HIROSHIMA MATH. J.
12 (1982), 43–58

Asymptotic theory of perturbed general disconjugate equations

William F. TRENCH
(Received May 20, 1981)

1. Introduction

There has been considerable recent interest in the asymptotic behavior of solutions of the equation

$$L_n u + Fu = 0, \quad 0 < t < \infty, \tag{1}$$

where L_n is the general disconjugate operator

$$L_n = \frac{1}{p_n}\frac{d}{dt}\frac{1}{p_{n-1}}\cdots\frac{1}{p_1}\frac{d}{dt}\frac{\cdot}{p_0} \qquad (n \geq 2), \tag{2}$$

with

$$p_i > 0 \quad \text{and} \quad p_i \in C[0, \infty), \quad 0 \leq i \leq n, \tag{3}$$

and F is some functional of u. As examples, we cite [1], [7], [8], [9], [10], [11], [13], and [17].

Here we are interested in comparing solutions of (1) with those of the unperturbed general disconjugate equation

$$L_n x = 0, \quad t > 0. \tag{4}$$

Willett [19] and the author [14] have observed that special attention should be paid to the asymptotic theory of equations of the form

$$L_n u + g(t, u, u', \dots, u^{(n-1)}) = 0,$$

where L_n is a *normal* disconjugate operator on $[0, \infty)$; that is, the equation

$$L_n x \equiv x^{(n)} + P_1(t)x^{(n-1)} + \cdots + P_n(t)x = 0,$$

with $P_1, \dots, P_n$ continuous, is disconjugate on $[0, \infty)$. Polya [12] showed that such an operator can be written as in (2), with (3) replaced by the stronger condition

$$p_i > 0 \quad \text{and} \quad p_i \in C^{(n-i)}[0, \infty), \quad 0 \leq i \leq n. \tag{5}$$

However, the additional smoothness conditions on $p_0, \dots, p_{n-1}$ which appear in

(5) are usually unnecessary, and it is more natural to formulate conditions on the perturbing functional F in terms of generalized derivatives associated with L_n, rather than in terms of ordinary derivatives. By taking this point of view it is possible to state one of the main results of [14] in a considerably improved form (Theorem 1, below).

In [14] the author suggested that the asymptotic theory of perturbed disconjugate equations can be based on integral smallness conditions on F which involve ordinary — rather than absolute — convergence of some of the improper integrals in question. Except for a result of Hartman and Wintner [3; Theorem 9.1, p. 379] for second order equations, this possibility seems to have been ignored before that, even in the case where $L_n u = u^{(n)}$. Since [14], the author has obtained results along these lines for linear homogeneous perturbations of the equation $u^{(n)} = 0$ $(n \geq 2)$, and of nonoscillatory second order equations [16], [18]. Theorems 1 and 2 below assume integral smallness on F which, in general form, do not require absolute convergence. This is not to say that it is unnecessary to assume absolute convergence of *some* integrals in order to obtain specific, usable, special cases; the point is that not *all* such integrals need be absolutely convergent, as has usually been assumed in the past. Theorem 3 illustrates this point for linear perturbations of $L_n u = 0$.

2. Preliminary definitions and lemmas

In connection with the operator L_n it is convenient (and customary) to define the generalized lower order derivatives $L_0, L_1, \ldots, L_{n-1}$ by

$$L_0 x = \frac{x}{p_0}, \quad L_r x = \frac{1}{p_r}(L_{r-1}x)', \quad 1 \leq r \leq n. \tag{6}$$

Henceforth we assume, in connection with the functional F in (1), that Fu is continuous on any interval over which $L_{n-1}u$ is continuous.

The following notation of Willett [19] is useful for representing solutions of $L_n x = 0$ and their generalized derivatives.

If $q_1, q_2, \ldots$ are locally integrable on $[0, \infty)$, define

$$I_0 = 1$$

and

$$I_j(t, s; q_j, \ldots, q_1) = \int_s^t q_j(\lambda) I_{j-1}(\lambda, s; q_{j-1}, \ldots, q_1) d\lambda, \quad s, t \geq 0, \ j \geq 1.$$

Willett [19; Lemma 2.2] has established the following identities, which will be useful below:

$$I_j(t, s; q_j, \ldots, q_1) = (-1)^j I_j(s, t; q_1, \ldots, q_j), \tag{7}$$

(8) $\sum_{j=0}^{k}(-1)^j I_j(t, a; q_k,\dots, q_{k-j+1}) I_{k-j}(s, a; q_1,\dots, q_{k-j}) = I_k(s, t; q_1,\dots, q_k)$.

