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Mean values and associated measures of superharmonic functions

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1. Introduction

Throughout this paper Ω will denote a non-empty open subset of the Euclidean space $\mathbf{R}^n$ ($n \geq 2$). For each point x of $\mathbf{R}^n$ and each positive number r, let $B(x, r)$ and $S(x, r)$ denote, respectively, the open ball and the sphere of centre x and radius r. We shall use v to denote a superharmonic function in Ω.

If the closure $\bar{B}(x, r)$ of $B(x, r)$ is contained in Ω, then $v(x) \geq \mathscr{M}(v, x, r)$, where $\mathscr{M}(v, x, r)$ is the spherical mean value of v given by

$$\mathscr{M}(v, x, r) = (s_n r^{n-1})^{-1} \int_{S(x,r)} v ds.$$

Here s denotes surface area measure on $S(x, r)$ and s_n is the surface area of the unit sphere in $\mathbf{R}^n$. It is well known that if $B(x, R) \subseteq \Omega$, then $\mathscr{M}(v, x, \cdot)$ is decreasing on $(0, R)$ and $\mathscr{M}(v, x, r) \to v(x)$ as $r \to 0+$.

The measure ν associated to v is a non-negative (Radon) measure in Ω such that

$$\int_\Omega \phi d\nu = -(p_n s_n)^{-1} \int_\Omega v(x) \Delta \phi(x) dx$$

for each infinitely differentiable function ϕ with compact support in Ω. Here Δ is the n-dimensional Laplacian operator and $p_n = \max\{1, n-2\}$.

We are concerned here with a comparison of the behaviour of $\mathscr{M}(u, x, r)/\mathscr{M}(v, x, r)$ and $\mu(\bar{B}(x, r))/\nu(\bar{B}(x, r))$ as $r \to 0+$, where u is a superharmonic function in Ω with associated measure μ and x is a point of Ω such that $v(x) = +\infty$. As applications, we shall obtain results which restrict the size of the set of points at which, for example,

$$\limsup_{r \to 0+} r^\alpha \mathscr{M}(u, x, r) > 0 \qquad (n \geq 3, 0 < \alpha \leq n-2)$$

and we shall improve some recent results of Kuran [6] on superharmonic and harmonic extensions.

For the latter application, we shall need to work, more generally, with the case where u is δ-superharmonic in an open subset ω of Ω. Recall that u is said to be δ-superharmonic in ω if there exist superharmonic functions u_1 and u_2 in

ω such that $u(x)=u_1(x)-u_2(x)$ whenever $x\in\omega$ and $u_1(x)$ and $u_2(x)$ are not both $+\infty$. Notice that the equation $u=u_1-u_2$ holds *q.p.* (that is, except on a polar set) in ω. *A fortiori*, it holds a.e. (s) on every sphere in ω (see [4; Theorem 7.5]). Hence, if $\bar{B}(x, r)\subset\omega$, then u is integrable on $S(x, r)$ and $\mathscr{M}(u, x, r)=\mathscr{M}(u_1, x, r)-\mathscr{M}(u_2, x, r)$. Now let μ_1 and μ_2 be the measures associated to u_1 and u_2. Since $\mu_1(\omega)$ and $\mu_2(\omega)$ may both be $+\infty$, it is not generally the case that $\mu_1-\mu_2$ defines a signed measure on ω. However, if we write $\mu(F)=\mu_1(F)-\mu_2(F)$ for each Borel subset F of ω for which the difference is well-defined, then the restriction of μ to any compact subset of ω is a finite signed measure. Throughout the paper u will be a δ-superharmonic (sometimes superharmonic) function in Ω or in some open subset of Ω, and we shall use μ to denote the set-function defined above. We call μ the measure associated to u. Clearly, if u is δ-superharmonic in ω, the superharmonic functions u_1 and u_2 such that $u=u_1-u_2$ *q.p.* in ω will not be unique. However, we have the following easy result.

LEMMA 1. *If u is δ-superharmonic in ω, then $\mu(F)$ is uniquely defined for any Borel set F whose closure is compact and is contained in ω.*

2. Main results

THEOREM 1. *Let u be δ-superharmonic in Ω. If $x\in\Omega$ and $v(x)=+\infty$, then*

$$\liminf_{r\to 0+}\frac{\mu(\bar{B}(x, r))}{v(\bar{B}(x, r))}\le\liminf_{r\to 0+}\frac{\mathscr{M}(u, x, r)}{\mathscr{M}(v, x, r)}$$

$$\le\limsup_{r\to 0+}\frac{\mathscr{M}(u, x, r)}{\mathscr{M}(v, x, r)}\le\limsup_{r\to 0+}\frac{\mu(\bar{B}(x, r))}{v(\bar{B}(x, r))}. \qquad (1)$$

By making suitable choices of v in Theorem 1, we obtain the following.

