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# Dirichlet finite solutions of Poisson equations on an infinite network

Dedicated to Professor Makoto Ohtsuka on his 60th birthday

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## §1. Introduction

Let $X$ be a countable set of nodes, $Y$ be a countable set of arcs, $K$ be the node-arc incidence function and $r$ be a strictly positive function on $Y$. The quartet $N=\{X, Y, K, r\}$ is called an infinite network if the graph $\{X, Y, K\}$ is connected, locally finite and has no self loop. For notation and terminologies, we mainly follow [4] and [5].

Let $L(X)$ be the set of all real functions on $X$ and $L^+(X)$ be the subset of $L(X)$ which consists of non-negative functions. For $u \in L(X)$, the Laplacian $\Delta u \in L(X)$ is defined by

$$\Delta u(x) = -\sum_{y\in Y} K(x, y)r(y)^{-1} \sum_{z\in X} K(z, y)u(z)$$

and the Dirichlet integral $D(u)$ of $u$ is defined by

$$D(u) = \sum_{y\in Y} r(y)^{-1}[\sum_{x\in X} K(x, y)u(x)]^2.$$

Denote by $\boldsymbol{D}(N)$ the set of all $u \in L(X)$ such that $D(u) < \infty$.

For $h \in L(X)$, we denote by $\boldsymbol{P}_h\boldsymbol{D}(N)$ the set of all Dirichlet finite solutions $u$ of the discrete Poisson equation $\Delta u = h$, i.e.

$$\boldsymbol{P}_h\boldsymbol{D}(N) = \{u \in \boldsymbol{D}(N); \Delta u = h\}.$$

We say that $h \in L(X)$ is distinguished if $h \neq 0$ and $\boldsymbol{P}_h\boldsymbol{D}(N) \neq \emptyset$. This notion was introduced by M. Nakai and L. Sario [2] in order to study the existence of Dirichlet finite non-harmonic biharmonic functions on Riemannian manifolds. Our main purpose of this paper is to obtain discrete analogues of results in [2] concerning conditions for a given $h$ to be distinguished. We shall also show that the distinguishedness of a given $h \in L(X)$ is related to the existence of flows with a current source.

In §2 we recall some facts of discrete Green potentials which play important roles in our study. Our main results are given in §3. Note that Theorem 3.4 has no counterpart in [2]. Relations between distinguishedness and existence of flows are discussed in §4. Results in this section have no counterparts in [2] either. In §5, representations of the $D$-minimum solutions of given distinguished functions are discussed.

## § 2. Green potentials

We first recall some facts from the theory of discrete Green potentials in [4] and [5].

Let $N'=\langle X', Y'\rangle$ be a finite subnetwork of $N$ and let $g_a^{N'}$ be the harmonic Green function of $N'$ with pole at $a\in X'$, i.e., $\Delta g_a^{N'}(x)=-\varepsilon_a(x)$ on $X'$ and $g_a^{N'}(x)=0$ on $X-X'$, where $\varepsilon_a\in L(X)$ is defined by $\varepsilon_a(a)=1$ and $\varepsilon_a(x)=0$ if $x\neq a$.

For every $f\in L(X)$, the Green potential $G_{N'}f$ and the energy $G_{N'}(f, f)$ of $f$ (with respect to $g^{N'}$) are defined by

$$G_{N'}f(x) = \sum_{z\in X'} g_z^{N'}(x)f(z),$$

$$G_{N'}(f, f) = \sum_{x\in X'}\sum_{z\in X'} g_z^{N'}(x)f(z)f(x).$$

Then we have $G_{N'}f(x)=0$ on $X-X'$ and

$$\Delta G_{N'}f(x) = -f(x) \text{ on } X'. \tag{2.1}$$

For $u, v\in L(X)$, we consider the inner product

$$(u, v) = \sum_{y\in Y} r(y)^{-1}[\sum_{z\in X} K(z, y)u(z)][\sum_{z\in X} K(z, y)v(z)]$$

of $u$ and $v$, if the sum is well defined.

