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Some commutativity theorems for rings

Dedicated to Professor F. Kasch on his 60th birthday

Yasuyuki HIRANO and Hisao TOMINAGA

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Throughout the present paper, R will represent an associative ring (with or without 1), and C the center of R. We denote by N and $D=D(R)$ the set of all nilpotent elements and the commutator ideal of R, respectively. Given $a, b\in R$, we set $[a, b]=ab-ba$ as usual, and formally write $a(1+b)$ (resp. $(1+b)a$) for $a+ab$ (resp. $a+ba$). Let m, n be fixed positive integers.

Following [7], a ring R is called *s-unital* if for each x in R, $x\in Rx\cap xR$. As stated in [7], if R is an s-unital ring, then for any finite subset F of R, there exists an element e in R such that $ex=xe=x$ for all x in F. Such an element e will be called a *pseudo-identity* of F.

We consider the following conditions:

1) There exist non-zero polynomials $\phi(t)$, $\psi(t)$ with integer coefficients whose constant terms are 0 and such that $[\phi(x), \psi(y)]=0$ for all $x, y\in R$.

$1)_n$ $[x^n, y^n]=0$ for all $x, y\in R$.

$1)'_n$ For each pair of elements x, y in R there exists a positive integer $i=i(x, y)$ such that $[x^{n^i}, y^n]=0$.

$2)_n$ $(xy)^n=x^ny^n$ and $(xy)^{n+1}=x^{n+1}y^{n+1}$ for all $x, y\in R$.

$3)_n$ $(xy)^n=(yx)^n$ for all $x, y\in R$.

$4)_n$ $[x, (xy)^n]=0$ for all $x, y\in R$.

$5)_n$ $[x^n, y]=0$ for all $x, y\in R$.

$5)'_n$ For each pair of elements x, y in R there exists a positive integer $i=i(x, y)$ such that $[x^{n^i}, y]=0$.

$6)_n$ $[x^n, y]=[x, y^n]$ for all $x, y\in R$.

$6)'_n$ There exists a polynomial $\psi(t)$ with integer coefficients such that $[x^2\psi(x), y]=[x, y^n]$ for all $x, y\in R$.

$6)''_n$ $[x, (x+y)^n-y^n]=0$ for all $x, y\in R$.

$7)_n$ For each pair of elements x, y in R there exists a polynomial $\rho(t)=\rho(x, y; t)$ with integer coefficients such that $[nx-x^2\rho(x), y]=0$.

$8)_n$ For each pair of elements x, y in R there exist a positive integer $i=i(x, y)$ with $(i, n)=1$ and a polynomial $\psi(t)=\psi(x, y; t)$ with integer coefficients such that $[ix-x^2\psi(x), y]=0$.

$9)_n$ For each pair of elements x, y in R, $n[x, y]=0$ implies $[x, y]=0$.

Needless to say, $1)_n$ implies 1) and $1)'_n$, and $5)_n$ does $6)_n$.

Recently, in [1], [3], [7], [8] and [9], the following commutativity theorems have been obtained.

A ([1, Theorem 1] and [9, Theorem 1]). *If R is an s-unital ring satisfying* $1)_n$ *and* $9)_n$, *then the following statements are equivalent*:

a) *R is commutative.*

b) $[x, (xy)^n-(yx)^n]=0$ *for all* $x, y \in R$.

c) $[x, \{x(1+u)\}^n-x^n(1+u)^n]=0$ *for all* $u \in N$ *and* $x \in R$.

B ([7, Theorem 3, 4)], [3, Theorem 5] and [9, Theorem 2]).

(1) *Let R be an s-unital ring satisfying* $2)_n$. *If N is n-torsion free, then R is commutative.*

(2) *Suppose $n>1$. If R is a ring with* 1 *satisfying* $6)_n$ *and* $9)_n$, *then R is commutative.*

(3) *Suppose $m \geq n$ and $mn>1$. Let R be an s-unital ring satisfying the identity* $[x^m, y]=[x, y^n]$. *If for each pair of elements x, y in R, $n![x, y]=0$ implies $[x, y]=0$, then R is commutative.*

C ([8, Theorem]). *Suppose $m>1$. Let R be a ring with* 1 *satisfying* $2)_n$. *If $(m, n)=1$ and $(x+y)^m=x^m+y^m$ for all $x, y \in R$, then R is commutative.*

D ([3, Theorem 6]). *Suppose $m>1$ and $n>1$. Let R be a ring with* 1 *satisfying* $6)_m$ and $6)_n$. *If $(m, n)=1$, then R is commutative.*

The present objective is to prove the following theorems.