It is easily verified that if a is in $[0, \infty)$, then the functions

(9) $$x_j(t) = p_0(t) I_{j-1}(t, a; p_1,\dots, p_{j-1}), \quad 1 \le j \le n,$$

are linearly independent solutions of (4), and that the functions

(10) $$y_j(t) = p_n(t) I_{n-j}(t, a; p_{n-1},\dots, p_j), \quad 1 \le j \le n,$$

are linearly independent solutions of $L_n^* y = 0$, where

$$L_n^* = \frac{1}{p_0}\frac{d}{dt}\frac{1}{p_1}\cdots\frac{1}{p_{n-1}}\frac{d}{dt}\frac{\cdot}{p_n}.$$

Moreover,

(11) $$L_r x_j(t) = 0, \quad j \le r,$$

and

(12) $$L_r x_j(t) = I_{j-r-1}(t, a; p_{r+1},\dots, p_{j-1}), \quad r+1 \le j \le n.$$

Throughout the rest of the paper, $x_1,\dots, x_n$ and $y_1,\dots, y_n$ will be as defined in (9) and (10), with $a \ge 0$.

The following lemma presents variation of parameters in a form suitable for treating (1) as a perturbation of (4).

LEMMA 1. *A function u is a solution of* (1) *if and only if*

(13) $$L_r u(t) = \sum_{j=r+1}^{n} c_j(t) L_r x_j(t), \quad 0 \le r \le n-1$$

(*recall* (11)), *where*

(14) $$c_j'(t) = (-1)^{n-j-1} y_j(t)(Fu)(t), \quad 1 \le j \le n.$$

PROOF. By the usual variation of parameters argument, it can be shown that if

$$u(t) = \sum_{j=1}^{n} c_j(t) x_j(t),$$

and

(15) $$\sum_{j=r+1}^{n} c_j'(t) L_r x_j(t) = 0, \quad 0 \le r \le n-2,$$

then u satisfies (1) if and only if

(16) $$c_n'(t) = -y_n(t)(Fu)(t).$$

(Note that $y_n = p_n$, from (10).) Now, (15) and (16) form a system of n equations in $c_1', c_2',\dots, c_n'$, with matrix

$$V = [L_{r-1}x_j]_{r,j=1}^n.$$

Since the right sides of the first $n-1$ equations (15) of this system vanish, (14) will follow if it is shown that the last column of V^{-1} is

$$\operatorname{col}\left[(-1)^{n-1}\frac{y_1}{y_n}, (-1)^{n-2}\frac{y_2}{y_n}, \dots, -\frac{y_{n-1}}{y_n}, 1\right].$$

This can be seen by setting $t=s$ in the identities

$$(17)\quad \textstyle\sum_{j=r+1}^n (-1)^{n-j}\frac{y_j(s)}{y_n(s)}L_r x_j(t) = I_{n-r-1}(t, s; p_{r+1}, \dots, p_{n-1}), \quad 0 \le r \le n-1,$$

which follow from (7), (8), (10), and (12).

The following lemma plays a crucial role in simplifying the asymptotic theory of (1).

LEMMA 2. *If*

$$(18)\qquad \int^\infty p_i(t)dt = \infty, \quad 1 \le i \le n-1,$$

then

$$(19)\qquad \left(\frac{L_r x_j}{L_r x_i}\right)' > 0 \quad on \quad (a, \infty), \quad and \quad \lim_{t\to\infty}\frac{L_r x_j(t)}{L_r x_i(t)} = \infty, \quad r < i < j \le n,$$

and

$$(20)\qquad \left(\frac{y_i}{y_j}\right)' > 0 \quad on \quad (a,\infty) \quad and \quad \lim_{t\to\infty}\frac{y_i(t)}{y_j(t)} = \infty, \quad 1 \le i < j \le n.$$

PROOF. From (10), (12) and Lemma 3.1 of Willett [19], the derivatives in (19) and (20) are positive if $t > a$. The assertions about the limits follow from (18) and l'Hospital's rule.

Notice that (18) places no restriction on p_0 or p_n. It is known [15] that (18) can be assumed without loss of generality; that is, if L_n as written in (2) does not satisfy (18), it can be rewritten as

$$L_n = \frac{1}{\tilde{p}_n}\frac{d}{dt}\frac{1}{\tilde{p}_{n-1}}\cdots\frac{d}{dt}\frac{1}{\tilde{p}_1}\frac{d}{dt}\frac{\cdot}{\tilde{p}_0},$$

where

$$\int^\infty \tilde{p}_i(t)dt = \infty, \quad 1 \le i \le n-1,$$

and $\tilde{p}_0, \tilde{p}_i, \ldots, \tilde{p}_n$ are unique up to positive multiplicative constants with product one. Therefore, we assume (18) henceforth, in which case L_n is said to be in canonical form at ∞ [15]. (For related results on canonical forms for disconjugate operators, see Granata [2].)