We can easily prove

LEMMA 2.1. *If $v\in L(X)$ vanishes on $X-X'$, then $(g_a^{N'}, v)=v(a)$.*

COROLLARY 1. *$D(G_{N'}f)=G_{N'}(f, f)$.*

COROLLARY 2. *If $G_{N'}(f, f)=0$, then $f=0$ on $X'$.*

Let $L_0(X)$ be the set of all $u\in L(X)$ with finite support and denote by $\boldsymbol{D}_0(N)$ the closure of $L_0(X)$ in $\boldsymbol{D}(N)$ with respect to the norm $\|u\|=[D(u)+u(x_0)^2]^{1/2}$ $(x_0\in X)$. An infinite network $N$ is said to be of hyperbolic (resp. parabolic) type if there exists (resp. does not exist) the harmonic Green function $g_a$ of $N$ with pole at $a$, i.e., $g_a\in \boldsymbol{D}_0(N)$ such that $\Delta g_a(x)=-\varepsilon_a(x)$ on $X$. Denote by $O_G$ the class of parabolic infinite networks. Let $\{N_n\}$ be an exhaustion of $N$ and let $g_a^{(n)}$ be the harmonic Green function of $N_n$ with pole at $a$. Then $g_a^{(n)}\leq g_a^{(n+1)}$. In case $N\in O_G$, $g_a^{(n)}(x)\to\infty$ as $n\to\infty$ for each $x\in X$. In case $N\notin O_G$, $g_a^{(n)}(x)\to g_a(x)$ for each $x\in X$ and $D(g_a^{(n)}-g_a)\to 0$ as $n\to\infty$.

In case $N\notin O_G$, the Green potential $Gf(x)$ and the energy $G(f, f)$ of $f\in L(X)$ are defined by

$$Gf(x) = \sum_{z \in X} g_z(x) f(z) = \sum_{z \in X} g_x(z) f(z),$$
$$G(f, f) = \sum_{x \in X} \sum_{z \in X} g_x(z) f(x) f(z)$$

whenever they are well defined.

In case $N \in O_G$, we have $\boldsymbol{D}(N) = \boldsymbol{D}_0(N)$. In case $N \notin O_G$, the Royden decomposition reads

$$\boldsymbol{D}(N) = \boldsymbol{D}_0(N) + \boldsymbol{HD}(N),$$

where $\boldsymbol{HD}(N) = \{u \in \boldsymbol{D}(N);\ \Delta u = 0\}$ and the sum is the vector space direct sum (cf. [5; Theorem 4.1]).

DEFINITION 2.1. For a distinguished $h$, a function $u$ in $\boldsymbol{P}_h\boldsymbol{D}(N)$ is called a $D$-minimum solution if $u \in \boldsymbol{D}_0(N)$.

By the above observation, we have

THEOREM 2.1. *Let $h$ be distinguished.*

(i) *If $N \notin O_G$, then there exists a unique D-minimum solution $u_h$ and* $\boldsymbol{P}_h\boldsymbol{D}(N) = \boldsymbol{HD}(N) + u_h$.

(ii) *If $N \in O_G$, then every $u \in \boldsymbol{P}_h\boldsymbol{D}(N)$ is a D-minimum solution.*

## § 3. Existence theorems

Denote by $FS(N)$ the set of all finite subnetworks of $N$. As a discrete analogue of [2; Theorem 1], we have

THEOREM 3.1. *In order that $h \in L(X)$ be distinguished it is necessary and sufficient that*

$$0 < \sup \{G_{N'}(h, h);\ N' \in FS(N)\} < \infty. \tag{3.1}$$

PROOF. First we assume that $h$ is distinguished. Take $u \in \boldsymbol{P}_h\boldsymbol{D}(N)$ and $N' = \langle X', Y' \rangle \in FS(N)$. Put $v = u + G_{N'}h$. Then $\Delta v = \Delta u - h = 0$ on $X'$ and hence

$$D(u) = D(v - G_{N'}h) = D(v) + D(G_{N'}h) - 2 \sum_{x \in X} [\Delta v(x)] [G_{N'}h(x)]$$
$$= D(v) + D(G_{N'}h) \geq D(G_{N'}h),$$

Since $G_{N'}h \in L_0(X)$. We have by Corollary 1 of Lemma 2.1

$$G_{N'}(h, h) = D(G_{N'}h) \leq D(u) < \infty.$$

Since $h \neq 0$ on $X$, there exists $N' \in FS(N)$ such that $G_{N'}(h, h) > 0$ by Corollary 2 of Lemma 2.1. Thus (3.1) holds.

