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An infinite-dimensional semisimple Lie algebra

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The following has been an open question which is asked at the end of [1]: If a Lie algebra L over a field of characteristic zero is locally finite and semisimple, is L necessarily locally finite-dimensional-and-semisimple? We will give a negative answer to this question and investigate an interesting property of the Lie algebra constructed for the above purpose. We should notice that the converse of this question is true.

First we consider a well known Lie algebra. Let V be a vector space of infinite dimension over a field of characteristic zero, S the set of all transformations of V considered as a Lie algebra, A the set of elements of S of trace zero (in the sense in §4 of [2]) and F the set of elements of S of finite rank. It is shown in [2] that A is infinite-dimensional and simple and that $C_S(A)=\{x \in S: [x, A]=\{0\}\}$ is the set of scalar multiplications. It is easy to see that $F^2=A$ and F is locally finite. Further the only ideals of F are $\{0\}$, A and F. Let $\sigma(F)$ be the locally solvable radical of F. Since $[\sigma(F), A]=\{0\}$ by the fact that A is not locally solvable, $\sigma(F) \subseteq C_S(A) \cap F=\{0\}$. Thus F is semisimple.

Next we construct an infinite-dimensional semisimple Lie algebra. Let $\mathfrak{k}$ be a field of characteristic zero and S_i be a Lie algebra over $\mathfrak{k}$ with basis $\{x_i, y_i, h_i\}$ and multiplication $[x_i, y_i]=h_i$, $[x_i, h_i]=2x_i$, $[y_i, h_i]=-2y_i$ for $i=1, 2,\dots$. Let z be a derivation of $\oplus_{i=1}^{\infty} S_i$ defined by $x_i \mapsto 2x_i$, $y_i \mapsto -2y_i$, $h_i \mapsto 0$ for $i=1, 2,\dots$. Consider the split extension $L=\oplus_{i=1}^{\infty} S_i \dotplus \mathfrak{k}z$.

Let $\sigma(L)$ be the locally solvable radical of L and take an element $w=\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m} b_j y_j + \sum_{k=1}^{p} c_k h_k + dz$ $(a_i, b_j, c_k, d \in \mathfrak{k})$ of $\sigma(L)$. Since $[w, S_{n+m+p+1}] \subseteq \sigma(L) \cap S_{n+m+p+1}=\{0\}$, we have $d=0$. Therefore $w \in \sigma(L) \cap \oplus_{i=1}^{\infty} S_i=\{0\}$. This implies that L is semisimple.

THEOREM. *There are locally finite and semisimple Lie algebras over a field of characteristic zero which are not locally finite-dimensional-and-semisimple.*

PROOF. Let M be a locally finite-dimensional-and-semisimple Lie algebra over a field of characteristic zero. Then for each element x of M there is a finite-dimensional and semisimple subalgebra F_x of M containing x. Since $M=\sum_{x \in M} F_x$ and $M^2 \supseteq \sum_{x \in M} F_x^2=\sum_{x \in M} F_x=M$, we must have $M=M^2$.

Let F and L be the Lie algebras given above. Then $F^2=A \neq F$ and $L^2=$

$\oplus_{i=1}^{\infty} S_i \neq L$. Therefore F and L are locally finite and semisimple but not locally finite-dimensional-and-semisimple. Q.E.D.

We will show an interesting proerty of our Lie algebra L.

PROPOSITION. *Every finite-dimensional subalgebra F of L containing z is of the form $S \oplus R$ where S is a semisimple subalgebra of F and R is the non-trivial solvable radical of F.*

PROOF. Throughout the proof all a_i, b_j and c_k will be elements of $\mathfrak{k}$. Let $X = \sum_{i=1}^{\infty} \mathfrak{k}x_i$, $Y = \sum_{i=1}^{\infty} \mathfrak{k}y_i$ and $H = \sum_{i=1}^{\infty} \mathfrak{k}h_i$.

(1) $F = (F \cap X) + (F \cap Y) + (F \cap H) + \mathfrak{k}z$.

Let $w = x + y + h$ be an element of F where $x \in X$, $y \in Y$ and $h \in H$. Then $[w, z] = 2(x - y) \in F$. Hence $2x + h \in F$ and $x = [2x + h, z]/4 \in F$. Therefore $h \in F$ and $y \in F$.

If $F \cap X = \{0\}$, then F is solvable. Hence we may assume that $F \cap X \neq \{0\}$. Further assume that $F \subseteq \oplus_{i=1}^{n} S_i + \mathfrak{k}z$ and put $A = \{1, \dots, n\}$. For $x = \sum_{i=1}^{n} a_i x_i \in (F \cap X) - \{0\}$ we put $A(x) = \{i \in A : a_i \neq 0\}$. Consider $\Sigma = \{A(x) : x \in (F \cap X) - \{0\}\}$ and Σ^* be the set of all minimal elements of Σ with respect to the order given by inclusion.

(2) If $A(x) = A(x') \in \Sigma^*$, then x and x' are linearly dependent.

Put $\Sigma^* = \{A(u_1), \dots, A(u_m)\}$. Then by (2) the u_i are determined up to scalar. We denote $A(u_i)$ by $A(i)$. Let $[u_i, F \cap Y] \neq \{0\}$ with $i = 1, \dots, p$ and $[u_i, F \cap Y] = \{0\}$ with $i = p+1, \dots, m$. When $[u_i, F \cap Y] = \{0\}$ for $i = 1, \dots, m$, $[F \cap X, F \cap Y] = \{0\}$ and hence F is solvable (see the proof of (5)).