For normal disconjugate equations, Hartman [4], [5], [6] established the existence of solutions $x_1, \ldots, x_n$ satisfying (19) with $r=0$, and Willett [19] showed that they could be represented in the form (9). The author [15] extended these results to the general disconjugate equation.

LEMMA 3. *If $q_1, q_2, \ldots,$ are continuous and positive on $[a, \infty)$ and $a<b$, then*

$$\frac{d}{dt}\left(\frac{I_j(t, b; q_j, \ldots, q_1)}{I_j(t, a; q_j, \ldots, q_1)}\right) > 0, \ t \geq b, j \geq 1. \tag{21}$$

PROOF. The proof is by induction. For convenience, let

$$f_j(t) = I_j(t, a; q_j, \ldots, q_1) \quad \text{and} \quad g_j(t) = I_j(t, b; q_j, \ldots, q_1).$$

Then

$$\left(\frac{g_j}{f_j}\right)' = \frac{q_j}{f_j^2}\,(f_j g_{j-1} - f_{j-1} g_j),$$

and so it suffices to show that

$$f_j g_{j-1} - f_{j-1} g_j > 0. \tag{22}$$

Since

$$f_1(t) g_0(t) - f_0(t) g_1(t) = \int_a^t q_1(\lambda) d\lambda - \int_b^t q_1(\lambda) d\lambda > 0,$$

if $a<b<t$, (21) follows for $j=1$. Now suppose $j \geq 2$ and (21) holds with j replaced by $j-1$. The left side of (22) can be written as

$$f_j(t) g_{j-1}(t) - f_{j-1}(t) g_j(t) = g_{j-1}(t) \int_a^b q_j(\lambda) f_{j-1}(\lambda) d\lambda$$
$$+ \int_b^t q_j(\lambda)\,[f_{j-1}(\lambda) g_{j-1}(t) - f_{j-1}(t) g_{j-1}(\lambda)] d\lambda.$$

The first term on the right is clearly positive, and the second can be rewritten as

$$f_{j-1}(t) \int_b^t q_j(\lambda) f_{j-1}(\lambda) \left[\frac{g_{j-1}(t)}{f_{j-1}(t)} - \frac{g_{j-1}(\lambda)}{f_{j-1}(\lambda)}\right] d\lambda,$$

which is positive by the inductive assumption if $t>b$. This establishes (22), and completes the proof.

LEMMA 4. *Suppose Q is continuous for $t \geq T \geq a$ and the integral $\int^{\infty} y_i(t)Q(t)dt$ converges for some i, $1 \leq i \leq n$. Let*

$$\rho(t) = \max_{\tau \geq t} \left| \int_{\tau}^{\infty} y_i(s)Q(s)ds \right| .$$

Then $\int^{\infty} y_j(s)Q(s)ds$ converges if $i \leq j \leq n$, and

$$\left| \int_t^{\infty} y_j(s)Q(s)\, ds \right| \leq 2\rho(t) \frac{y_j(t)}{y_i(t)}, \quad t \geq T \geq a. \tag{23}$$

PROOF. Obviously (23) holds with $i=j$, in which case the two on the right may be replaced by one. If $j>i$, let

$$c(t) = \int_t^{\infty} y_i(s)Q(s)ds, \tag{24}$$

and suppose $T \leq t < t_1$. Then

$$\begin{aligned} \int_t^{t_1} y_j(s)Q(s)ds &= -\int_t^{t_1} \frac{y_j(s)}{y_i(s)} c'(s)ds \\ &= -\frac{y_j(t_1)}{y_i(t_1)} c(t_1) + \frac{y_j(t)}{y_i(t)} c(t) + \int_t^{t_1} \left(\frac{y_j(s)}{y_i(s)} \right)' c(s)ds. \end{aligned} \tag{25}$$

From (20) and the boundedness of $c(t)$, the first term on the right of (25) approaches zero, and the integral on the right converges absolutely, as $t_1 \to \infty$; hence the integral on the left converges as $t_1 \to \infty$, and

$$\int_t^{\infty} y_j(s)Q(s)ds = \frac{y_j(t)}{y_i(t)} c(t) + \int_t^{\infty} \left(\frac{y_j(s)}{y_i(s)} \right)' c(s)ds.$$

This implies (23), again because of (20).