Conversely assume that (3.1) holds. Let $\{N_n\}$ $(N_n = \langle X_n, Y_n \rangle)$ be an exhaustion of $N$ and $x_0 \in X_1$ and put $v_n = -G_{N_n}h$. We may assume that $G_{N_1}(h, h) > 0$. Then $h \neq 0$ on $X_1$. For $m \geq n$, we have by (2.1) and Lemma 2.1

$$(v_n, v_m) = -\sum_{x\in X}[\Delta v_m(x)]v_n(x) = -\sum_{x\in X}h(x)v_n(x)$$
$$= -\sum_{x\in X}[\Delta v_n(x)]v_n(x) = D(v_n),$$

so that $D(v_n - v_m) = D(v_m) - D(v_n)$. Put $u_n = v_n - v_n(x_0)$. Then $\Delta u_n = h$ on $X_n$ and $\{u_n\}$ is a Cauchy sequence in the Hilbert space $\boldsymbol{D}(N)$ with the norm $\|u\|$. Hence there exists $\bar{u} \in \boldsymbol{D}(N)$ such that $\|u_n - \bar{u}\| \to 0$ as $n \to \infty$. Then $\{u_n(x)\}$ converges to $\bar{u}(x)$ for each $x \in X$ by [5; Lemma 1.1], and hence $\Delta \bar{u} = h$ on $X$. Namely $h$ is distinguished.

By applying this theorem, we have

PROPOSITION 3.1. *Let $N \in O_G$. Then any $h \in L^+(X)$ is not distinguished.*

In case $N \notin O_G$, denote by $E(G)$ the set of all $f \in L^+(X)$ such that $G(f, f) < \infty$. Then we have

PROPOSITION 3.2. *Let $N \notin O_G$ and let $h \in L^+(X)$. Then $h$ is distinguished if and only if $h \neq 0$ and $h \in E(G)$.*

COROLLARY 1. *Let $N \notin O_G$ and $h_1$ and $h_2$ be elements of $E(G)$ such that $h_1 \neq h_2$. Then $h = h_1 - h_2$ is distinguished.*

Denote by $O_{QD}$ the class of all infinite networks $N$ for which $\{u \in \boldsymbol{D}(N); \Delta u = -1\} = \emptyset$. We proved in [7; Theorem 3.1] that $N \notin O_{QD}$ if and only if $1 \in E(G)$. Thus we can easily prove

COROLLARY 2. *Let $N \notin O_{QD}$. If $h$ is nonzero and bounded on $X$, then $h$ is distinguished.*

As a discrete analogue of [2; Theorem 2], we have

THEOREM 3.2. *In order that $h \in L(X)$ be distinguished it is necessary and sufficient that*

(3.2) $$0 < \sup\{[\sum_{x\in X} h(x)f(x)]^2/D(f); f \in L_0(X), f \neq 0\} < \infty.$$

PROOF. Assume that $u \in \boldsymbol{P}_h\boldsymbol{D}(N)$. For any $f \in L_0(X)$, we have

$$\sum_{x\in X} h(x)f(x) = \sum_{x\in X}[\Delta u(x)]f(x) = -(u, f),$$

so that

$$[\sum_{x\in X} h(x)f(x)]^2 = |(u, f)|^2 \leq D(u)D(f).$$

Since $h \neq 0$, there exists $f \in L_0(X)$ such that $\sum_{x\in X} h(x)f(x) \neq 0$. Thus the supremum in (3.2) is positive and dominated by $D(u)$.

Conversely assume that (3.2) holds. Let $N'\in FS(N)$ and put $f=G_{N'}h$. Then $f\in L_0(X)$ and

$$\sum_{x\in X} h(x)f(x)=G_{N'}(h,\ h)=D(f).$$

Thus (3.2) implies (3.1).