THEOREM 1. *If R is an s-unital ring satisfying* $1)_n$ *and* $9)_n$, *then the following statements are equivalent*:

a) *R is commutative.*

b) *Every $u \in N$ with $u^2=0$ is central.*

c) $[x, \{x^n(1+u)\}^n-\{x^{n-1}(1+u)x\}^n]=0$ *for all $u \in N$ with $u^2=0$ and $x \in R$.*

d) $[x, \{x(1+u)\}^n-x^n(1+u)^n]=0$ *for all $u \in N$ with $u^2=0$ and $x \in R$.*

THEOREM 2. *Let R be an s-unital ring satisfying* $9)_n$.

(1) *If any of the conditions* $2)_n$, $3)_n$, $4)_n$, $5)_n$, $5)'_n$ *and* $6)'_n$ *is satisfied, then R is commutative.*

(2) *Suppose $n>1$. If R satisfies the condition* $6)_n$ *or* $6)''_n$, *then R is commutative.*

(3) *The conditions* $1)_n$ *and* $1)'_n$ *are equivalent.*

THEOREM 3. *Suppose $m>1$ and $(m, n)=1$. Let R be an s-unital ring satisfying* $6)''_m$. *If R satisfies one of the conditions* $2)_n$, $3)_n$, $4)_n$, $5)_n$, $5)'_n$ *and* $6)'_n$, *then R is commutative.*

THEOREM 4. *If R is an s-unital ring satisfying* 1), $7)_n$ *and* $9)_n$, *then R is commutative.*

THEOREM 5. *Let R be an s-unital ring satisfying* $6)'_m$ *and* $6)'_n$. *If* $(m, n)=1$, *then R is commutative.*

Obviously, Theorem 1 covers Theorem A. Moreover, in view of Theorem 2 (3), Theorem 1 also improves [4, Theorem 1]. Theorems 2 and 5 improve Theorems B and D, and Theorem 3 contains Theorem C.

In preparation for the proof of our theorems, we establish the following lemmas and propositions.

LEMMA 1. *Let R be a ring satisfying a polynomial identity $f=0$, where the coefficients of f are integers with highest common factor* 1. *If there exists no prime p for which the ring of 2×2 matrices over $GF(p)$ satisfies $f=0$, then D is a nil ideal and there exists a positive integer h such that $[x, y]^h=0$ for all $x, y \in R$.*

PROOF. By [2, Theorem 1], D is a nil ideal. Consider the direct product $R^{R\times R}$. Since the ring $R^{R\times R}$ satisfies the same identity $f=0$, $D(R^{R\times R})$ is also nil. Let $X=(x)_{(x,y)\in R\times R}$, $Y=(y)_{(x,y)\in R\times R}$, and $[X, Y]^h=0$. Then it is immediate that $[x, y]^h=0$ for all $x, y\in R$.

LEMMA 2. *If an s-unital ring R satisfies* $1)'_n$ *and* $9)_n$, *then $[u, x^n]=0$ for all $u\in N$ and $x\in R$, and N is a commutative nil ideal containing D.*

PROOF. Obvious by [6, Theorem] and the proof of [4, Lemma 5].

LEMMA 3. *If R is an s-unital ring satisfying* 1), *then there exists a positive integer k such that $kD=0$.*

PROOF. Let $\phi(t)=p_1t+p_2t^2+\cdots+p_mt^m$. Suppose $p_1=0$. Obviously, $\phi'(t)=2p_2t+3p_3t^2+\cdots+mp_mt^{m-1}$ is non-zero, and so there exists an integer t_1 such that $q_1=\phi'(t_1)\neq 0$. Then $\phi_1(t)=\phi(t_1+t)=q_1t+\cdots+p_mt^m$, and $[\phi_1(x), \psi(y)]=0$ for all $x, y\in R$. (Note that R is s-unital.) Because of the above observation, we may assume that $p_1\neq 0$. Now, replacing x by ix in the identity

$$[p_1x, \psi(y)] + \cdots + [p_mx^m, \psi(y)] = [\phi(x), \psi(y)] = 0,$$

we have

$$i[p_1x, \psi(y)] + \cdots + i^m[p_mx^m, \psi(y)] = 0 \qquad (i = 1,\dots, m).$$

Hence, $d[p_1x, \psi(y)]=0$, where $d(\neq 0)$ is the determinant of the matrix of integer coefficients in the last equations. Finally, repeating the above procedure for $\psi(y)$, we obtain the conclusion.