We will use (17) again. In this connection it is convenient to define

$$g_r(t, s) = y_n(s) I_{n-r-1}(t, s; p_{r+1}, \ldots, p_{n-1}), \quad 0 \leq r \leq n-1. \tag{26}$$

LEMMA 5. *Under the hypotheses of Lemma 4, the integrals*

$$\int_t^{\infty} g_r(t, s)Q(s)ds, \quad i-1 \leq r \leq n, \tag{27}$$

converge, and

$$\left| \int_t^{\infty} g_r(t, s)Q(s)ds \right| \leq 2\rho(t) \frac{y_{r+1}(t)}{y_i(t)}, \quad t \geq T,\ i-1 \leq r \leq n-1. \tag{28}$$

PROOF. From (17) and (26),

$$g_r(t, s) = \sum_{j=r+1}^{n} (-1)^{n-j} y_j(s) L_r x_j(t),$$

so Lemma 4 implies that the integrals (27) converge. Since $g_{n-1}(t, s)=y_n(s)$ (see (26)), (23) with $j=n$ implies (28) with $r=n-1$; hence, we need only consider (28) with $r\leq n-2$. For convenience, define

$$G_r(t) = \int_t^\infty g_r(t, s)Q(s)ds.$$

From (24), we can rewrite this as

$$G_r(t) = -\int_t^\infty H_{ir}(t, s)c'(s)ds, \tag{29}$$

where

$$H_{ir}(t,s) = \frac{y_n(s)I_{n-r-1}(t, s; p_{r+1},\dots, p_{n-1})}{y_i(s)},$$

which, from (7) and (10), can be rewritten as

$$H_{ir}(t, s) = (-1)^{n-r-1} \frac{y_{r+1}(s)}{y_i(s)} \frac{I_{n-r-1}(s, t; p_{n-1},\dots, p_{r+1})}{I_{n-r-1}(s, a; p_{n-1},\dots, p_{r+1})}. \tag{30}$$

If $0\leq r\leq n-2$, then

$$0 < \frac{I_{n-r-1}(s, t; p_{n-1},\dots, p_{r+1})}{I_{n-r-1}(s, a; p_{n-1},\dots, p_{r+1})} < 1, \quad a < t < s, \tag{31}$$

and $H_{ir}(t, t)=0$; hence, since $\lim_{t\to\infty} c(t)=0$, integrating (29) by parts yields

$$G_r(t) = \int_t^\infty c(s) \frac{\partial H_{ir}}{\partial s}(t, s)ds, \tag{32}$$

provided we can show that the integral on the right converges. From (30),

$$(-1)^{n-r-1} \frac{\partial H_{ir}}{\partial s}(t, s) = \left(\frac{y_{r+1}(s)}{y_i(s)}\right)' \frac{I_{n-r-1}(s, t; p_{n-1},\dots, p_{r+1})}{I_{n-r-1}(s, a; p_{n-1},\dots, p_{r+1})} + \left(\frac{y_{r+1}(s)}{y_i(s)}\right) \frac{\partial}{\partial s}\left(\frac{I_{n-r-1}(s, t; p_{n-1},\dots, p_{r+1})}{I_{n-r-1}(s, a; p_{n-1},\dots, p_{r+1})}\right).$$

From Lemma 3, the partial derivative in the second term on the right is positive if $s>t\geq a$; moreover, $(y_{r+1}/y_i)'\leq 0$ since $r\geq i-1$. Therefore, from (31),

$$\left|\frac{\partial H_{ir}(t, s)}{\partial s}\right| \leq -\left(\frac{y_{r+1}(s)}{y_i(s)}\right)' + \frac{y_{r+1}(t)}{y_i(t)} \frac{\partial}{\partial s}\left(\frac{I_{n-r-1}(s, t; p_{n-1},\dots, p_{r+1})}{I_{n-r-1}(s, a; p_{n-1},\dots, p_{r+1})}\right)$$

if $s>t\geq a$. This and (31) imply that the integral in (32) converges, and also that (28) holds. (We may drop the 2 in (28) if $r=i-1$, but this is not important.)

3. Main results

Suppose u is a solution of (1) for which the parameter functions $c_1, c_2,\dots, c_n$ of Lemma 1 converge to finite limits as $t\to\infty$; say

$$\lim_{t\to\infty} c_j(t) = a_j,$$

and let

$$q(t) = \textstyle\sum_{j=1}^n a_j x_j(t). \tag{33}$$

Then clearly there is an asymptotic relationship between the generalized derivatives $L_0u,\dots, L_{n-1}u$ and $L_0q,\dots, L_{n-1}q$: from (13) and (33),

$$L_ru(t) - L_rq(t) = \textstyle\sum_{j=r+1}^n (c_j(t)-a_j)L_rx_j(t),$$

which, from (19), yields the obvious estimate

$$L_ru(t) - L_rq(t) = o(L_rx_n(t)),\quad 0\leq r\leq n-1. \tag{34}$$

However, this is by no means the best available estimate, as we will now see.

