexive. In this case we have $x^* \in Jx$ if and only if $x \in J^* x^*$, where J^* denotes the duality mapping of X^*, and so $J^* = J^{-1}$. Now $\omega_R(|x-y|) \leq |x^* - y^*|$ for $x^*, y^* \in \bar{B}_R^*(0)$, $x \in J^* x^*$, $y \in J^* y^*$, hence J^* is single-valued and uniformly continuous on bounded subsets of X^*. Therefore, $X = (X^*)^*$ is uniformly convex and the proof is complete. q.e.d.

3. Local boundedness of accretive operators

Let A be a (possibly multivalued) operator in X, i.e. a mapping $A\colon X \to 2^X$. The sets $D(A) = \{x \in X\colon Ax \neq \emptyset\}$ and $R(A) = \bigcup_{x \in X} Ax$ are called the *domain* and the *range* of A, respectively. If $M \subset X$ we let $A(M) = \bigcup_{x \in M} Ax$. A is said to be *locally bounded* at $x \in X$, if there is a ball $\bar{B}_r(x)$ such that $A(\bar{B}_r(x))$ is a bounded subset of X. Recall that an operator A is *accretive* if for every $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ there is $x^* \in J(x-y)$ such that $(u - v, x^*) \geq 0$ holds.

Fitzpatrick, Hess and Kato [5] proved that an accretive operator A is already locally bounded on $\operatorname{int} D(A)$, the interior of $D(A)$, provided X^* is uniformly convex. Kenmochi [7] obtained this result in case X, X^* are reflexive and strictly convex and J, J^{-1} continuous.

As an application of Theorem 1 we shall prove

THEOREM 2. *Let X be uniformly convex and A an accretive operator on X. Then A is locally bounded on* $\operatorname{int} D(A)$.

PROOF. It is sufficient to assume $0 \in \operatorname{int} D(A)$ and to prove that $A(\bar{B}_\delta(0))$ is bounded for some ball $\bar{B}_\delta(0) \subset \operatorname{int} D(A)$, $\delta > 0$. So let $\bar{B}_r(0) \subset \operatorname{int} D(A)$ for some $r > 0$ and define $S_n = \{x \in \bar{B}_r(0)\colon Ax \cap \bar{B}_n(0) \neq \emptyset\}$. Then we have $\bar{B}_r(0) = \bigcup_n \bar{S}_n$ and from Baire's Category Theorem we get $x_0 \in D(A)$, $\rho > 0$, $n_0 \in \boldsymbol{N}$ such that $\bar{B}_\rho(x_0) \subset \bar{B}_r(0) \cap \bar{S}_{n_0}$. Let $\delta = \rho/4$ and choose $x \in \bar{B}_\delta(0)$, $u \in Ax$. Then accretiveness of A implies

$$(u, y^*) \leq (v, y^*) \leq |v|\,|y - x| \leq n_0(r + \delta)$$

for all $y \in \bar{B}_\rho(x_0) \cap S_{n_0}$ and some $y^* \in J(y - x)$, where $v \in Ay$ had been chosen in such a manner that $|v| \leq n_0$. But since J is weakly* upper semicontinuous, even to each $y \in \bar{B}_\rho(x_0)$ we find $y^* \in J(y - x)$ such that

$$(u, y^*) \leq n_0(r+\delta) \tag{2}$$

holds. We also choose $u_0 \in A(-x_0)$ to obtain

$$-(u, x_0^*) \leq -(u_0, x_0^*) \leq |u_0|(r+\delta) \tag{3}$$

for some $x_0^* \in J(x+x_0)$. Since X is uniformly convex, by Theorem 1 we may choose $\omega_r \in \mathscr{C}$ such that estimate (1) holds on $\bar{B}_r(0)$. Put $y=(x+x_0)+\delta|u|^{-1}u+x$ and let $y^* \in J(y-x)$ such that (2) holds. Then estimates (1), (2) and (3) imply $|u|\omega_r(\delta) \leq (u, y^*-x_0^*) \leq (n_0+|u_0|)(r+\delta)$, hence $|u| \leq C_0$ for each $u \in A(\bar{B}_\delta(0))$, where $C_0=\omega_r(\delta)^{-1}(r+\delta)(n_0+|u_0|)$ is independent of x and u. q.e.d.

4. The sum of m-accretive operators

In this section, A and B always denote *m-accretive* operators in X, i.e. A, B are accretive and $R(I+A)=R(I+B)=X$ holds. Recall $A+B$ is defined by

$$(A+B)(x) = \{u \in X : u = v+w \quad \text{where} \quad v \in Ax,\, w \in Bx\};$$

hence we have $D(A+B)=D(A) \cap D(B)$. In general $A+B$ is not m-accretive, even in the case of singlevalued and linear operators A, B in Hilbert space. But in application one has to deal with such sums very often and so it is an important problem to single out conditions for A and/or B which ensure m-accretiveness of the sum $A+B$. Much work has been done concerning this problem; let us mention Brezis [1], [2] in case X is a Hilbert space, and Kato [6], Webb [11], Kobayashi/Kobayasi [8] for more general spaces.