COROLLARY 1. *Let R be a ring satisfying* $9)_n$. *If there exists a polynomial $\psi(t)$ with integer coefficients such that $[nx-x^2\psi(x), y]=0$ for all $x, y\in R$, then R is commutative.*

PROOF. As is easily seen from the proof of Lemma 3, there exists a positive integer k such that $kD=0$. Combining this with $9)_n$, we can see that there exists a polynomial $\gamma(t)$ with integer coefficients such that $[x-x^2\gamma(x), y]=0$ for all x, $y \in R$. Then R is commutative by [5, Theorem 3].

PROPOSITION 1. *If R is an s-unital ring satisfying $1)_n$ and $9)_n$, then $DN=0$, and in particular, $D^2=0$.*

PROOF. According to Lemma 2, N is a commutative nil ideal containing D and $[u, x^n]=0$ for all $u \in N$ and $x \in R$. Now, let $u \in N$, and x, $y \in R$. Then

$$\begin{aligned} 0 &= [xu, y^n] = x[u, y^n] + [x, y^n]u = [x, y^n]u \\ &= \textstyle\sum_{i=0}^{n-1} y^i[x, y]y^{n-i-1}u = \sum_{i=0}^{n-1} y^i(y^{n-i-1}u)[x, y] = ny^{n-1}[x, y]u. \end{aligned}$$

Hence, by [1, Lemma 1 (2)], we obtain $n[x, y]u=0$. On the other hand, by Lemma 3 and $9)_n$, $k[x, y]u=0$ with a positive integer k such that $(n, k)=1$. Now, it is immediate that $[x, y]u=0$, proving $DN=0$.

PROPOSITION 2. *If R is an s-unital ring, then there hold the following implications*: $2)_n \Rightarrow 3)_n \Rightarrow 4)_n \Leftrightarrow 5)_n \Rightarrow 5)'_n$.

PROOF. Since $2)_n$ together with $5)_n$ implies $3)_n$ and $5)_n$ does $4)_n$ and $5)'_n$, it is enough to show that $2)_n \Rightarrow 4)_n$ and $3)_n \Rightarrow 4)_n \Rightarrow 5)_n$.

$2)_n \Rightarrow 4)_n$. Since $xyx^ny^n=(xy)^{n+1}=x^{n+1}y^{n+1}$, we have $x[x^n, y]y^n=0$, and therefore $x[x^n, y]=0$ by [1, Lemma 1 (2)]. In particular, $x[x^n, y^n]=0$. Hence, $[x, (xy)^n]=x\{(xy)^n-(yx)^n\}=x[x^n, y^n]=0$.

$3)_n \Rightarrow 4)_n$. It is immediate that $[x, (xy)^n]=x\{(xy)^n-(yx)^n\}=0$.

$4)_n \Rightarrow 5)_n$. As a consideration of $x=E_{12}$ and $y=E_{21}$ shows, D is a nil ideal (Lemma 1). Let T be the (s-unital) subring of R generated by all n-th powers of elements of R. Let $u \in N$, and u' the quasi-inverse of u. If a is an arbitrary element of R, and e a pseudo-identity of $\{u, a\}$, then $[u, a]^n=[e+u, \{(e+u)(e+u')a\}^n]=0$. In particular, every nilpotent element of T is in the center of T. Now, let s, $t \in T$. Since $s^nt^n-(st)^n$ is in the nil ideal $D(T)$, we get $s^n[s, t^n]=[s, s^nt^n]=[s, (st)^n]=0$. Then, $[s, t^n]=0$ by [1, Lemma 1 (2)]. This implies that $[x^n, y^{n^2}]=0$ for all x, $y \in R$. So, according to Lemma 3, we can find a positive integer k such that $kD=0$. Then, recalling that $[x^n, [x^n, y]]=0$, we see that $[x^{nk}, y]=kx^{n(k-1)}[x^n, y]=0$. This enables us to see that $x^{n^2k}[x, y^n]=[x, x^{n^2k}y^n]=[x, (x\cdot x^{nk-1}y)^n]=0$. Hence, $[x, y^n]=0$ again by [1, Lemma 1(2)].

