HIROSHIMA MATH. J.
11 (1981), 525–531

On the decomposition of homogeneous systems with nondegenerate Killing-Ricci tensor

Michihiko KIKKAWA

(Received May 2, 1981)

§ 0. Introduction

The notion of homogeneous systems has been introduced in [4]. For the investigation of various properties of homogeneous Lie loops and their tangent Lie triple algebras introduced in [1], it seems to be more convenient to treat them as homogeneous systems. In the present paper, the decomposition theorem of analytic homogeneous systems is shown in such a case that the canonical connection satisfies a tensor equation (2.4) and that the Killing-Ricci form of the tangent Lie triple algebra is nondegenerate (Theorem and Corollary 1 in § 2). For the symmetric homogeneous systems we get the de Rham-Wolf decomposition of the pseudo Riemannian structure defined by the Ricci tensor (Corollary 2 in § 2).

§ 1. Preliminaries

An *analytic homogeneous system* (G, η) is a connected analytic manifold G together with an analytic ternary operation $\eta: G \times G \times G \to G$ satisfying (1) $\eta(x, y, x) = \eta(x, x, y) = y$, (2) $\eta(x, y, \eta(y, x, z)) = z$ and (3) $\eta(x, y, \eta(u, v, w)) = \eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w))$. Let (G, η) be an analytic homogeneous system. For $x, y \in G$, the analytic diffeomorphism $\eta(x, y)$ of G defined by $\eta(x, y)z = \eta(x, y, z)$ is called the *displacement* of (G, η) from x to y. Let G be an analytic *homogeneous Lie loop*, i.e., a loop with an analytic multiplication xy satisfying the conditions (1) the left translations and right translations are all analytic diffeomorphisms of G, (2) there exists a two sided identity element e, (3) for each x there exists a two sided inverse element x^{-1} of x such that $L_x^{-1} = L_{x^{-1}}$, where L_x denotes the left translation by x, and (4) the *left inner mapping* $L_{x,y} = L_{xy}^{-1} L_x L_y$ is an automorphism of the loop G for any $x, y \in G$. For such a homogeneous Lie loop G an analytic homogeneous system η can be defined on G by

$$(1.1) \qquad \eta(x, y, z) = x((x^{-1}y)(x^{-1}z)) \qquad \text{for} \quad x, y, z \in G.$$

Any analytic homogeneous system (G, η) with a base point e is a local homogeneous Lie loop under the multiplication

$$(1.2) \qquad xy = \eta(e, x, y)$$

in a neighborhood U of e, and in fact it is a (global) analytic homogeneous Lie loop if and only if the right translations are the diffeomorphisms of G (cf. [4], [5]). In the following the terminologies, notations and results concerning homogeneous systems and homogeneous loops are referred to the papers [1]–[5].

Let (G, η) be an analytic homogeneous system with an arbitrarily fixed base point e. By using the multiplication (1. 2) on G, the *left inner mapping group* A_e generated by all left inner mappings $L_{x,y}$ is defined at e which is the same as the group of diffeomorphisms generated by all of $\eta(y, e)\eta(x, y)\eta(e, x)$, $x, y \in G$. Denote by ∇ the canonical connection of (G, η). Its torsion S and curvature R are parallel tensor fields on G and they define on the tangent space $\mathfrak{G} = T_e(G)$ at e a Lie triple algebra under the bilinear and trilinear operations

$$(1.3) \qquad XY = S_e(X, Y), \quad [X, Y, Z] = D(X, Y)Z = R_e(X, Y)Z$$

for $X, Y, Z \in \mathfrak{G}$. It should be noted that the signs of S and R are adopted oppositely to those defined usually by other authors. Any displacement of (G, η) is an affine transformation of ∇ and especially the left inner mapping group A_e is a subgroup of the affine transformation group $\mathrm{Aff}(\nabla)$ of ∇. Let K_e denote the closure of A_e in $\mathrm{Aff}(\nabla)$. Then the product manifold $A = G \times K_e$ has a Lie group structure with which (G, ∇) can be identified with the reductive homogeneous space A/K_e of K. Nomizu [9] with the canonical connection of 2nd kind. A pseudo Riemannian metric B on G will be said to be *invariant* if it is an invariant metric of the homogeneous space A/K_e, i.e., if each displacement of (G, η) is an isometry of B. (G, η) is called a *geodesic* homogeneous system if, for any geodesic curve $x(t)$ $(t \in I)$ of the canonical connection, $\eta(x(t_1), x(t_2))$ induces the parallel displacement of tangent vectors from $x(t_1)$ to $x(t_2)$ along the geodesic for $t_1, t_2 \in I$. A homogeneous system is said to be *regular* if it is geodesic and the group K_e induces on $\mathfrak{G}$ the holonomy group of ∇ at e. In [5–II] we have shown the following

