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Positive solutions of semilinear second order elliptic equations in exterior domains

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1. Introduction

This paper is concerned with positive solutions of the semilinear elliptic equation

$$\Delta u - \phi(x)|u|^{\gamma} \operatorname{sgn} u = 0 \tag{1}$$

in exterior domains, where ϕ is a positive continuous function, γ is a positive constant and $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator in Euclidean space R^n. Equation (1) is called superlinear or sublinear according as $\gamma > 1$ or $0 < \gamma < 1$.

Recently equations including

$$\Delta u + \phi(x)|u|^{\gamma} \operatorname{sgn} u = 0 \tag{2}$$

have been considered by Noussair and Swanson [6, 8] and effective conditions for (2) to have positive solutions in exterior domains have been established. For other related results with regard to (2) the reader is referred to Kitamura and Kusano [3], Noussair [4] and Noussair and Swanson [5, 7].

Our purpose here is to discuss the existence and asymptotic behavior of positive solutions, defined in exterior domains, of equation (1) which has little been studied in the literature. Employing the techniques of Noussair and Swanson, we reduce the multi-dimensional problem under study to the problem of one dimension and make extensive use of known results on the existence and asymptotic behavior of positive solutions of ordinary differential equations of the form

$$y'' - p(t)|y|^{\gamma} \operatorname{sgn} y = 0, \qquad p(t) > 0. \tag{3}$$

In Section 2 we prove a basic existence theorem for positive solutions of equation (1). We distinguish the superlinear case (Section 3) and the sublinear case (Section 4), and establish in each case effective sufficient conditions under which equation (1) possesses positive solutions having various asymptotic properties as $|x| \to \infty$.

2. Reduction to ordinary differential equations

Let $|x|$ denote the Euclidean norm of a point $x=(x_1,\dots, x_n)$ in R^n and put

$$G_a = \{x \in R^n\colon |x| > a\}, \quad \bar{G}_a = \{x \in R^n\colon |x| \geqq a\},$$

where a is some positive constant. For convenience we assume that $a>1$. For a constant $\alpha \in (0, 1)$ and the closure $\bar{M}$ of a bounded domain $M \subset G_a$, let $C^\alpha(\bar{M})$ and $C^{2+\alpha}(\bar{M})$ denote the usual Hölder spaces.

We now consider equation (1) in G_a under the following assumptions:

(I) The function $\phi\colon \bar{G}_a \to (0, \infty)$ is continuous and there exists a constant $\alpha \in (0, 1)$ such that $\phi \in C^\alpha(\bar{M})$ for all bounded domains $M \subset G_a$.

(II) There exist two continuous functions $p, P\colon [a, \infty) \to (0, \infty)$ such that $p, P \in C^\alpha(\bar{I})$ for some $\alpha \in (0, 1)$ and for every bounded interval $I \subset [a, \infty)$ and

$$p(|x|) \leqq \phi(x) \leqq P(|x|), \qquad x \in \bar{G}_a. \tag{4}$$

By a solution of (1) in G_a is meant a function $u \in C^{2+\alpha}(\bar{M})$ for some $\alpha \in (0, 1)$ and for every bounded subdomain $M \subset G_a$ such that $u(x)$ satisfies (1) at every point $x \in G_a$. Supersolutions and subsolutions of (1) in G_a, i.e. functions $v(x)$ and $w(x)$ satisfying

$$\Delta v - \phi(x)|v|^\gamma \operatorname{sgn} v \leqq 0 \quad\text{and}\quad \Delta w - \phi(x)|w|^\gamma \operatorname{sgn} w \geqq 0$$

in G_a, respectively, are defined similarly.

The result of Noussair and Swanson [8, Theorem 3.3] is applied to our problem.

THEOREM 0. *Let assumption* (I) *be satisfied. If there exist a supersolution* $v(x)$ *and a subsolution* $w(x)$ *of* (1) *in* G_a *such that* $0 < w(x) \leqq v(x)$ *in* $\bar{G}_a$, *then* (1) *has a positive solution* $u(x)$ *in* G_a *such that* $w(x) \leqq u(x) \leqq v(x)$ *in* $\bar{G}_a$.

The following existence theorem is basic to our considerations in the subsequent sections.

THEOREM 1. *Let assumptions* (I) *and* (II) *be satisfied. Suppose that the ordinary differential equations*

$$\frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) - r^{n-1}p(r)\rho^\gamma = 0, \quad r > a, \tag{5}$$

$$\frac{d}{dr}\left(r^{n-1}\frac{d\eta}{dr}\right) - r^{n-1}P(r)\eta^\gamma = 0, \quad r > a, \tag{6}$$

have positive solutions $\rho(r)$, $\eta(r)$ *respectively such that* $\rho(r) \geqq \eta(r)$ *for all* $r \geqq a$.