Now we shall give necessary conditions for $h$ to be distinguished. A subset $B$ of $X$ is said to be wide at the ideal boundary of $N$ if there exists a sequence $\{x_n\}$ of nodes in $B$ such that $x_n\notin X(x_m)$ (i.e., $K(x_n, y)K(x_m, y)=0$ for all $y\in Y$) if $n\neq m$ and

$$\limsup_{n\to\infty} n^2[\textstyle\sum_{k=1}^n \sum_{y\in Y} |K(x_k, y)|r(y)^{-1}]^{-1} = \infty. \tag{3.3}$$

We have

THEOREM 3.3. *If there exists a positive number* $\varepsilon>0$ *such that* $\{x\in X;\ h(x)\geq\varepsilon\}$ *is wide at the ideal boundary of* $N$, *then* $\boldsymbol{P}_h\boldsymbol{D}(N)=\emptyset$.

PROOF. Let $B=\{x\in X;\ h(x)\geq\varepsilon\}$ and choose a sequence $\{x_n\}$ in $B$ such that $x_n\notin X(x_m)$ if $n\neq m$ and (3.3) holds. Put $f_n=\sum_{k=1}^n \varepsilon_{x_k}$. Then $f_n\in L_0(X)$,

$$\sum_{x\in X} h(x)f_n(x) = \sum_{k=1}^n h(x_k) \geq n\varepsilon,$$

$$D(f_n) = \sum_{k=1}^n D(\varepsilon_{x_k}) = \sum_{k=1}^n \sum_{y\in Y} |K(x_k, y)|r(y)^{-1}.$$

It follows that

$$\liminf_{n\to\infty} [\textstyle\sum_{x\in X} h(x)f_n(x)]^2/D(f_n) \geq \limsup_{n\to\infty} n^2\varepsilon^2/D(f_n) = \infty.$$

Hence $\boldsymbol{P}_h\boldsymbol{D}(N)=\emptyset$ by Theorem 3.2.

LEMMA 3.1. *Let* $N\in O_G$ *and* $u\in\boldsymbol{D}(N)$. *If* $\sum_{x\in X}|\Delta u(x)|<\infty$, *then* $\sum_{x\in X}\Delta u(x)=0$.

PROOF. Since $N\in O_G$, there exists a sequence $\{f_n\}$ in $L_0(X)$ such that $0\leq f_n\leq 1$ on $X$ and $\|f_n-1\|\to 0$ as $n\to\infty$ (cf. [4; Theorem 3.1]). Then $(u, f_n)\to(u, 1)=0$ as $n\to\infty$. We may assume that $\Delta u\neq 0$. For any $\varepsilon>0$, we can find a finite subset $X'$ of $X$ such that $\sum_{x\in X-X'}|\Delta u(x)|<\varepsilon$. Since $\{f_n\}$ converges pointwise to 1, there exists $n_0$ such that $|f_n(x)-1|<\varepsilon/c$ on $X'$ for all $n\geq n_0$ with $c=\sum_{x\in X}|\Delta u(x)|$. We have

$$\begin{aligned}|\textstyle\sum_{x\in X}\Delta u(x) + (u, f_n)| &= |\textstyle\sum_{x\in X}[\Delta u(x)][1-f_n(x)]| \\ &\leq \textstyle\sum_{x\in X'}|\Delta u(x)||1-f_n(x)| + 2\sum_{x\in X-X'}|\Delta u(x)| \\ &< \textstyle\sum_{x\in X'}|\Delta u(x)|\varepsilon/c + 2\varepsilon < 3\varepsilon\end{aligned}$$

for all $n \geq n_0$. Thus we have $\sum_{x\in X} \Delta u(x)=0$.

THEOREM 3.4. *Let $N \in O_G$ and $h$ be distinguished. If $\sum_{x\in X} |h(x)|<\infty$, then $\sum_{x\in X} h(x)=0$.*

PROOF. There exists $u \in \boldsymbol{D}(N)$ such that $\Delta u=h$ on $X$. Our assertion is an immediate consequence of the above lemma.