Theorem 1. *Suppose u is a solution of* (1) *on* $[T,\infty)$ *such that the integral* $\int^\infty y_i(s)(Fu)(s)ds$ *converges for some* i, $1\leq i\leq n$. *Then the parameter functions* $c_i,\dots, c_n$ *associated with* u *in Lemma* 1 *converge to finite limits as* $t\to\infty$:

$$\lim_{t\to\infty} c_j(t) = a_j,\quad i\leq j\leq n. \tag{35}$$

Moreover, if

$$\rho(t) = \max_{\tau\geq t}\left|\int_\tau^\infty y_i(s)(Fu)(s)ds\right|,$$

then

$$\left| L_ru(t) - \textstyle\sum_{j=r+1}^n a_jL_rx_j(t)\right| \leq 2\rho(t)\frac{y_{r+1}(t)}{y_i(t)},\quad i-1\leq r\leq n-1, \tag{36}$$

and, if $i\geq 2$,

$$L_ru(t) = \textstyle\sum_{j=i}^n a_jL_rx_j(t) + o(L_rx_i(t)),\quad 0\leq r\leq i-2. \tag{37}$$

Proof. By Lemma 4 and our assumption, the integrals $\int^\infty y_j(t)(Fu)(t)dt$ converge for $i\leq j\leq n$; therefore, from (14), the limits in (35) exist and

$$c_j(t) = a_j + (-1)^{n-j}\int_t^\infty y_j(s)(Fu)(s)ds, \quad i \le j \le n.$$

Substituting this in (13) and using (17) and (26) yield

$$(38) \quad L_r u(t) = \textstyle\sum_{j=r+1}^n a_j L_r x_j(t) + \int_t^\infty g_r(t, s)(Fu)(s)ds, \quad i-1 \le r \le n,$$

and now Lemma 5 (specifically, (28)) implies (36). This completes the proof if $i=1$. If $i \ge 2$, set $r=i-1$ in (38) to obtain

$$(39) \quad L_{i-1}u(t) = \textstyle\sum_{j=i}^n a_j L_{i-1}x_j(t) + \varepsilon(t),$$

where

$$(40) \quad \varepsilon(t) = \int_t^\infty g_{i-1}(t, s)(Fu)(s)ds = o(1).$$

From (6) and (39), integration yields

$$(41) \quad L_{i-2}u(t) = k_{i-2} + \textstyle\sum_{j=i}^n a_j L_{i-2}x_j(t) + I_1(t, T; \varepsilon p_{i-1}),$$

where k_{i-2} is a constant. Now $\lim_{t\to\infty} L_{i-2}x_i(t) = \infty$, and

$$\lim_{t\to\infty} \frac{I_1(t, T; \varepsilon p_{i-1})}{L_{i-2}x_i(t)} = \lim_{t\to\infty} \frac{I_1(t, T; \varepsilon p_{i-1})}{I_1(t, a; p_{i-1})} = 0$$

(see (12)), where the last limit is zero because of (40). This proves (37) with $r=i-2$. If $i \ge 3$, then (6) and repeated integration, starting from (41), yield

$$L_r u(t) = \textstyle\sum_{j=r}^{i-2} k_j L_r x_{j+1}(t) + \sum_{j=i}^n a_j L_r x_j(t) + I_{i-r-1}(t, T; p_{r+1}, \dots, p_{i-2}, \varepsilon p_{i-1}),$$
$$0 \le r \le i-3,$$

where $k_r, \dots, k_{i-2}$ are constants of integration. Now

$$L_r x_{j+1}(t) = o(L_r x_i(t)), \quad 0 \le j \le i-2,$$

and

$$\lim_{t\to\infty} \frac{I_{i-r-1}(t, T; p_{r+1}, \dots, p_{i-2}, \varepsilon p_{i-1})}{L_r x_i(t)} = 0,$$

again because of (12) and (40). This completes the proof.