PROPOSITION. *Let (G, η) be a simply connected regular analytic homogeneous system. The tangent Lie triple algebra $\mathfrak{G}$ at some base point e is decomposed into a direct sum of ideals $\mathfrak{G}_i$ of $\mathfrak{G}$ as*

$$\mathfrak{G} = \mathfrak{G}_1 + \cdots + \mathfrak{G}_k$$

if and only if (G, η) is isomorphic to the product of invariant subsystems (G_i, η_i) of (G, η) as

$$(G, \eta) \cong (G_1, \eta_1) \times \cdots \times (G_k, \eta_k),$$

where each G_i contains e with the tangent Lie triple algebra $\mathfrak{G}_i$ at e.

Here, a subsystem (H, η_H) of (G, η) is said to be *invariant* if $\eta(x, y)xH =$

yH for any $x, y \in G$, where $xH = \eta(H, x, H)$, and it is *normal* if $\eta(xH, yH, zH) = \eta(x, y, z)H$ for $x, y, z \in G$. A subsystem is normal if and only if it is the kernel of a homomorphism of homogeneous systems (cf. [5–III]).

§ 2. Decomposition theorems

Let (G, η) be a regular analytic homogeneous system with a base point e. We apply the formulas and results obtained in [7] to the tangent Lie triple algebra $\mathfrak{G}$ of (G, η) at e. Choose an arbitrary chart (U, x^i) around e and denote by $E_i = \left.\frac{\partial}{\partial x^i}\right|_e$, $1 \le i \le n = \dim G$, the natural basis at e. By using this basis the operations of Lie triple algebra $\mathfrak{G}$ are expressed as

$$\begin{aligned} XY &= X^i Y^j S_{ij}^k(e) E_k, \\ D(X, Y)Z &= X^i Y^j Z^k R_{ijk}^l(e) E_l \end{aligned} \tag{2.1}$$

for $X, Y, Z \in \mathfrak{G}$, where S_{ij}^k and R_{ijk}^l are respectively the components of the torsion S and the curvature R of the canonical connection ∇ with respect to the chart. In [7] we have defined the *Killing-Ricci form* β of a Lie triple algebra $\mathfrak{G}$ to be a symmetric bilinear form on $\mathfrak{G}$ obtained by restricting the Killing form α of the standard enveloping Lie algebra $\mathfrak{A} = \mathfrak{G} + D(\mathfrak{G}, \mathfrak{G})$ to $\mathfrak{G} \times \mathfrak{G}$, where $D(\mathfrak{G}, \mathfrak{G})$ denotes the inner derivation algebra of $\mathfrak{G}$ generated by all inner derivations $D(X, Y)$. By the formula (2.2) in [7] we get

$$\beta_{ij} = S_{ij}(e) + R_{ij}(e) + R_{ji}(e), \tag{2.2}$$

where $\beta_{ij} = \beta(E_i, E_j)$ and $S_{ij} = S_{im}^k S_{jk}^m$, $R_{ij} = R_{kij}^k$ are the components of covariant tensor fields of degree 2 obtained by contraction of indices. In [7] we also have introduced a trilinear form γ on $\mathfrak{G}$ as $\gamma(X, Y, Z) = \mathrm{tr.}\, D(X, Y)L(Z)$ for $X, Y, Z \in \mathfrak{G}$, where $L(Z)$ is an endomorphism of $\mathfrak{G}$ given by $L(Z)W = ZW$. It is easy to see that the coefficients $\gamma_{ijk} = \gamma(E_i, E_j, E_k)$ of γ is given by

$$\gamma_{ijk} = R_{ijm}^l(e) S_{kl}^m(e). \tag{2.3}$$

Since any displacement $\eta(e, x)$ preserves S and R, the tensor equations

$$R_{ijm}^l S_{kl}^m = 0 \tag{2.4}$$

hold on G if and only if $\gamma = 0$ at e. We concider the covariant tensor field B of degree 2 whose components with respect to any chart in G are given by

$$B_{ij} = S_{ij} + R_{ij} + R_{ji} \tag{2.5}$$

and we call it the *Killing-Ricci tensor* of (G, η).