## §4. Relations with a flow problem

Let us consider the following flow problem on an infinite network $N=\{X, Y, K, r\}$:
$(FP_h)$ Given $h \in L(X)$, find $w \in L(Y)$ such that

$$\sum_{y\in Y} K(x, y)w(y) = -h(x) \quad \text{on} \quad X,\ H(w) = \sum_{y\in Y} r(y)w(y)^2 < \infty.$$

This problem was studied by H. Flanders [1] in the case where $h \in L_0(X)$. In [1], $h$ is called a current source.

We have

THEOREM 4.1. *If $h$ is distinguished, then problem $(FP_h)$ has a solution.*

PROOF. Let $u \in \boldsymbol{P}_h\boldsymbol{D}(N)$ and define $w \in L(Y)$ by $w(y)=r(y)^{-1} \sum_{x\in X} K(x, y)u(x)$. Then we have $H(w)=D(u)$ and

$$\sum_{y\in Y} K(x, y)w(y) = -\Delta u(x) = -h(x).$$

Thus $w$ is a solution of problem $(FP_h)$.

In order to study the converse of this theorem, we recall the notion of paths defined in [3]. For $a, x \in X$, a path $P$ from $a$ to $x$ is the triple $(C_X(P), C_Y(P), p)$ of a finite ordered set $C_X(P)=\{x_0, x_1,\dots, x_n\}$ of nodes, a finite ordered set $C_Y(P)=\{y_1, y_2,\dots, y_n\}$ of arcs and a function $p$ on $Y$ called the path index of $P$ such that

$$x_0 = a,\ x_n = x,\ x_i \neq x_k (i \neq k),\ e(y_i) = \{x_{i-1}, x_i\},$$
$$p(y_i) = -K(x_{i-1}, y_i) \text{ and } p(y) = 0 \text{ if } y \notin C_Y(P).$$

THEOREM 4.2. *If $h \neq 0$ and problem $(FP_h)$ has a solution, then $h$ is distinguished.*

PROOF. Denote by $F_h(Y)$ the set of all solutions of problem $(FP_h)$ and consider the following extremum problem: Minimize $H(w)$ subject to $w \in F_h(Y)$. Let $\alpha$ be the value of this problem and let $\{w_n\}$ be a sequence in $F_h(Y)$ such that $H(w_n) \to \alpha$ as $n \to \infty$. Since $(w_n+w_m)/2 \in F_h(Y)$, we have

$$\alpha \le H((w_n+w_m)/2) \le H((w_n+w_m)/2)+H((w_n-w_m)/2)=[H(w_n)+H(w_m)]/2,$$

so that $H(w_n-w_m)\to 0$ as $n, m\to\infty$. By the relation

$$r(y)(w_n(y)-w_m(y))^2 \le H(w_n-w_m),$$

we see that $\{w_n(y)\}$ is a Cauchy sequence for each $y\in Y$. Let $\bar{w}(y)$ be the limit of $\{w_n(y)\}$. Then it is easily seen that $\bar{w}\in F_h(Y)$ and $H(\bar{w})=\alpha$. Let $w'\in L(Y)$ be a finite cycle, i.e., the support of $w'$ is a finite set and $\sum_{y\in Y} K(x, y)w'(y)=0$ on $X$. For any real number $t$, we have $\bar{w}+tw'\in F_h(Y)$, so that $H(\bar{w})\le H(\bar{w}+tw')$. It follows that

$$\sum_{y\in Y} r(y)\bar{w}(y)w'(y) = 0. \tag{4.1}$$

Let $a$, $x\in X$ and define $\bar{u}\in L(X)$ by

$$\bar{u}(a) = 0 \quad\text{and}\quad \bar{u}(x) = \sum_{y\in Y} r(y)p(y)\bar{w}(y)\ (x\neq a),$$

where $p(y)$ is the path index of a path $P$ from $a$ to $x$. By (4.1), we see that $\bar{u}$ is independent of the choice of $P$ and uniquely determined by $\bar{w}$. We shall prove that

$$\sum_{x\in X} K(x, y)\bar{u}(x) = r(y)\bar{w}(y) \quad\text{on}\quad Y. \tag{4.2}$$