Then (1) *has a positive solution* $u(x)$ *in* G_a *such that* $\eta(|x|) \leqq u(x) \leqq \rho(|x|)$, $x \in \bar{G}_a$.

PROOF. Put $v(x)=\rho(r)$ and $w(x)=\eta(r)$ where $r=|x|$. Using the first inequality in (4), we have

$$\Delta v - \phi(x)v^{\gamma} \leqq r^{1-n}\left\{\frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) - r^{n-1}p(r)\rho^{\gamma}\right\} = 0$$

for $x \in G_a$, and using the second inequality in (4), we have $\Delta w - \phi(x)w^{\gamma} \geqq 0$ in G_a. This implies that $v(x)$ is a supersolution of (1) in G_a and $w(x)$ is a subsolution of (1) in G_a. Since $0 < w(x) \leqq v(x)$ for all $x \in \bar{G}_a$, the conclusion follows from Theorem 0.

EXAMPLE 1. Let $\phi(x)$ satisfy assumption (I). Suppose that there exist constants $c_2 > c_1 > 0$ such that

$$c_1(|x|\log|x|)^{-2} \leqq \phi(x) \leqq c_2(|x|\log|x|)^{-2}, \qquad n = 2,$$
$$c_1|x|^{-2} \leqq \phi(x) \leqq c_2|x|^{-2}, \qquad n \geqq 3,$$

for $x \in G_a$. Then the linear equation

$$\Delta u - \phi(x)u = 0$$

has a positive solution $u(x)$ in G_a such that

$$(\log|x|)^{-\delta_2} \leqq u(x) \leqq (\log|x|)^{-\delta_1}, \qquad x \in G_a, \quad n = 2,$$
$$|x|^{-(n-2)(1+\delta_2)} \leqq u(x) \leqq |x|^{-(n-2)(1+\delta_1)}, \quad x \in G_a, \quad n \geqq 3,$$

where $\delta_i = \{(1+4c_i)^{1/2} - 1\}/2$ $(i=1, 2)$ for $n=2$ and $\delta_i = \{(1+4c_i(n-2)^{-2})^{1/2} - 1\}/2$ $(i=1,2)$ for $n \geqq 3$.

We change the variables in order to transform (5), (6) into simpler forms. Substituting $r=e^t$, $y(t)=\rho(e^t)$ for $n=2$ and $r=\beta(t) \equiv [t/(n-2)]^{1/(n-2)}$, $y(t)=t\rho(\beta(t))$ for $n \geqq 3$ in (5), we have

$$y'' - \tilde{p}(t)y^{\gamma} = 0, \qquad t > t_0, \tag{7}$$

where $\tilde{p}(t)=e^{2t}p(e^t)$, $t_0=\log a$ for $n=2$ and $\tilde{p}(t)=t^{-3-\gamma}[\beta(t)]^{2n-2}p(\beta(t))$, $t_0=(n-2)a^{n-2}$ for $n \geqq 3$. Similarly, substituting $z(t)=\eta(e^t)$ for $n=2$ and $z(t)=t\eta(\beta(t))$ for $n \geqq 3$ in (6), we have

$$z'' - \tilde{P}(t)z^{\gamma} = 0, \qquad t > t_0, \tag{8}$$

where $\tilde{P}(t)=e^{2t}P(e^t)$ for $n=2$ and $\tilde{P}(t)=t^{-3-\gamma}[\beta(t)]^{2n-2}P(\beta(t))$ for $n \geqq 3$. From assumption (II) it follows that $\tilde{p}$, $\tilde{P} \in C^{\alpha}(\bar{I})$ for all bounded intervals $I \subset [t_0, \infty)$ and satisfy

(9) $$\tilde{p}(t) \leqq \tilde{P}(t), \qquad t \geqq t_0.$$

It is convenient to use (7), (8) instead of (5), (6).

3. Superlinear equations

We now discuss the superlinear case, thus we assume that $\gamma > 1$ in (1) throughout this section.

LEMMA 1. *Suppose* (9) *holds and let* $y(t)$ *and* $z(t)$ *be positive decreasing solutions of* (7) *and* (8) *on* $[t_0, \infty)$, *respectively. If* $y(t_0) = z(t_0)$, *then* $y(t) \geqq z(t)$ *for all* $t \geqq t_0$.