One method to attack this problem is to consider the following approximate equation (4) to $v \in Ax+Bx+x$:

$$v \in Ax_\lambda + B_\lambda x_\lambda + x_\lambda \quad \text{or} \quad u_\lambda \equiv v - x_\lambda - B_\lambda x_\lambda \in Ax_\lambda, \tag{4}$$

where B_λ denotes the Yosida-approximation of B, i.e.

$$B_\lambda = \lambda^{-1}(I-R_\lambda) \quad \text{and} \quad R_\lambda = (I+\lambda B)^{-1}, \qquad \lambda > 0. \tag{5}$$

Since B_λ is Lipschitz-continuous and accretive it is easy to see that, given $v \in X$ and $\lambda > 0$, (4) admits exactly one solution $x_\lambda \in D(A)$. Now we have

Lemma 1. *Let A, B be m-accretive with $D(A) \cap D(B) \neq \emptyset$ and let x_λ be defined by* (4) *with* (5). *Then,* (i) *$\{x_\lambda\}_{\lambda>0}$ is bounded*; (ii) *the uniform convexity of X^* and the boundedness of $\{B_\lambda x_\lambda\}_{\lambda>0}$ imply $v \in R(I+A+B)$.*

Since a simple proof of Lemma 1 is already available, we refer to [9].

In case X is a Hilbert space it is known that $(\operatorname{int} D(A)) \cap D(B) \neq \emptyset$ implies boundedness of $\{B_\lambda x_\lambda\}_{\lambda>0}$ for every $v \in X$; cp. [1]. Hence by Lemma 1 we get $R(I+A+B)=x$, and so $A+B$ is m-accretive. For more general spaces no such result has been obtained up to now. As another application of Theorem 1 we shall prove

THEOREM 3. *Let X, X^* be uniformly convex, A, B be m-accretive and let*

$$(\operatorname{int} D(A)) \cap D(B) \neq \emptyset. \tag{6}$$

Then $A+B$ is m-accretive.

PROOF. We may assume $0 \in (\operatorname{int} D(A)) \cap D(B)$. Then from Theorem 2 we get M, $r>0$ such that $\bar{B}_r(0) \subset D(A)$ and $|w| \leq M$ for all $w \in A(\bar{B}_r(0))$. Let $x \in D(A)$, $u \in Ax$, $z \in \bar{B}_r(0)$, $w \in Az$; from accretiveness of A we obtain

$$(u, J(z-x)) \leq (w, J(z-x)) \leq M(|x|+r). \tag{7}$$

Let $R \geq 2r$ be fixed; by Theorem 1, there is $\omega_R \in \mathscr{C}$ such that estimate (1) holds. Put $z=r|u|^{-1}u$. Then from (1) and (7) we deduce

$$|u|\omega_R(r) \leq (u, J(-x+z)-J(-x)) \leq (u, Jx)+M(|x|+r), \tag{8}$$

provided $|x| \leq R/2$. Now, for any fixed $v \in X$ let x_λ and u_λ be defined by (4). Since $\{x_\lambda\}_{\lambda>0}$ is bounded, it is possible to choose $R>0$ such that $|x_\lambda| \leq R/2$ also holds for $\lambda>0$. The estimate (8) yields

$$|u_\lambda|\omega_R(r) \leq (u_\lambda, Jx_\lambda)+M(|x_\lambda|+r) \leq -(B_\lambda x_\lambda, Jx_\lambda)+C_0$$

for some constant C_0. Since B_λ is accretive, too, we obtain

$$-(B_\lambda x_\lambda, Jx_\lambda) \leq -(B_\lambda 0, Jx_\lambda) \leq (R/2)\inf\{|\eta| : \eta \in B0\}.$$

Thus $\{u_\lambda\}_{\lambda>0}$ is bounded and so $\{B_\lambda x_\lambda\}_{\lambda>0}$ is bounded, too. Therefore, Lemma 1 applies and $v \in R(I+A+B)$; since $v \in X$ had been chosen arbitrarily, we have $R(I+A+B)=X$, i.e. $A+B$ is m-accretive. q.e.d.

In case $D(A)$ is open, Kenmochi [7] proved that maximal accretiveness of A on $D(A)$ implies m-accretiveness of A, provided X^* is uniformly convex. Especially, a single valued demicontinuous accretive operator A with open domain is m-accretive if and only if

$$x_n \to x \quad \text{implies} \quad |Ax_n| \to \infty \quad \text{for every} \quad x \in \partial D(A),\ (x_n) \subset D(A), \tag{9}$$

where $\partial D(A)$ denotes the boundary of $D(A)$. Hence, by Theorem 3 we obtain:

COROLLARY. *Let X, X^* be uniformly convex, A be a singlevalued demicontinuous accretive operator with open domain such that* (9) *holds, and let B be m-accretive with domain dense in X. Then $A+B$ is m-accretive.*

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