LEMMA 4. *Assume that for each $u \in N$ and $x \in R$ there exists a positive integer $i=i(u, x)$ such that $[(1+u)^{n^i}, x]=0$. Then for each $u \in N$ and $x \in R$ there exists a positive integer l such that $[n^lu, x]=0$.*

PROOF. Let $u \in N$, and $x \in R$. By hypothesis, there exists a positive integer i such that $[(1+u)^{n^i}, x]=0$. If $u^2=0$, then $[n^iu, x]=[(1+u)^{n^i}, x]=0$. Suppose now that if $u^h=0$ with $h<k$ then $[n^j u, x]=0$ for some positive integer j, and consider u with $u^k=0$. Then, we can find a positive integer j such that $[n^j u^2, x]=\cdots=[n^j u^{n^i}, x]=0$. Obviously, $[n^{i+j}u, x]=n^j[(1+u)^{n^i}, x]=0$. This completes the proof.

LEMMA 5. *Let R be a ring satisfying the identity $[[x, y], z]=0$. If $n>1$, then $6)_n$ implies $5)_{n^6}$.*

PROOF. First, we claim that R satisfies the identity

$$(x^{(n-1)^2}-1)[x, y^{n^3}] = 0.$$

Indeed,

$$0 = [x^{n^2}, y^n] - [x^n, y^{n^2}] = nx^{n(n-1)}[x^n, y^n] - nx^{n-1}[x, y^{n^2}]$$

$$= n(x^{(n-1)^2}-1)x^{n-1}[x, y^{n^2}] = (x^{(n-1)^2}-1)[x^n, y^{n^2}] = (x^{(n-1)^2}-1)[x, y^{n^3}].$$

Since every ring is a subdirect sum of subdirectly irreducible rings, we may assume that R itself is a subdirectly irreducible ring with heart $S(\neq 0)$. Now, let a be an arbitrary element in the right annihilator $r(S)$ of S. If $[a, r^{n^3}]$ is non-zero for some $r\in R$, then, by the claim at the opening, the left ideal $I=\{x\in R\mid xa^{(n-1)^2}=x\}$ contains the non-zero central element $[a, r^{n^3}]$, so that $I\supseteq S$. But then $s=sa^{(n-1)^2}=0$ for all $s\in S$. This is a contradiction. We have thus seen that $[a, y^{n^3}]=0$ for all $y\in R$. Next, we prove that R satisfies the identity $[x^{n^3}, y^{n^3}]=0$. If $[x, y^{n^3}]=0$ for all $x, y\in R$, there is nothing to prove. Now, assume that $[b, d^{n^3}]\neq 0$ for some $b, d\in R$. Then, again by the opening claim, the left annihilator $l(b^{(n-1)^2+1}-b)$ contains the non-zero central element $[b, d^{n^3}]$, and so contains S. Then, since $b^{(n-1)^2+1}-b$ is in $r(S)$, it follows from what was just shown above that $[b^{(n-1)^2+1}-b, d^{n^3}]=0$. Thus, at any rate, R satisfies the identity $[x^{(n-1)^2+1}-x, y^{n^3}]=0$, and so the subring generated by all n^3-th powers of elements of R is commutative by [5, Theorem 3]. Consequently, R satisfies the identity $[x^{n^3}, y^{n^3}]=0$. Now, by $6)_n$, it is immediate that $[x^{n^6}, y]=[x^{n^3}, y^{n^3}]=0$.

PROPOSITION 3. *If $n>1$, then $6)_n$, $6)'_n$ and $6)''_n$ are equivalent, and $6)_n$ implies $5)_{n^\alpha}$ for some positive integer α.*

PROOF. Obviously, $6)_n$ implies $6)'_n$. If $6)'_n$ is satisfied, then

$$[x, (x+y)^n - y^n] = [x^2\psi(x), (x+y) - y] = [x^2\psi(x), x] = 0.$$

Next, if $6)''_n$ is satisfied then

$$[x, y^n] - [x^n, y] = [x, (x+y)^n] - [(x+y)^n, y] = [x+y, (x+y)^n] = 0.$$

We have thus seen the equivalence of $6)_n$, $6)'_n$ and $6)''_n$.