For the proof of this lemma see Taliaferro [9, Lemma 1.2].

THEOREM 2. *Let assumptions* (I) *and* (II) *be satisfied.* (A) *If*

(10) $$\int_a^\infty r(\log r)p(r)dr = \infty, \qquad n = 2,$$

(11) $$\int_a^\infty r^{n-1-\gamma(n-2)}p(r)dr = \infty, \qquad n \geqq 3,$$

then (1) *has a positive solution* $u(x)$ *in* G_a *satisfying*

(12) $$u(x) \longrightarrow 0 \quad as \quad |x| \longrightarrow \infty, \qquad n = 2,$$

(13) $$|x|^{n-2}u(x) \longrightarrow 0 \quad as \quad |x| \longrightarrow \infty, \qquad n \geqq 3.$$

(B) *If*

(14) $$\int_a^\infty r(\log r)P(r)dr < \infty, \qquad n = 2,$$

(15) $$\int_a^\infty r^{n-1-\gamma(n-2)}P(r)dr < \infty, \qquad n \geqq 3,$$

then (1) *has a positive solution* $u(x)$ *in* G_a *satisfying*

(16) $$m \leqq u(x) \leqq M, \qquad n = 2,$$

(17) $$m|x|^{2-n} \leqq u(x) \leqq M|x|^{2-n}, \qquad n \geqq 3,$$

for some positive constants m *and* M.

PROOF. (A) By the change of variables of the previous section, equation (5) becomes (7) and conditions (10), (11) reduce to

(18) $$\int_{t_0}^\infty t\tilde{p}(t)dt = \infty.$$

Then, by Taliaferro's theorem [9, Theorem 1.1], for each $\lambda>0$ there exists a positive decreasing solution $y(t)$ of (7) such that $y(t_0)=\lambda$ and $\lim_{t\to\infty} y(t)=0$. Therefore equation (5) has a positive solution $\rho(r)$ which satisfies

$$(19)\qquad \lim_{r\to\infty} \rho(r) = 0 \quad (n = 2), \qquad \lim_{r\to\infty} r^{n-2}\rho(r) = 0 \quad (n \geqq 3).$$

Similarly, since

$$\int_{t_0}^{\infty} t\tilde{P}(t)dt = \infty$$

by (9) and (18), there exists a positive decreasing solution $z(t)$ of (8) such that $z(t_0)=y(t_0)$ and $\lim_{t\to\infty} z(t)=0$. Accordingly, there exists a positive solution $\eta(r)$ of (6) such that

$$(20)\qquad \lim_{r\to\infty} \eta(r) = 0 \quad (n = 2), \qquad \lim_{r\to\infty} r^{n-2}\eta(r) = 0 \quad (n \geqq 3).$$

On the other hand, by Lemma 1 the inequality $y(t)\geqq z(t)$ holds for all $t\geqq t_0$, so that we have $\rho(r)\geqq\eta(r)$ for all $r\geqq a$. Applying Theorem 1 and taking (19), (20) into account, we see that equation (1) has a positive solution $u(x)$ satisfying (12) or (13).

(B) The conditions (14), (15) reduce to

$$(21)\qquad \int_{t_0}^{\infty} t\tilde{P}(t)dt < \infty.$$

Then, again by Taliaferro's theorem [9], for a positive constant λ there exists a positive decreasing solution $z(t)$ of (8) which satisfies $z(t_0)=\lambda$ and tends to some positive constant as $t\to\infty$. The solution $\eta(r)$ of (6) corresponding to $z(t)$ has the following asymptotic behavior:

$$(22)\qquad \lim_{r\to\infty} \eta(r) = K_1 \quad (n = 2), \qquad \lim_{r\to\infty} r^{n-2}\eta(r) = K_1' \quad (n \geqq 3),$$

where K_1 and K_1' are some positive constants. On the other hand, by (9) and (21) we obtain

$$(23)\qquad \int_{t_0}^{\infty} t\tilde{p}(t)dt < \infty.$$

Therefore, we see that there exists a positive solution $\rho(r)$ of (5) such that $\rho(a)=\eta(a)$ and

$$(24)\qquad \lim_{r\to\infty} \rho(r) = K_2 \quad (n = 2), \qquad \lim_{r\to\infty} r^{n-2}\rho(r) = K_2' \quad (n \geqq 3),$$

where K_2 and K_2' are some positive constants. By Lemma 1 we have $\rho(r)\geqq\eta(r)$ for all $r\geqq a$. From Theorem 1 and (22), (24) it follows that equation (1) has a positive solution $u(x)$ satisfying (16), (17).