With $i=1$, and q as defined in (33), (36) implies that

$$(42) \quad L_r u(t) - L_r q(t) = o\left(\frac{y_{r+1}(t)}{y_1(t)}\right), \quad 0 \le r \le n-1,$$

which is considerably sharper than the obvious estimate (34). The difference between (34) and (42) is perhaps most striking in the case where $L_n u = u^{(n)}$, in which case (34) becomes

$$u^{(r)}(t) - q^{(r)}(t) = o(t^{n-r-1}), \quad 0 \le r \le n-1,$$

while (42) becomes

$$u^{(r)}(t) - q^{(r)}(t) = o(t^{-r}), \quad 0 \le r \le n-1.$$

For the special case where $L_n u = u^{(n)}$, Theorem 1 was given in [14], which also contains results for perturbations of more general normal disconjugate equations; however, those results are not so precise as (36) and (37).

Theorem 1 has the following obvious corollary.

COROLLARY 1. *If u is an oscillatory solution of* (1) *for which the integral* $\int^{\infty} y_i(t)(Fu)(t)dt$ *converges, then*

$$L_r u = \begin{cases} o(L_r x_i), & 0 \le r \le i-2, \\ o(y_{r+1}/y_i), & i-1 \le r \le n-1. \end{cases}$$

We now give conditions under which (1) has solutions which behave asymptotically like a given solution of $L_n x = 0$. In this connection the following definition is useful.

DEFINITION 1. Suppose $1 \le i \le n$, and let $H_i(T)$ be the space of functions h such that $L_{n-1}h$ is continuous for $t \ge T > a$, and

$$L_r h = \begin{cases} O(L_r x_i), & 0 \le r \le i-2, \\ O\left(\frac{y_{r+1}}{y_i}\right), & i-1 \le r \le n-1, \end{cases} \quad t \ge T.$$

For each such h, let

$$N_i(T; h) = \sup_{t \ge T} \left\{ \sum_{r=0}^{i-2} \frac{|L_r h(t)|}{L_r x_i(t)} + \sum_{r=i-1}^{n-1} \frac{y_i(t)}{y_{r+1}(t)} |L_r h(t)| \right\}. \tag{43}$$

THEOREM 2. *Let*

$$q(t) = \sum_{j=i}^{m} a_j x_j(t) \tag{44}$$

where $1 \le i \le m \le n$ and $a_i, \ldots, a_m$ are constants. Suppose there is a constant M and a nonincreasing function σ_1 defined for $t \ge a$ such that

$$\lim_{t \to \infty} \sigma_1(t) = 0$$

and $\int^{\infty} y_i(t)(Fv)(t)dt$ *exists and satisfies*

$$\max_{\lambda \geq T} \left| \int_{\lambda}^{\infty} y_i(s)(Fv)(s)ds \right| \leq \sigma_1(T) \tag{45}$$

whenever $L_{n-1}v$ *is continuous on* $[T, \infty)$, $v-q \in H_i(T)$, *and*

$$N_i(T; v-q) \leq M.$$

Suppose further that there is a nonincreasing function σ_2 *defined for* $t \geq a$ *such that* $\lim_{t\to\infty} \sigma_2(t) = 0$ *and*

$$\max_{\lambda \geq T} \left| \int_{\lambda}^{\infty} y_i(s)[(Fv_1)(s)-(Fv_2)(s)]ds \right| < \sigma_2(T) N_i(T; v_1 - v_2) \tag{46}$$

whenever v_1 *and* v_2 *both satisfy the above stated conditions on* v. *Then there is a solution of* (1), *defined for sufficiently large* t, *such that*

$$|L_r u(t) - L_r q(t)| \leq 2\sigma_1(t) \frac{y_{r+1}(t)}{y_i(t)}, \quad i-1 \leq r \leq n-1, \tag{47}$$

and, if $i \geq 2$,

$$L_r u(t) = L_r q(t) + o(L_r x_i(t)), \quad 0 \leq r \leq i-2. \tag{48}$$

PROOF. Choose T so that

$$\sigma_1(T) \leq M/2n \quad \text{and} \quad \sigma_2(T) = \gamma < 1/2n, \tag{49}$$

and assume henceforth that $t \geq T$. For brevity, let

$$\|h\| = N_i(T; h)$$

for $h \in H_i(T)$. Let $\tilde{H}_i(T)$ be the subset of $H_i(T)$ for which $\|h\| \leq M$.