Now we state the main theorem:

THEOREM. *Let (G, η) be a regular analytic homogeneous system on a connected and simply connected manifold G. Assume that the torsion S and the curvature R of the canonical connection satisfy the equation* (2.4). *If the Killing-Ricci form β of the tangent Lie triple algebra $\mathfrak{G}$ of (G, η) at a base point e is nondegenerate, then we have the following* (1)–(3):

(1) *The Lie triple algebra $\mathfrak{G}$ is decomposed into a direct sum of simple ideals as*

$$\mathfrak{G} = \mathfrak{G}_1 + \cdots + \mathfrak{G}_k.$$

(2) *(G, η) is isomorphic with the product homogeneous systems as*

$$(G, \eta) \cong (G_1, \eta_1) \times \cdots \times (G_k, \eta_k),$$

where each (G_r, η_r), $r=1,\dots, k$, is a normal subsystem of (G, η) containing e whose tangent Lie triple algebra is $\mathfrak{G}_r$ in (1).

(3) *The Killing-Ricci tensor B (resp. B_r, $r=1,\dots, k$) of the homogeneous system (G, η) (resp. (G_r, η_r)) defines an invariant pseudo Riemannian structure on G (resp. G_r) and the isomorphism in* (2) *induces an isometry of pseudo Riemannian manifolds.*

Moreover, such decompositions in (1) *and* (2) *are unique up to the order.*

PROOF. (1) (For the details cf. [7].) Applying Proposition 2 of [7] to the Killing-Ricci form β of the tangent Lie triple algebra $\mathfrak{G}$ at e with the assumption $\gamma_{ijk}=0$, we get

$$\begin{aligned} &\beta(XY, Z) + \beta(Y, XZ) = 0, \\ &\beta(X, D(Y, Z)W) + \beta(D(X, W)Y, Z) = 0 \end{aligned} \tag{2.6}$$

for $X, Y, Z, W \in \mathfrak{G}$. Since β is assumed to be nondegenerate, $\alpha(D(X, Y), Z)=\gamma(X, Y, Z)=0$ and $\alpha(D(X, Y), D(Z, W))=\beta(Y, D(Z, W)X)$ imply that the Killing form α of the standard enveloping Lie algebra $\mathfrak{A}=\mathfrak{G}+D(\mathfrak{G}, \mathfrak{G})$ is also nondegenerate, i.e., $\mathfrak{A}$ is a semi-simple Lie algebra. If $\mathfrak{H}$ is an ideal of $\mathfrak{G}$, then $\mathfrak{H}^\perp=\{X \in \mathfrak{G};\ \beta(X, \mathfrak{H})=0\}$ is an ideal of $\mathfrak{G}$ since, by (2.6), $\beta(\mathfrak{G}\mathfrak{H}^\perp, \mathfrak{H}) \subset \beta(\mathfrak{H}^\perp, \mathfrak{G}\mathfrak{H}) \subset \beta(\mathfrak{H}^\perp, \mathfrak{H})=(0)$ and $\beta(D(\mathfrak{G}, \mathfrak{H}^\perp)\mathfrak{G}, \mathfrak{H}) \subset \beta(D(\mathfrak{G}, \mathfrak{H})\mathfrak{G}, \mathfrak{H}^\perp) \subset \beta(\mathfrak{H}, \mathfrak{H}^\perp)=(0)$ which imply $\mathfrak{G}\mathfrak{H}^\perp \subset \mathfrak{H}^\perp$ and $D(\mathfrak{G}, \mathfrak{H}^\perp)\mathfrak{G} \subset \mathfrak{H}^\perp$. Set $\mathfrak{M}=\mathfrak{H} \cap \mathfrak{H}^\perp$. Since $\mathfrak{M}$ is an ideal of $\mathfrak{G}$, $\mathfrak{B}=\mathfrak{M}+D(\mathfrak{G}, \mathfrak{M})$ is an ideal of the Lie algebra $\mathfrak{A}$ and it satisfies $\alpha([\mathfrak{B}, \mathfrak{B}], \mathfrak{A}) \subset \alpha(\mathfrak{B}, \mathfrak{B}) \subset \alpha(\mathfrak{M}, \mathfrak{M})+\alpha(\mathfrak{M}, D(\mathfrak{G}, \mathfrak{M}))+\alpha(D(\mathfrak{G}, \mathfrak{M}), D(\mathfrak{G}, \mathfrak{M})) \subset \alpha([\mathfrak{G}, \mathfrak{M}], D(\mathfrak{G}, \mathfrak{M})) \subset \beta(\mathfrak{M}, \mathfrak{M})=(0)$ from which we obtain $\mathfrak{B}^{(1)}=(0)$. Hence $\mathfrak{B}=(0)$ by semi-simplicity of $\mathfrak{A}$ and we see that $\mathfrak{M}=(0)$. Thus, if $\mathfrak{H}$ is a proper ideal of $\mathfrak{G}$, then we have a direct sum $\mathfrak{G}=\mathfrak{H}+\mathfrak{H}^\perp$ of ideals and we get the simple ideal decomposition (1) of the theorem. The decomposition of (G, η) in (2) is an immediate consequence of (1) and Proposition in § 1, where each invariant subsystem (G_r, η_r) is a normal subsystem since it is the kernel of the projection homomorphism. The uniqueness of the decompositions in (1) and (2) follows from simplicity of each ideal $\mathfrak{G}_r$ of $\mathfrak{G}$.