In the remainder of this section we replace assumption (II) by the following more restrictive one:

(III) There exists a continuous function $p: [a, \infty) \to (0, \infty)$ such that $p \in C^{\alpha}(\bar{I})$ for some $\alpha \in (0, 1)$ and for every bounded interval $I \subset [a, \infty)$ and that

$$p(|x|) \leqq \phi(x) \leqq Cp(|x|), \qquad x \in \bar{G}_a, \tag{25}$$

for some positive constant $C \geqq 1$.

Then, equation (6) becomes

$$\frac{d}{dr}\left(r^{n-1}\frac{d\eta}{dr}\right) - Cr^{n-1}p(r)\eta^{\gamma} = 0, \qquad r > a, \tag{26}$$

and its reduced equation (8) becomes

$$z'' - C\tilde{p}(t)z^{\gamma} = 0, \qquad t > t_0. \tag{27}$$

The relationship between the solutions of (7) and (27) is described in the following

LEMMA 2. *Let $C > 0$. Then $y(t)$ is a solution of* (7) *if and only if $C^{1/(1-\gamma)}y(t)$ is a solution of* (27).

THEOREM 3. *Let assumptions* (I) *and* (III) *be satisfied. If*

$$\int_a^{\infty} r(\log r)^{\gamma} p(r)dr < \infty, \qquad n = 2, \tag{28}$$

$$\int_a^{\infty} rp(r)dr < \infty, \qquad n \geqq 3, \tag{29}$$

then (1) *has a positive solution $u_1(x)$ in G_a satisfying*

$$m \log|x| \leqq u_1(x) \leqq M \log|x|, \qquad n = 2, \tag{30}$$

$$m \leqq u_1(x) \leqq M, \qquad n \geqq 3, \tag{31}$$

for some positive constants m and M. Moreover, (1) *has a positive solution $u_2(x)$ in G_a such that*

$$u_2(x)/\log|x| \longrightarrow \infty \quad as \quad |x| \longrightarrow \infty, \qquad n = 2, \tag{32}$$

$$u_2(x) \longrightarrow \infty \quad as \quad |x| \longrightarrow \infty, \qquad n \geqq 3. \tag{33}$$

PROOF. The conditions (28), (29) imply

$$\int_{t_0}^{\infty} t^{\gamma}\tilde{p}(t)dt < \infty. \tag{34}$$

According to Taliaferro's theorems [9, Theorems 2.4 and 3.2], if (34) holds, then (7) has positive solutions $y_1(t)$ and $y_2(t)$ on $t \geqq t_0$ satisfying

$$0 < \lim_{t\to\infty} y_1'(t) < \infty \quad \text{and} \quad \lim_{t\to\infty} y_2'(t) = \infty,$$

respectively. To the functions $y_1(t)$ and $y_2(t)$ there correspond positive solutions $\rho_1(r)$ and $\rho_2(r)$ of (5) satisfying

$$\lim_{r\to\infty} \rho_1(r)/\log r = K \quad (n = 2), \qquad \lim_{r\to\infty} \rho_1(r) = K' \quad (n \geqq 3),$$

and

$$\lim_{r\to\infty} \rho_2(r)/\log r = \infty \quad (n = 2), \qquad \lim_{r\to\infty} \rho_2(r) = \infty \quad (n \geqq 3),$$

respectively, where K and K' are some positive constants. On the other hand, by Lemma 2, $z_i(t) = C^{1/(1-\gamma)} y_i(t)$ $(i=1, 2)$ are solutions of (27), and since $\gamma > 1$ and $C \geqq 1$, we have $y_i(t) \geqq z_i(t)$ $(i=1, 2)$ for all $t \geqq t_0$. Consequently we see that (5) and (26) have positive solutions $\rho_i(r)$ and $\eta_i(r) = C^{1/(1-\gamma)} \rho_i(r)$, respectively, such that $\rho_i(r) \geqq \eta_i(r)$ $(i=1, 2)$ for all $r \geqq a$. Now the conclusion follows from Theorem 1.

REMARK 1. Under assumptions (I) and (III), if (28), (29) hold, then equation (1) also has a positive solution $u(x)$ in G_a satisfying (16), (17). In fact, it is obvious that conditions (28) and (29) imply (14) and (15) with $P(r) = Cp(r)$, respectively.

