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# The pure braid groups and the Milnor $\bar{\mu}$-invariants of links

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## 1. The statement of results

In this note, we study a relation between the pure braid groups $P_n$ and the Milnor $\bar{\mu}$-invariants of links, and shall prove the *mod p* residual nilpotence of $P_n$.

Let

$$X_n = \{(x_1,\dots, x_n) \in \boldsymbol{C}^n \mid x_i \neq x_j \text{ if } i \neq j\}$$

be the configuration space of $\boldsymbol{C}$. Then the symmetric group $S_n$ of degree $n$ acts freely on $X_n$ by the permutation of the coordinates. Let $Y_n = X_n/S_n$ be the quotient space by the action of $S_n$. Then we have

$$\pi_i(X_n) = \pi_i(Y_n) = 0 \quad (i \geq 2)$$

and the exact sequence

$$1 \longrightarrow \pi_1(X_n) \longrightarrow \pi_1(Y_n) \longrightarrow S_n \longrightarrow 1.$$

DEFINITION 1. $\pi_1(Y_n)$ (resp. $\pi_1(X_n)$) is said to be the *braid group* (resp. the *pure braid group*) of degree $n$, and is denoted by $B_n$ (resp. $P_n$).

In fact, $B_n$ coincides with Artin's braid group of the equivalence classes of braids (see [1]).

For any braid $b \in B_n$, let $\hat{b}$ be the closed braid of $b$ (see [1]). If $b \in P_n$, then $\hat{b}$ is a link of $n$ components in $S^3$.

DEFINITION 2. Put

$$P_{n,q} = \{b \in P_n \mid \bar{\mu}(i_1 \cdots i_k)(\hat{b}) = 0 \quad \text{for any} \quad k \leq q\},$$
$$P_{n,q}^{(p)} = \{b \in P_n \mid \bar{\mu}(i_1 \cdots i_k)(\hat{b}) \equiv 0 \quad mod\, p \quad \text{for any} \quad k \leq q\}$$

where $\bar{\mu}$ is the Milnor $\bar{\mu}$-invariant of links and $p$ is a prime (see [2]).

Then we can prove the following

THEOREM 1. (i) *$P_{n,q}$ is a normal subgroup of $B_n$ and therefore of $P_n$.*

(ii) $[P_{n,q}, P_{n,r}] \subset P_{n,q+r}$ ($[\ ,\ ]$ *denotes the commutator group*).

(iii) $\cap_q P_{n,q} = \{1\}$.

THEOREM 2. (i) *$P_{n,q}^{(p)}$ is a normal subgroup of $B_n$ and therefore of $P_n$.*

(ii) $[P_{n,q}^{(p)}, P_{n,r}^{(p)}] \subset P_{n,q+r}^{(p)}$.

(iii) $b \in P_{n,q}^{(p)} \Rightarrow b^p \in P_{n,pq}^{(p)}$.

(iv) $\bigcap_q P_{n,q}^{(p)} = \{1\}$.

By these theorems, we see immediately the following

COROLLARY. *$P_n$ is residually nilpotent and moreover, mod $p$ residually nilpotent, i.e. $P_n$ is embeddable into the product of finite $p$-groups for any prime $p$.*

## 2. Some known results

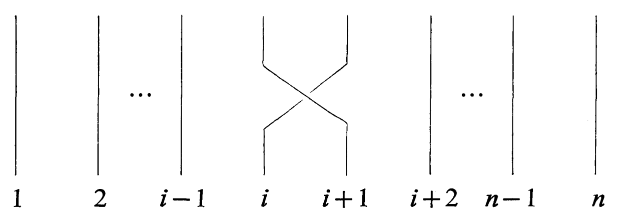
Let $F_n$ be the free group of rank $n$ with free generators $x_1, \ldots, x_n$. Then

FACT 1. *We have a monomorphism $\phi_n$: $B_n \to Aut(F_n)$ given by*

$$\phi_n(\sigma_i)(x_i) = x_{i+1}, \qquad \phi_n(\sigma_i)(x_{i+1}) = x_{i+1}^{-1}x_i x_{i+1},$$
$$\phi_n(\sigma_i)(x_j) = x_j \quad (j \notin \{i, i+1\}),$$