Suppose now that $6)_n$ is satisfied. By Lemma 1, there exists a positive integer h such that $[x, y]^h=0$ for all $x, y\in R$. Choose a positive integer κ such that $n^\kappa\geq h$. Let T be the subring of R generated by all n^κ-th powers of elements of

R. Since $[[x, y], z^{n^{\kappa}}]=[[x, y]^{n^{\kappa}}, z]=0$ for all $x, y, z \in R$, we get $[s^{n^6}, t]=0$ for all $s, t \in T$ (Lemma 5). It therefore follows that $[x^{n^{2\kappa+6}}, y]=[x^{n^{\kappa+6}}, y^{n^{\kappa}}]=0$ for all $x, y \in R$.

The next is a slight generalization of [2, Theorem 2].

COROLLARY 2. *Suppose $n>1$. Let T be the subring of R generated by all n-th powers of elements of R. If R satisfies $6)_n$ and the centralizer of T in R coincides with C, then R is commutative.*

PROOF. According to Proposition 3, there exists a positive integer α such that $[x^{n^{\alpha-1}}, y]=[x^{n^{\alpha}}, y]=0$ for all $x, y \in R$. Then, $[x^{n^{\alpha-1}}, y]=0$ by hypothesis. We can repeat the above process to obtain the conclusion $[x, y]=0$.

LEMMA 6. *The condition $8)_n$ implies $9)_n$.*

PROOF. Suppose $n[a, b]=0$ $(a, b \in R)$. Let R' be the subring of R generated by $\{a, b\}$. Then it is easy to see that $n[x, y]=0$ for all $x, y \in R'$. Combining this with $8)_n$, we can show that for each pair of elements x, y in R' there exists a polynomial $\gamma(t)=\gamma(x, y; t)$ with integer coefficients such that $[x-x^2\gamma(x), y]=0$. Hence, R' is commutative by [5, Theorem 3], and so $[a, b]=0$.

We now proceed to prove our theorems.

PROOF OF THEOREM 1. a)$\Rightarrow$c) and d). Trivial.

b)$\Rightarrow$a). By Proposition 1, every commutator squares to 0, and hence is central. Then $n^2x^{n-1}y^{n-1}[x, y]=nx^{n-1}[x, y^n]=[x^n, y^n]=0$. Now, by [1, Lemma 1 (2)], it follows that $n^2[x, y]=0$, and so $[x, y]=0$.

c)$\Rightarrow$b). Let $u^2=0$. Since $[x^n, u]=0$ by Lemma 2, we have

$$\begin{aligned} 0 &= [x, \{x^n(1+u)\}^n - \{x^{n-1}(1+u)x\}^n] \\ &= [x, x^{n^2}(1+u)^n - x^{n^2-1}(1+u)^n x] \\ &= x^{n^2-1}[x, [x, (1+u)^n]] = nx^{n^2-1}[x, [x, u]]. \end{aligned}$$

Now, by making use of [1, Lemma 1 (2)] and $9)_n$, we obtain $[x, [x, u]]=0$. This yields $nx^{n-1}[x, u]=[x^n, u]=0$. Hence, we get $[x, u]=0$ again by [1, Lemma 1 (2)] and $9)_n$.

d)$\Rightarrow$b). Let $u^2=0$. Since $[\{(1+u)x\}^n, 1+u]=0$ by Lemma 2, we see that

$$\begin{aligned} 0 &= x(1+u)^{-1}[\{(1+u)x\}^n, 1+u] = [x, \{x(1+u)\}^n] \\ &= [x, x^n(1+u)^n] = nx^n[x, u]. \end{aligned}$$

Then, by [1, Lemma 1 (2)], we obtain $n[x, u]=0$, and hence $[x, u]=0$.