EXAMPLE 2. Suppose that $\phi(x)$ satisfies assumption (I) and there exists a number σ such that in G_a

$$c_1 |x|^{-2} (\log |x|)^\sigma \leqq \phi(x) \leqq c_2 |x|^{-2} (\log |x|)^\sigma, \qquad n = 2,$$
$$c_1 |x|^\sigma \leqq \phi(x) \leqq c_2 |x|^\sigma, \qquad n \geqq 3,$$

for some positive constants c_1, c_2. If $\sigma \geqq -n + \gamma(n-2)$, then equation (1) has a positive solution $u(x)$ in G_a satisfying (12), (13). If $\sigma < -n + \gamma(n-2)$, then equation (1) has a positive solution $u(x)$ in G_a satisfying (16), (17). If $\sigma < -1 - \gamma$ for $n=2$ and $\sigma < -2$ for $n \geqq 3$, then equation (1) has positive solutions $u_1(x)$ satisfying (30), (31), $u_2(x)$ satisfying (32), (33) and $u(x)$ satisfying (16), (17) in G_a.

4. Sublinear equations

In this section we consider the sublinear case, i.e. equation (1) with $0 < \gamma < 1$.

THEOREM 4. *Let assumptions* (I) *and* (II) *be satisfied.*

(A) *If* (14), (15) *hold, then* (1) *has a positive solution* $u(x)$ *in* G_a *satisfying* (16), (17).

(B) *If*

(35) $$\int_a^\infty r(\log r)^\gamma P(r)dr < \infty, \qquad n = 2,$$

(36) $$\int_a^\infty rP(r)dr < \infty, \qquad n \geqq 3,$$

then (1) *has a positive solution* $u_1(x)$ *in* G_a *satisfying* (30), (31).

PROOF. (A) Conditions (14), (15) imply (21). It is well-known (see, for example [2]) that for any positive constant c there is a positive decreasing solution $z(t)$ of (8) such that $\lim_{t\to\infty} z(t)=c$. Since (23) holds by (9) and (21), there is a positive decreasing solution $y(t)$ of (7) such that $\lim_{t\to\infty} y(t)=z(t_0)>0$. Then it is clear that $y(t)\geqq z(t)$ for all $t\geqq t_0$. Arguing as in the part (B) of Theorem 2, we see that the conclusion of (A) holds.

(B) In view of (35), (36) we have

(37) $$\int_{t_0}^\infty t^\gamma \tilde{P}(t)dt < \infty.$$

By the proof of Čanturija's theorem [1, Theorem 6], if (37) holds, then every unbounded positive solution of (8) is asymptotically linear as $t\to\infty$, that is, if $z(t)$ is an unbounded positive solution, then there exists a positive constant c such that $\lim_{t\to\infty} z'(t)\leqq c$. Since $z''(t)>0$, $z'(t)$ is increasing, so we have $z'(t)\leqq c$ for all $t\geqq t_0$. Integrating over $[t_0, t]$, we get

(38) $$z(t) \leqq c(t - t_0) + z(t_0)$$

for $t\geqq t_0$. If $y(t)$ is an unbounded positive solution of (7) satisfying

$$y(t_0) = z(t_0) > 0, \quad y'(t_0) \geqq c,$$

then $y'(t)\geqq c$ for all $t\geqq t_0$. Therefore it is easily derived from (38) that $z(t)\leqq y(t)$ for all $t\geqq t_0$. On the other hand, since (34) holds by virtue of (9) and (37), $y(t)$ must be asymptotically linear as $t\to\infty$. Proceeding as in the proof of Theorem 3, we get the conclusion of (B).

REMARK 2. Conditions (14) and (15) imply (35) and (36), respectively. Therefore, if (14), (15) hold, then (1) has a positive solution $u_1(x)$ in G_a satisfying (30), (31) as well as a positive solution $u(x)$ in G_a satisfying (16), (17).

EXAMPLE 3. Suppose that $\phi(x)$ satisfies assumption (I) and there exists a number σ such that in G_a

$$0 < \phi(x) \leqq c_2|x|^{-2}(\log|x|)^\sigma, \qquad n = 2,$$
$$0 < \phi(x) \leqq c_2|x|^\sigma, \qquad n \geqq 3,$$

for some positive constant c_2. If $\sigma < -1-\gamma$ for $n=2$ and $\sigma < -2$ for $n \geqq 3$, then (1) has a positive solution $u_1(x)$ in G_a satisfying (30), (31). If $\sigma < -n+\gamma(n-2)$, then (1) has positive solutions $u(x)$ and $u_1(x)$ in G_a satisfying (16), (17) and (30), (31), respectively.

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