From the direct sum decomposition of $\mathfrak{G}$ in (1) it follows that the Killing-Ricci form β_r of each simple ideal $\mathfrak{G}_r$ is obtained by the restriction of β to $\mathfrak{G}_r$. The Killing-Ricci tensor B is a covariant symmetric tensor field of degree 2. Since any displacement of the homogeneous system is an affine transformation of the canonical connection, we see that the Killing-Ricci form β of the tangent Lie triple algebra $\mathfrak{G}$ at e is nondegenerate if and only if the Killing-Ricci tensor B is nondegenerate on G. Also we see that the tensor field B is invariant under any displacement of (G, η). Therefore, B defines an invariant pseudo Riemannian metric on G. In the same manner, we have a pseudo Riemannian metric on each G_r by the Killing-Ricci tensor B_r of (G_r, η_r) which is coincident with the induced metric from B into the submanifold G_r. On the other hand, an analytic isomorphism of homogeneous systems is an affine diffeomorphism of the canonical connection. Thus the isomorphism of homogeneous systems in (2) preserves the Killing-Ricci tensor, i.e., it is an isometry of the pseudo Riemannian structures. (3) is thereby shown. q. e. d.

Let (G, μ) be a geodesic homogeneous Lie loop on a connected analytic manifold G and $\mathfrak{G}$ be its tangent Lie triple algebra at the identity element e of G. Under the map η: $G\times G\times G\to G$ defined by (1.1) in § 1, G is an analytic homogeneous system. From their definitions it follows that the canonical connection of the homogeneous system and of the homogeneous Lie loop are coincident and that the homogeneous system is also geodesic. A geodesic homogeneous Lie loop will be said to be *regular* if its homogeneous system is regular. Let (H, η_H) be an invariant (analytic) subsystem of (G, η) with $e\in H$. Then H is an invariant Lie subloop of (G, μ) under the multiplication $uv=\eta_H(e, u, v)=\eta(e, u, v)$ for $u, v\in H$. In fact, any left inner map $L_{x,y}=\eta(xy, e)\eta(x, xy)\eta(e, x)$ preserves H since H is an invariant subsystem of G. Conversely, any invariant Lie subloop of the homogeneous Lie loop (G, μ) gives an invariant subsystem of (G, η) containing e. Therefore, the theorem given above implies the following;