Let $y\in Y$ and $e(y)=\{x_1, x_2\}$. Let $P$ be a path from $a$ to $x_1$. In case $x_2\in C_X(P)$, we have $p(y)=-K(x_2, y)=K(x_1, y)$ and $\bar{u}(x_1)=\bar{u}(x_2)+r(y)p(y)\bar{w}(y)$, so that

$$\sum_{x\in X} K(x, y)\bar{u}(x) = K(x_1, y)[\bar{u}(x_1)-\bar{u}(x_2)] = r(y)\bar{w}(y).$$

In case $x_2\notin C_X(P)$, let $P'$ be the path from $a$ to $x_2$ generated by the path $P$ and the arc $y$ and let $p'$ be the path index of $P'$. Then $p'(y)=-K(x_1, y)=K(x_2, y)$ and $\bar{u}(x_2)=\bar{u}(x_1)+r(y)p'(y)\bar{w}(y)$, so that (4.2) holds. It follows that $D(\bar{u})=H(\bar{w})<\infty$ and $\Delta\bar{u}(x)=h(x)$ on $X$. Namely $h$ is distinguished.

As an application of this result, we have

THEOREM 4.3. *Let $h\in L_0(X)$, $h\neq 0$ and $\sum_{x\in X} h(x)=0$. Then $h$ is distinguished.*

PROOF. Let $A$ be the support of $h$, i.e., $A=\{x\in X;\ h(x)\neq 0\}$ and let $b\notin A$. Define $w_b\in L(Y)$ by

$$w_b(y) = -\sum_{x\in A} h(x)p_x(y),$$

where $p_x$ denotes the path index of a path from $b$ to $x$ $(x\neq b)$. Since the support of $p_x$ and the set $A$ are finite sets, we have $H(w_b)<\infty$. Since $\sum_{y\in Y} K(z, y)p_x(y)=\varepsilon_x(z)$ $(z\neq b)$, $\sum_{y\in Y} K(b, y)p_x(y)=-1$ and $\sum_{x\in X} h(x)=0$, we see that $w_b$ is a solution of problem $(FP_h)$. Thus $h$ is distinguished by Theorem 4.2.

## §5. Representation of $\boldsymbol{D}$-minimum solutions

In order to represent the $D$-minimum solution $u_h$ for a distinguished $h$ as the limit of $\{G_{N_n}h\}$, we first study the discrete analogue of harmonizable functions.

For $f \in L(X)$ and $N' = \langle X', Y' \rangle \in FS(N)$, let $h_f^{N'} = f + G_{N'}(\Delta f)$. This is the unique function such that $\Delta h_f^{N'}(x) = 0$ on $X'$ and $h_f^{N'}(x) = f(x)$ on $X - X'$.

We say that $f \in L(X)$ is harmonizable if $\{h_f^{N_n}\}$ is convergent for every exhaustion $\{N_n\}$ of $N$.

We have

LEMMA 5.1. *If $N \notin O_G$, then every $f \in \boldsymbol{D}_0(N)$ is harmonizable and $h_f^{N_n}(x) \to 0$ as $n \to \infty$ for every exhaustion $\{N_n\}$ of $N$.*

PROOF. Let $N_n = \langle X_n, Y_n \rangle$ and put $f_n = h_f^{N_n}$. Since $f_n = f + G_{N_n}(\Delta f)$, $f_n \in \boldsymbol{D}_0(N)$. By an argument similar to the last half of the proof of Theorem 3.1, we see that $D(f_n - f_m) \to 0$ as $n, m \to \infty$. It follows from [5; Theorem 3.3] that $\{f_n\}$ converges pointwise to a function $v \in \boldsymbol{D}_0(N)$. Since $\Delta f_n = 0$ on $X_n$, we have $\Delta v = 0$ on $X$. Thus, [5; Lemma 1.3] implies $v = 0$.

The present proof of this lemma is due to Professor $F$–$Y$. Maeda of Hiroshima University.