PROOF OF THEOREM 2. (1) First, we prove that if R satisfies $5)'_n$, then R

is commutative. Let $a, b \in R$, and e a pseudo-identity of $\{a, b\}$. Then $[a^{n^i}, b]=0$ with some positive integer i. Since $[a, b] \in N$ (Lemma 2), $[a, [a, b]]=0$ by Lemma 4. Hence we get $n^i a^{n^i-1}[a, b]=[a^{n^i}, b]=0$. Similarly, $n^j(a+e)^{n^j-1}[a, b]=0$ with some positive integer j. From these we obtain $n^k a^{n^k-1}[a, b]=0=n^k(a+e)^{n^k-1}[a, b]$, where $k=\max\{i, j\}$. Then, by [1, Lemma 1 (2)] there holds that $n^k[a, b]=0$, and hence $[a, b]=0$.

If any of the conditions $2)_n$, $3)_n$, $4)_n$ and $5)_n$ is satisfied, R is commutative by Proposition 2 and what was just shown above. If $6)'_1$ is satisfied then R is commutative by [5, Theorem 3]. On the other hand, in case $n>1$ and $6)'_n$ is satisfied, R satisfies $5)_{n^\alpha}$ for some positive integer α (Proposition 3). Thus, again by the the above, R is commutative.

(2) This is only a combination of (1) and Proposition 3.

(3) It suffices to show that $1)'_n$ implies $1)_n$. Let T be the (s-unital) subring of R generated by all n-th powers of elements of R. Then T satisfies $5)'_1$, and hence T is commutative by (1). That is, R satisfies $1)_n$.

Combining Theorem 2 with Lemma 6, we obtain

COROLLARY 3. *Let R be an s-unital ring satisfying* $8)_n$.

(1) *If any of the conditions* $2)_n$, $3)_n$, $4)_n$, $5)_n$, $5)'_n$ *and* $6)'_n$ *is satisfied, then R is commutative.*

(2) *Suppose $n>1$. If R satisfies the condition* $6)_n$ *or* $6)''_n$, *then R is commutative.*

PROOF OF THEOREM 3. Let $x, y \in R$, and e a pseudo-identity of $\{x, y\}$. Then

$$[x^m, y] = [x^m, y+e] = [(x+y+e)^m, y+e]$$
$$= [(x+y+e)^m, y] = [(x+e)^m, y].$$

Thus we have

$$[mx + \tbinom{m}{2}x^2 + \cdots + mx^{m-1}, y] = [(x+e)^m - x^m, y] = 0,$$

and so R satisfies $8)_n$. Hence, R is commutative by Corollary 3.

PROOF OF THEOREM 4. By Lemma 3, there exists a positive integer k such that $kD=0$. In view of $9)_n$, we may assume that $(k, n)=1$. Combining this with $7)_n$, we see that for each pair of elements x, y in R there exists a polynomial $\gamma(t)=\gamma(x, y; t)$ with integer coefficients such that $[x-x^2\gamma(x), y]=0$. Hence, R is commutative by [5, Theorem 3].

PROOF OF THEOREM 5. If $m=1$ or $n=1$, then R is commutative by [5, Theorem 3]. Henceforth, we assume that $m>1$ and $n>1$. Then, by Proposition 3,

$$[x, my + \binom{m}{2}y^2 + \cdots + my^{m-1}] = 0 \text{ and } [x, ny + \binom{n}{2}y^2 + \cdots + ny^{n-1}] = 0$$

(see the proof of Theorem 3). Since $(m, n)=1$, the last two identities imply that there exists a polynomial $\gamma(t)$ with integer coefficients such that $[x, y-y^2\gamma(y)]=0$ for all $x, y \in R$. Hence, again by [5, Theorem 3], R is commutative.

Finally, we prove the following

COROLLARY 4. *Suppose* $mn>1$ *and* $(m, n)=1$. *If* R *is an s-unital ring satisfying the identity* $[x^n, y]=[x, y^m]$, *then* R *is commutative.*

PROOF. We may assume that $n>1$. If $m=1$, then R is commutative by [5, Theorem 3]. Thus, henceforth, we assume that $m>1$. Then, by Proposition 3, R satisfies $5)_{m^\alpha}$ for some positive integer α. This also implies that $[x, y^{n^\alpha}]=[x^{m^\alpha}, y]=0$. Since $(m^\alpha, n^\alpha)=1$, R is commutative by Theorem 5.

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Department of Mathematics,
Faculty of Science,
*Hiroshima University**)
and
Department of Mathematics,
Faculty of Science,
Okayama University

* The present address of the first named author is as follows: Department of Mathematics, Faculty of Science, Okayama University.