COROLLARY 1. *Let (G, μ) be a geodesic regular homogeneous Lie loop defined on a connected and simply connected analytic manifold G. Assume that the canonical connection satisfies the tensor equation* (2.4) *and that the Killing-Ricci form of the tangent Lie triple algebra $\mathfrak{G}$ of G is nondegenerate. Then G is decomposed into a product loop of invariant Lie subloops $G_r(1\leq r\leq k)$ such that the tangent Lie triple algebra of each G_r is a simple ideal of the tangent Lie triple algebra $\mathfrak{G}$.*

An analytic homogeneous system (G, η) is said to be *symmetric* ([5–II]) if the map S_e of G into G defined by $S_e(x)=\eta(x, e, e)$ is an automorphism of (G, η) at some (hence every) point e. A homogeneous Lie loop G is symmetric if and only if its homogeneous system is symmetric. We have seen in [5-II] that G is an affine symmetric space if (G, η) is symmetric, and in this case the tangent Lie triple algebra is reduced to Lie triple system. As a matter of fact, the torsion tensor of the canonical connection vanishes identically for a symmetric homogeneous system (G, η), and the equation (2.4) is trivially satisfied. Moreover, the Killing-Ricci tensor of a symmetric homogeneous system is reduced to (twice) the Ricci tensor $B_{ij}=R_{ij}+R_{ji}$. Thus the following is obtained from the theorem:

COROLLARY 2. *Let* (G, η) *be a symmetric regular homogeneous system whose Ricci tensor is nondegenerate at some point e. Then we have the following* (1)–(3):

(1) *The tangent Lie triple system* $\mathfrak{G}$ *at e is decomposed into a direct sum of simple ideals of* $\mathfrak{G}$ *in a unique manner* (*up to the order*).

(2) *If G is connected and simply connected,* (G, η) *is decomposed into a product of symmetric normal subsystems according to the direct sum decomposition of the tangent Lie triple systems in* (1).

(3) *The canonical connection of* (G, η) *is equal to the pseudo Riemannian connection of the pseudo Riemannian metric defined by the Ricci tensor B, and the decomposition in* (2) *gives the de Rham-Wolf decomposition of the pseudo Riemannian structure* (G, B).

PROOF. Only to prove is the last statement (3). Since $\nabla B=0$, the coefficients $\{^{k}_{ij}\}$ of the pseudo Riemannian connection defined by the metric tensor B are equal to those of the canonical connection ∇ of (G, η). On the other hand, we know that the Lie triple algebra $\mathfrak{G}$ is semi-simple in the sense of [6] if its Killing-Ricci form is nondegenerate (cf. [7]). Therefore, the Lie triple system $\mathfrak{G}$ is of reductive type in the sense of J. Wolf [10] and the decomposition in (2) is reduced to the decomposition of Theorem 5.9 in [10] without the flat part. q.e.d.

References

[1] M. Kikkawa, Geometry of homogeneous Lie loops, Hiroshima Math. J., **5** (1975), 141–179.

[2] ———, A note on subloops of a homogeneous Lie loop and subsystems of its Lie triple algebra, Hiroshima Math. J., **5** (1975), 439–446.

[3] ———, On some quasigroups of algebraic models of symmetric spaces III, Mem. Fac. Lit. Sci., Shimane Univ. Nat. Sci., **9** (1975), 7–12.

[4] ———, On the left translations of homogeneous loops, Mem. Fac. Lit. Sci., Shimane Univ. Nat. Sci., **10** (1976), 19–25.

[5] ———, On homogeneous systems I, II, III, Mem. Fac. Lit. Sci., Shimane Univ. Nat. Sci., **11** (1977), 9–11; Mem. Fac. Sci. Shimane Univ., **12** (1978), 5–13; **14** (1980), 41–46.

[6] ———, Remarks on solvability of Lie triple algebras, Mem. Fac. Sci., Shimane Univ., **13** (1979), 17–22.

[7] M. Kikkawa, On Killing-Ricci forms of Lie triple algebras, Pacific J. Math., **95** (1981), to appear.
[8] O. Loos, Symmetric Spaces I, Benjamin, 1969.
[9] K. Nomizu, Invariant affine connections on homogeneous spaces, Amer. J. Math., **76** (1954), 33–65.
[10] J. Wolf, On the geometry and classification of absolute parallelisms I, II, J. Diff. Geom., **6** (1972), 317–342; **7** (1972/1973), 19–44.

Department of Mathematics,
Faculty of Science,
Shimane University