COROLLARY. *If $N \notin O_G$, then every $f \in \boldsymbol{D}(N)$ is harmonizable.*

PROOF. Let $f \in \boldsymbol{D}(N)$. There exist $u \in \boldsymbol{HD}(N)$ and $v \in \boldsymbol{D}_0(N)$ such that $f = u + v$. For every exhaustion $\{N_n\}$ of $N$, we have by Lemma 5.1

$$h_f^{N_n}(x) = h_u^{N_n}(x) + h_v^{N_n}(x) = u(x) + h_v^{N_n}(x) \longrightarrow u(x)$$

as $n \to \infty$.

THEOREM 5.1. *Let $N \notin O_G$ and $h$ be distinguished. Then the unique $D$-minimum solution $u_h$ is given by $u_h = -\lim_{n\to\infty} G_{N_n}h$, where $\{N_n\}$ is an exhaustion of $N$.*

PROOF. Since $h_{u_h}^{N_n} = u_h + G_{N_n}h$ and $u_h \in \boldsymbol{D}_0(N)$, our assertion follows from Lemma 5.1.

THEOREM 5.2. *Let $N \in O_G$ and $h$ be distinguished. For each $u \in \boldsymbol{P}_h\boldsymbol{D}(N)$ and an exhaustion $\{N_n\}$ of $N$, there is a sequence $\{c_n\}$ of real numbers such that $u = \lim_{n\to\infty}(-G_{N_n}h + c_n)$.*

PROOF. Put $u_n = G_{N_n}h - G_{N_n}h(x_0)$. Then there exists $\bar{u} \in \boldsymbol{D}(N)$ such that $\|u_n - \bar{u}\| \to 0$ as $n \to \infty$ (cf. the proof of Theorem 3.1). Let $u \in \boldsymbol{P}_h\boldsymbol{D}(N)$. Since

$N \in O_G$, we have $u = -\bar{u} + c$ with a real constant $c$. Let us put $c_n = G_{N_n}h(x_0) + c$. Since $\{u_n\}$ converges pointwise to $\bar{u}$, we have $u(x) = -\lim_{n\to\infty}(-G_{N_n}h(x) + c_n)$.

In case $N \notin O_G$, we say that a function $f \in L(X)$ is $G$-absolute if $G|f|(a) = \sum_{x\in X} g_a(x)|f(x)| < \infty$ at one and hence, by Harnack's principle (cf. [6; Lemma 1.3]), at all $a \in X$.

We have

THEOREM 5.3. *Let $N \notin O_G$. If $h$ is distinguished and $G$-absolute, then $u_h = -Gh$.*

PROOF. Since $h$ is $G$-absolute, $G_{N_n}|h|(x) \le G|h|(x) < \infty$ and $u_h(x) = -\lim_{n\to\infty} G_{N_n}h(x) = -Gh(x)$ by Theorem 5.1.

As a corollary, we have the following Riesz-type decomposition:

COROLLARY 1. *Let $N \notin O_G$ and $u \in \boldsymbol{D}(N)$. If $\Delta u$ is $G$-absolute, then $u$ is decomposed in the form: $u = \pi u - Gh$, where $h = \Delta u$ and $\pi u$ is the harmonic part of $u$ in the Royden decomposition.*

PROOF. We may assume that $h = \Delta u \ne 0$. Then $h$ is distinguished. Let $v = u - \pi u$. Then $v$ is the $D$-minimum solution for $h$. Since $h$ is $G$-absolute, we have $v = -Gh$ by Theorem 5.3.

Since any $h \in E(G)$ is $G$-absolute, we have by Proposition 3.2

COROLLARY 2. *Let $N \notin O_G$ and $h \in L^+(X)$. If $h$ is distinguished, then $u_h = -Gh$.*

Denote by $O_{QP}$ the class of all infinite networks $N$ for which $\{u \in L^+(X); \Delta u = -1\} = \emptyset$. We proved in [7; Theorem 3.1] that $N \notin O_{QP}$ if and only if 1 is $G$-absolute. Thus if $N \notin O_{QP}$, then any bounded function $h \in L(X)$ is $G$-absolute. In order to obtain a similar result in the case where $h$ is not bounded, we study the growth of a distinguished function.