(3) For each $u_i = \sum_{j \in A(i)} a_j x_j$ $(1 \leq i \leq p)$ there exists $v_i = \sum_{j \in A(i)} b_j y_j \in F \cap Y$ such that $a_j b_j = 1$ for any $j \in A(i)$. We put $t_i = [u_i, v_i] = \sum_{j \in A(i)} h_j$ for $i = 1, \dots, p$.

Let $y' = \sum_{k=1}^{n} c_k y_k$ be an element of F such that $[u_i, y'] \neq 0$. Then $[u_i, [u_i, y']] = 2\sum_{j \in A(i)} a_j^2 c_j x_j \in (F \cap X) - \{0\}$. By (2) there exists a non-zero element a in $\mathfrak{k}$ such that $a_j^2 c_j = a a_j$ for any $j \in A(i)$. Now $v_i = y'/a$ satisfies the condition.

(4) $A(i) \cap A(j) = \emptyset$ for $1 \leq i < j \leq p$.

Let us write $u_j = \sum_{k \in A(j)} a_k x_k$. If $A(i) \cap A(j) \neq \emptyset$, then $[u_j, t_i] \neq 0$. Hence $A(i) \cap A(j) \in \Sigma$. By minimality of $A(j)$ we have $A(i) \cap A(j) = A(j)$. This implies that $A(i) = A(j)$. Therefore $i = j$ by (2).

(5) Let $B = A - \cup_{i=1}^{p} A(i)$. Then $F \cap X = \sum_{i=1}^{p} \mathfrak{k}u_i + (F \cap \sum_{i \in B} \mathfrak{k}x_i)$ and $[F \cap \sum_{i \in B} \mathfrak{k}x_i, F \cap Y] = \{0\}$.

Let $x = \sum_{i=1}^{n} a_i x_i$ be an element of $F \cap X$. Since $[x, t_i] = 2\sum_{j \in A(i)} a_j x_j \in F \cap X$, $\sum_{j \in A(i)} a_j x_j = b_i u_i$ for some $b_i \in \mathfrak{k}$ by (2). Therefore by (4) we have $x = \sum_{i=1}^{p} (\sum_{j \in A(i)} a_j x_j) + \sum_{j \in B} a_j x_j = \sum_{i=1}^{p} b_i u_i + \sum_{j \in B} a_j x_j$, which implies the first assertion. To prove the second assertion we assume that there exist $x = \sum_{j \in B} a_j x_j$ and $y = \sum_{k=1}^{n} c_k y_k$ in F such that $[x, y] \neq 0$. Let $C = \{j \in B : a_j c_j \neq 0\}$. Then $C \neq \emptyset$ by our assumption. Since $[x, [x, y]] \in (F \cap X) - \{0\}$, $C \in \Sigma$. We can take q with $p<q\leq m$ such that $A(q)\subseteq C$. But since $[u_q, y]\neq 0$. We have $1\leq q\leq p$. This is a contradiction.

(6) $F\cap Y=\sum_{i=1}^{p}\mathfrak{k}v_i+(F\cap\sum_{i\in B}\mathfrak{k}y_i)$ and $[F\cap\sum_{i\in B}\mathfrak{k}y_i, F\cap X]=\{0\}$ can be shown as in (5).

(7) $F\cap H=\sum_{i=1}^{p}\mathfrak{k}t_i+(F\cap\sum_{i\in B}\mathfrak{k}h_i)$.

Let $h=\sum_{k=1}^{n}c_kh_k$ be an element of $F\cap H$ and write $u_i=\sum_{j\in A(i)}a_jx_j$ $(1\leq i\leq p)$. Then $[u_i, h]=2\sum_{j\in A(i)}a_jc_jx_j\in F\cap X$. By (2) there exists $b_i\in\mathfrak{k}$ such that $a_jc_j=b_ia_j$ for any $j\in A(i)$. Therefore by (4) $h=\sum_{i=1}^{p}(\sum_{k\in A(i)}c_kh_k)+\sum_{k\in B}c_kh_k=\sum_{i=1}^{p}b_it_i+\sum_{k\in B}c_kh_k$.

(8) Conclusion. Let $T_i=\mathfrak{k}u_i+\mathfrak{k}v_i+\mathfrak{k}t_i$ for $i=1,\dots, p$. Since $[u_i, v_i]=t_i$, $[u_i, t_i]=2u_i$ and $[v_i, t_i]=-2v_i$ by (3), T_i is simple. By (1), (5), (6) and (7) we have $F=\sum_{i=1}^{p}T_i+(F\cap\sum_{k\in B}\mathfrak{k}x_i)+(F\cap\sum_{i\in B}\mathfrak{k}y_i)+(F\cap\sum_{i\in B}\mathfrak{k}h_i)+\mathfrak{k}z$. We now put $S=\sum_{i=1}^{p}T_i$ and $R=(F\cap\sum_{i\in B}\mathfrak{k}x_i)+(F\cap\sum_{i\in B}\mathfrak{k}y_i)+(F\cap\sum_{i\in B}\mathfrak{k}h_i)+\mathfrak{k}(\sum_{i=1}^{p}t_i-z)$. By simple computation we have $[S, \sum_{i=1}^{p}t_i-z]=\{0\}$. Thus $[S, R]=\{0\}$ by the definition of B, and we have $F=S\oplus R$. Since $\sum_{i=1}^{p}T_i=\oplus_{i=1}^{p}T_i$ by (4), S is semisimple. On the other hand $R^{(2)}=\{0\}$ by (5) and (6), R is solvable. Q. E. D.

References

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