From (45), (49), and Lemma 5,

$$\left| \int_t^{\infty} g_r(t, s)(Fv)(s)ds \right| \leq (M/n) \frac{y_{r+1}(t)}{y_i(t)}, \quad i-1 \leq r \leq n-1, \tag{50}$$

whenever $v-q \in \tilde{H}_i(T)$. If $i \geq 2$, and $v-q \in \tilde{H}_i(T)$, define

$$G_{i-1}(t; v) = \int_t^{\infty} g_{i-1}(t, s)(Fv)(s)ds$$

and note that

$$|G_{i-1}(t; v)| \leq M/n,$$

from (50) with $r = i-1$. Therefore,

$$(51) \qquad |I_1(t, T; p_{i-1}G_i(\cdot\,; v))| \leq (M/n)I_1(t, a; p_{i-1}) = (M/n)L_{i-2}x_i$$

(see (12)), and, if $i \geq 3$,

$$(52) \quad |I_{i-r-1}(t, T; p_{r+1},\dots, p_{i-2}, p_{i-1}G_{i-1}(\cdot\,; v))| \\ \leq (M/n)I_{i-r-1}(t, a; p_{r+1},\dots, p_{i-1}) = (M/n)L_r x_i, \quad 0 \leq r \leq i-3,$$

(again see (12)).

Now define a sequence $\{v_k\}$ of functions on $[T, \infty)$, with $v_0(t) = q(t)$, and, for $k \geq 1$,

(a) if $i=1$,

$$(53) \qquad v_k(t) = q(t) + p_0(t)\int_t^\infty g_0(t, s)(Fv_{k-1})(s)ds;$$

(b) if $i=2$,

$$(54) \qquad v_k(t) = q(t) + p_0(t)I_1(t, T; p_1G_1(\cdot\,; v_{k-1}));$$

(c) if $3 \leq i \leq n$,

$$(55) \qquad v_k(t) = q(t) + p_0(t)I_{i-1}(t, T; p_1,\dots, p_{i-2}, p_{i-1}G_{i-1}(\cdot\,; v_{k-1})).$$

If $v_{k-1} - q \in \tilde{H}_i(T)$, then the integrals in (53), (54), and (55) all exist, and so v_k is defined in each of the cases (a), (b), and (c). Moreover, by calculating $L_0v_k,\dots, L_{n-1}v_k$ from whichever of (53), (54), or (55) is applicable and invoking (50), (51), and (52), it can be seen that

$$|L_rv_k(t) - L_rq(t)| \leq (M/n)L_rx_i(t), \quad 0 \leq r \leq i-2,$$

and

$$|L_rv_k(t) - L_rq(t)| \leq (M/n)\frac{y_{r+1}(t)}{y_i(t)}, \quad i-1 \leq r \leq n-1.$$

Therefore

$$\|v_k - q\| \leq M;$$

that is, $v_k - q \in \tilde{H}_i(T)$ if $v_{k-1} - q \in \tilde{H}_i(T)$. Since $q - v_0 \in \tilde{H}_i(T)$, it follows by induction that $q - v_k \in \tilde{H}_i(T)$ for all $k \geq 0$.

We will now show that $\{v_k\}$ converges. From (53), (54), and (55),

$$L_r(v_k(t) - v_{k-1}(t)) = \int_t^\infty g_r(t, s)[(Fv_{k-1})(s) - (Fv_{k-2})(s)]ds, \\ i-1 \leq r \leq n-1.$$

Therefore, from (46) and Lemma 5,

(56) $$|L_r(v_k(t) - v_{k-1}(t))| \leq 2\|v_{k-1} - v_{k-2}\|\sigma_2(t)\frac{y_{r+1}(t)}{y_i(t)}$$
$$< 2\gamma\|v_{k-1}-v_{k-2}\|\frac{y_{r+1}(t)}{y_i(t)}, \quad i-1 \leq r \leq n-1,$$

because of (49). If $i \geq 2$, an argument based on (54) or (55), and similar to that used in obtaining (51) and (52), implies that

(57) $$|L_r(v_k(t)-v_{k-1}(t))| \leq 2\gamma\|v_{k-1}-v_{k-2}\|L_r x_i(t), \quad 0 \leq r \leq i-2.$$

Now (56) and (57) imply that

(58) $$\|v_k-v_{k-1}\| \leq 2n\gamma\|v_{k-1}-v_{k-2}\|.$$

If we let

$$w_k = v_k - q$$

so that $w_k \in \tilde{H}_i(T)$, then (58) implies that

(59) $$\|w_k-w_{k-1}\| \leq 2n\gamma\|w_{k-1} - w_{k-2}\|.$$

Since $2n\gamma < 1$, an elementary argument based on (59) shows that $\{w_k\}$ is a Cauchy sequence in the Banach space $H_i(T)$ under the norm $\|\ \|$, and so $\{w_k\}$ converges in this norm to a limit function w, which is also in $H_i(T)$; in fact, since each w_k is in $\tilde{H}_i(T)$, so is w. A routine argument now shows that the function $u=q+w$ is a solution of (1) on $[T, \infty)$. Moreover, since $u-q \in \tilde{H}_i(T)$, (45) holds with $v=u$, and so Theorem 1 implies (47) and (48).