*where $\sigma_i\,(1 \leq i \leq n-1)$ is the generator of $B_n$ defined by the following braid*

$\sigma_i$:




DEFINITION 3. For a group $G$, let $\Gamma_* G$ (resp. $\Gamma_*^{(p)} G$) be the ordinary (resp. *mod p*, or, restricted) *lower central series of* $G$ ($p$: a prime). This sequence is characterized by the property that this is the minimal sequence $\{G_i\}$ of subgroups of $G$ which satisfies the following conditions (i) and (ii) (resp. (i), (ii) and (iii)):

$$\text{(i)}\quad G_1 = G, \qquad \text{(ii)}\quad [G_m, G_n] \subset G_{m+n},$$
$$\text{(iii)}\quad x \in G_n \Rightarrow x^p \in G_{np}.$$

FACT 2. *For any $b \in P_n$ there are words $f_i = f_i(x_1, \ldots, x_n) \in F_n$ $(i = 1, \ldots, n)$ such that*

$$\phi_n(b)(x_i) = x_i^{f_i(x_1, \ldots, x_n)} \quad (x^f = f^{-1}xf)$$

*and the sum of the exponents of $x_i$ in $f_i$ is zero. Such an $f_i$ is unique.*

*The above equality is called the "standard presentation" of $b$ or $\phi_n(b)$. Moreover, for any $b \in P_n$,*

$$b \in P_{n,p} \Longleftrightarrow f_i(x_1,\dots, x_n) \in \Gamma_q F_n \text{ for any } i,$$

$$b \in P_{n,q}^{(p)} \Longleftrightarrow f_i(x_1,\dots, x_n) \in \Gamma_q^{(p)} F_n \text{ for any } i.$$

This follows from the definition of the $\bar{\mu}$-invariant since the link group $G=\pi_1(S^3-\hat{b})$ for $b \in P_n$ has the presentation

$$G = \{x_1,\dots, x_n \mid (x_i, f_i) = 1 \ (i = 1,\dots, n)\}$$

and $x_i$ and $f_i$ are the meridean and the longitude of the $i$-th component of $b$.

Let $Q=U(\boldsymbol{Z}_p[[[v_1,\dots, v_n]]])$ be the unit group of the non-commutative formal power series ring on variables $v_1,\dots, v_n$ over $\boldsymbol{Z}_p$, and $\Psi\colon F_n \to Q$, $\Psi(x_i)=1+v_i$, be the *mod* $p$-Magnus expansion. Then we see the following

FACT 3 (*Zassenhaus* [3]). *For any* $x \in F_n$, $x \in \Gamma_q^{(p)} F_n \Leftrightarrow \Psi(x)=1+$*(terms of degree* $\geq q$*).*

## 3. The proof of Theorems

We shall only prove Theorem 2 since the proof of Theorem 1 is similar to and more simpler than the proof of Theorem 2.

PROOF OF (i) IN THEOREM 2. The normality is clear since the closed braids of $b$ and $b^a$ are equivalent for any $a$ and $b \in B_n$.

Let $b$, $c \in P_{n,q}^{(p)}$, $\phi_n(b)=B$, $\phi_n(c)=C$, and $B(x_i)=x_i^{f_i}$, $C(x_i)=x_i^{g_i}$ be the standard presentations of $b$ and $c$. Then $BC(x_i)=B(x_i^{g_i})=x_i^{f_i C(g_i)}$. The multiplicative closedness of $P_{n,q}^{(p)}$ follows from Facts 2 and 3 since $\Gamma_q^{(p)} G$ is a characteristic subgroup of $G$. Let $B^{-1}(x_i)=x_i^{h_i}$ be also the standard presentation. Then

$$x_i = BB^{-1}(x_i) = x_i^{f_i B(h_i)}, \quad h_i = B^{-1}(f_i)$$

and hence $b^{-1} \in P_{n,q}^{(p)}$.