LEMMA 5.2. *Let $N \notin O_G$ and let $h$ be distinguished and $h \in \boldsymbol{D}(N)$. For any $\varepsilon > 0$, put $A_\varepsilon = \{z \in X; |h(z)| > \varepsilon\}$ and $B_\varepsilon = \{z \in X; |h(z)| \le \varepsilon\}$. Then*

$$\sum_{z\in A_\varepsilon} g_x(z)|h(z)| < \infty, \tag{5.1}$$

$$\sum_{z\in B_\varepsilon} g_x(z)h(z)^2 < \infty \tag{5.2}$$

*for some and hence for all $x \in X$.*

PROOF. Let $v(x) = \min[\max(h(x), -\varepsilon), \varepsilon]$. Then $v$ is bounded and $D(v) \le D(h) < \infty$. Note that $v(x)h(x)$ is equal to $\varepsilon|h(x)|$ if $x \in A_\varepsilon$ and to $h(x)^2$ if $x \in B_\varepsilon$.

Let $\{N_n\}$ be an exhaustion of $N$ and let $g_x^{(n)}$ be the harmonic Green function of $N_n$ with pole at $x$. Since $g_x^{(n)}$ and $g_x$ are bounded, we see that $g_x^{(n)}v$ and $g_xv$ are bounded and belong to $\boldsymbol{D}(N)$. Let $u_h$ be the $D$-minimum solution for $h$. Then we have

$$(5.3)\qquad \sum_{z\in X} g_x^{(n)}(z)v(z)h(z) = \sum_{z\in X} g_x^{(n)}(z)v(z)\,[\Delta u_h(z)] = -(g_x^{(n)}v,\, u_h)\,.$$

There exists a constant $c>0$ independent of $n$ such that $D(g_x^{(n)}v)\leq c[D(g_x^{(n)})+D(v)]$. Since $D(g_x^{(n)}-g_x)\to 0$ as $n\to\infty$, $\{D(g_x^{(n)}v)\}$ is bounded. By using the fact that $\{g_x^{(n)}v\}$ converges pointwise to $g_xv$ and $u_h\in\boldsymbol{D}_0(N)$, we have

$$\lim_{n\to\infty}(g_x^{(n)}v,\, u_h) = (g_xv,\, u_h)\,.$$

Recall that $vh\geq 0$ on $X$ and $\{g_x^{(n)}\}$ converges increasingly to $g_x$. We have by (5.3)

$$\sum_{z\in X} g_x(z)v(z)h(z) = -\,(g_xv,\, u_h)\,.$$

From the relation

$$\sum_{z\in X} g_x(z)v(z)h(z) = \varepsilon\sum_{z\in A_\varepsilon} g_x(z)|h(z)| + \sum_{z\in B_\varepsilon} g_x(z)h(z)^2,$$

our assertion follows.

THEOREM 5.4. *Let $N\notin O_{QP}$ and $h$ be distinguished. If $h\in\boldsymbol{D}(N)$, then $h$ is G-absolute, so that $u_h=-Gh$.*

PROOF. Since $N\notin O_{QP}$, we see by [4; Theorem 3.2] that $N\notin O_G$. Let $A_\varepsilon$ and $B_\varepsilon$ be the same as in Lemma 5.2. We have

$$\sum_{z\in B_\varepsilon} g_x(z)|h(z)| \leq \varepsilon\sum_{z\in B_\varepsilon} g_x(z) \leq \varepsilon G1(x) < \infty.$$

It follows from (5.1) that $h$ is G-absolute.

The above proof shows that we can slightly sharpen this result as follows:

THEOREM 5.5. *Let $N\notin O_G$ and $h$ be distinguished. If $h\in\boldsymbol{D}(N)$ and if there exists $\varepsilon>0$ such that $\sum_{z\in B_\varepsilon} g_x(z)<\infty$, then $h$ is G-absolute.*

Theorems 5.4 and 5.5 are discrete analogues of [2; Theorem 7].

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