THEOREM 3. *Suppose $P_1,\dots, P_n$ and f are continuous on $[0, \infty)$. Let $1 \leq i \leq m \leq n$, and suppose*

(60) $$\int^\infty y_i(t)\,|P_{n-r}(t)|L_r x_i(t)dt < \infty, \quad 0 \leq r \leq i-2,$$

(61) $$\int^\infty y_{r+1}(t)\,|P_{n-r}(t)|dt < \infty, \quad i-1 \leq r \leq n-1,$$

and that the integrals

(62) $$\int^\infty y_i(t)f(t)dt$$

and

(63) $$\int^\infty y_i(t)P_{n-r}(t)L_r x_m(t)dt, \quad 0 \leq r \leq m-1,$$

converge. Let q be as in (44). *Then the equation*

$$(64) \qquad L_n u + P_1(t)L_{n-1}u + \cdots + P_n(t)L_0 u = f(t)$$

has a solution u such that

$$L_r u(t) = L_r q(t) + o(L_r x_i(t)), \quad 0 \leq r \leq i-2,$$

and

$$L_r u(t) = L_r q(t) + o\left(\frac{y_{r+1}(t)}{y_i(t)}\right), \quad i-1 \leq r \leq n-1.$$

PROOF. We can rewrite (64) in the form (1), with

$$Fu = -f + \sum_{r=0}^{n-1} P_{n-r} L_r u.$$

If $v = q + h$, then

$$(65) \qquad Fv = -f + \sum_{r=0}^{m-1} P_{n-r} L_r q + \sum_{r=0}^{n-1} P_{n-r} L_r h;$$

moreover, if $v_1 = q + h_1$ and $v_2 = q + h_2$, then

$$(66) \qquad Fv_1 - Fv_2 = \sum_{r=0}^{n-1} P_{n-r} L_r(h_1 - h_2).$$

From (60), (61), and (66), the function

$$\sigma_2(t) = \sum_{r=0}^{i-2} \int_t^\infty y_i(s)\,|P_{n-r}(s)| L_r x_i(s) ds + \sum_{r=i-1}^{n-1} \int_t^\infty y_{r+1}(s)\,|P_{n-r}(s)| ds$$

satisfies the requirements of Theorem 2. (To verify (46), recall (43)). From (19) and Dirichlet's theorem for convergent improper integrals, the convergence of (63) implies that the integrals

$$\int^\infty y_i(t) P_{n-r}(t) L_r q(t) dt, \quad 0 \leq r \leq m-1,$$

converge. This and the convergence of (62) imply that the function

$$c(t) = \int_t^\infty y_i(s)(f(s) - \sum_{r=0}^{m-1} P_{n-r}(s) L_r q(s)) ds$$

is defined for $t \geq 0$. Moreover, from (65), the function

$$\sigma_1(t) = M\sigma_2(t) + \max_{\tau \geq t} |c(\tau)|$$

satisfies the requirements of Theorem 2, for any constant $M > 0$. Therefore (64) has a solution u which satisfies (47) and (48), and this completes the proof.

If $L_n x = x^{(n)}$, then we can take

$$x_j(t) = t^{j-1}/(j-1)! \quad \text{and} \quad y_j(t) = t^{n-j}/(n-j)!, \quad 1 \leq j \leq n.$$

Therefore, Theorem 3 has the following corollary.

COROLLARY 2. *Suppose* $P_1, \dots, P_n$ *and* f *are continuous on* $[0, \infty)$ *and*

$$\int^{\infty} t^{k-1}|P_k(t)|dt < \infty, \quad 1 \le k \le n.$$

Let

$$q(t) = \sum_{j=1}^{m} A_j t^{j-1},$$

where $1 \le i \le m \le n$, *and* $A_1, \dots, A_m$ *are constants. Suppose the integrals*

$$\int^{\infty} t^{n-i} f(t)dt$$

and

$$\int^{\infty} P_k(t) t^{k+(m-i)-1} dt, \quad n-m+1 \le k \le n,$$

converge. Then the equation

$$y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = f(t)$$

has a solution y *such that*

$$y^{(r)}(t) = q^{(r)}(t) + o(t^{i-r-1}), \quad 0 \le r \le n-1.$$

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Department of Mathematical Sciences,
Drexel University,
Philadelphia, Pennsylvania,
U.S.A.