PROOF OF (ii) OF THEOREM 2. Let $b \in P_{n,q}^{(p)}$, $c \in P_{n,r}^{(p)}$, and $B$, $C$, $f_i$, $g_i$ be as above, and $(B, C)(x_i)=x_i^{h_i}$, where $(B, C)=B^{-1}C^{-1}BC$, be the standard presentation. Then we have

$$x_i^{f_i B(gi)} = B(x_i^{g_i}) = BC(x_i) = CB(x_i^{h_i}) = C(x_i^{f_i B(h_i)}) = x_i^{g_i C(f_i) CB(h_i)},$$

and hence $f_i B(g_i) = g_i C(f_i) CB(h_i)$,

$$CB(h_i) = C(f_i^{-1}) g_i^{-1} f_i B(g_i) = C(f_i)^{-1} f_i (f_i, g_i) g_i^{-1} B(g_i).$$

Since $(f_i, g_i) \in \Gamma_{q+r}^{(p)}F_n$, we have only to show that $C(f_i)^{-1}f_i \in \Gamma_{q+r}^{(p)}F_n$. Let $\tilde{C}$ be a lifting of the automorphism $C$ of $F_n$ to a ring automorphism of the Magnus algebra $\boldsymbol{Z}_p[[[v_1,\dots, v_n]]]$. In fact, $\tilde{C}$ is a substitution of $v_i$+(terms of degree$\geq r+1$) for $v_i$. Since $\Psi(f_i)=1+$(terms of degree$\geq q$), $\Psi(f_i)\equiv\Psi(C(f_i))\equiv\tilde{C}(\Psi(f_i))$ *mod* (deg $\geq q+r$), and therefore $C(f_i)^{-1}f_i \in \Gamma_{q+r}^{(p)}F_n$.

PROOF OF (iii) IN THEOREM 2. For $b \in P_{n,q}^{(p)}$, let $B$ and $f_i$ be as above and let $B^p(x_i)=x_i^{g_i}$ be the standard presentation. Then we have the following by induction on $j$:

$$B^j(x_i) = x_i^{f_i B(f_i) B^2(f_i)\cdots B^{j-1}(f_i)},$$

which shows $g_i=f_iB(f_i)\cdots B^{p-1}(f_i)$. Therefore we have (iii) by the following implication:

$$f_i \in \Gamma_q^{(p)}F_n \Longrightarrow g_i \in \Gamma_{pq}^{(p)}F_n.$$

This is proved as follows: If $\tilde{B}$ is a lifting of $B$ to the automorphism of the Magnus algebra, then we can show that

$$\Psi(B^j(f_i)) = \tilde{B}^j(\Psi(f_i)) = 1+c_1+\binom{j}{1}c_2+\cdots+\binom{j}{j}c_{j+1} \quad (\deg c_k \geq qk)\,,$$

for $f_i \in \Gamma_q^{(p)}F_n$, by induction on $j$. Therefore the above implication follows from the following combinatorial lemma.

LEMMA. *Let $c_i$ be a homogeneous element of degree $i$ of a graded algebra over $\boldsymbol{Z}_p$ (not necessarily commutative). Then the homogeneous part of degree $k$ $(0<k<p)$ of*

$$\prod_{i=0}^{p-1}\left(1+c_1+\binom{i}{1}c_2+\cdots+\binom{i}{i}c_{i+1}\right)$$

*vanishes.*

This lemma is proved by an elementary computation of binomial coefficients.

PROOF OF (iv) IN THEOREM 2. We shall prove (iv) by induction on $n$. It is true for $n=1$. Assume that it is true for $n-1$. For any $b \in \cap_q P_{n,q}^{(p)}$, let $b_0$ be a restriction of $b$ to $P_{n-1}$. By the inductive assumption, $b_0=1 \in P_{n-1}$ is clear. Then the $n$-th component of $\hat{b}$ represents an element $\alpha$ of $\pi_1(S^3-\hat{b}_0)\approx F_{n-1}$. If $\alpha$ is not straight, then there is some non-zero *mod p* $\bar{\mu}$-invariants since $\cap_q \Gamma_q^{(p)}F_{n-1}=\{1\}$. Therefore $\alpha=1$, and $b$ is trivial.

## References

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