THEOREM 2. *Let α be a positive real number and let f be a non-negative, continuous, increasing (in the wide sense) function on $[0, \alpha]$ such that f is differentiable on $(0, \alpha)$ and*

$$\int_0^\alpha t^{1-n}f(t)dt=+\infty.$$

Put

$$\hat{f}(r)=p_n\int_r^\alpha t^{1-n}f(t)dt \qquad (0<r<\alpha).$$

If u is δ-superharmonic in Ω and if $x\in\Omega$, then

$$\liminf_{r\to 0+}\{\mu(\bar{B}(x, r))/f(r)\}\le\liminf_{r\to 0+}\{\mathscr{M}(u, x, r)/\hat{f}(r)\}$$

$$\le\limsup_{r\to 0+}\{\mathscr{M}(u, x, r)/\hat{f}(r)\}\le\limsup_{r\to 0+}\{\mu(\bar{B}(x, r))/f(r)\}.$$

COROLLARY. *Let u and x be as in Theorem 2. If $n\geq 3$ and $0\leq q<n-2$, then*

$$(n-2)\liminf_{r\to 0+} r^{-q}\mu(\bar{B}(x, r)) \leq (n-q-2)\liminf_{r\to 0+} r^{n-q-2}\mathscr{M}(u, x, r)$$
$$\leq (n-q-2)\limsup_{r\to 0+} r^{n-q-2}\mathscr{M}(u, x, r) \leq (n-2)\limsup_{r\to 0+} r^{-q}\mu(\bar{B}(x, r)).$$

Further (corresponding to the case $q=n-2$), if $n\geq 2$,

$$p_n \liminf_{r\to 0+} r^{2-n}\mu(\bar{B}(x, r)) \leq \liminf_{r\to 0+} \{\mathscr{M}(u, x, r)/\log(1/r)\}$$
$$\leq \limsup_{r\to 0+} \{\mathscr{M}(u, x, r)/\log(1/r)\} \leq p_n \limsup_{r\to 0+} r^{2-n}\mu(\bar{B}(x, r)).$$

We come now to the first application of these results. Applying a technique of Watson [8] to the above corollary, we obtain the following.

THEOREM 3. *Suppose that $n\geq 3$ and that $0<\beta\leq n-2$. If u is superharmonic in Ω and*

$$S_\beta = \{x\in\Omega\colon \limsup_{r\to 0+} r^\beta\mathscr{M}(u, x, r) = +\infty\}$$

and

$$T_\beta = \{x\in\Omega\colon \limsup_{r\to 0+} r^\beta\mathscr{M}(u, x, r) > 0\},$$

then $m_{n-2-\beta}(S_\beta)=0$ and $m_\gamma(T_\beta)=0$ for all $\gamma>n-2-\beta$, where m_γ denotes γ-dimensional Hausdorff measure.

Finally, we come to the results on superharmonic and harmonic extensions. Some preliminary explanations are necessary. We shall use E to denote a polar set, closed in the topology of Ω. If u is superharmonic in $\Omega\backslash E$ and $\bar{B}(x, r)\subset\Omega$, then u is defined a.e. (s) on $S(x, r)$ and is measurable, but not necessarily integrable, on $S(x, r)$. If such a function u possesses a (possibly infinite) integral over $S(x, r)$, we shall continue to denote its mean value over $S(x, r)$ by $\mathscr{M}(u, x, r)$. In forming the quotient of two extended real-valued functions ϕ and ψ, both defined at x, we adopt the convention that $\phi(x)/\psi(x)=0$ if $\phi(x)>-\infty$ and $\psi(x)=+\infty$. With these understandings, we have the following lemma, whose proof is left to the reader.

LEMMA 2. *Suppose that $v>0$ on E and let u be superharmonic in $\Omega\backslash E$. If $y\in E$ and*

$$\liminf_{x\to y, x\in\Omega\backslash E} \{u(x)/v(x)\} = k > -\infty$$

then $\mathscr{M}(u, y, r)$ exists for all sufficiently small r and

$$\liminf_{r\to 0+} \frac{\mathscr{M}(u, y, r)}{\mathscr{M}(v, y, r)} \geq k.$$

The main result on superharmonic extensions is as follows. Its proof depends on Theorem 1, a result of Kuran [6; Theorem 1] (quoted as Theorem A in § 8), and a measure theoretic result of Watson [9; Theorem 1] (quoted as Theorem B in § 8).

THEOREM 4. *Suppose that $v>0$ on E and let u be superharmonic in $\Omega\backslash E$. If*

$$\liminf_{x\to y, x\in\Omega\backslash E} \{u(x)/v(x)\} > -\infty \tag{2}$$

for each y in E and if

$$\limsup_{r\to 0+} \frac{\mathscr{M}(u, y, r)}{\mathscr{M}(v, y, r)} \geq 0 \tag{3}$$

for v-almost all y in E, then u has a superharmonic extension to Ω.

COROLLARY. *Suppose that $v>0$ on E and that u is superharmonic in $\Omega\backslash E$. If*

$$\liminf_{x\to y, x\in\Omega\backslash E} \{u(x)/v(x)\}$$

is greater than $-\infty$ for each y in E and is non-negative for v-almost all y in E, then u has a superharmonic extension to Ω.

I am grateful to Professor F-Y. Maeda for pointing out that this corollary is essentially contained in a recent result of Brelot [2; Theorem 5] and for mentioning that Brelot's assumption that $v(E\cap\omega)>0$ for every open set ω such that $E\cap\omega$ is non-empty is superfluous.

As an application of Theorem 4, we obtain the following results on harmonic continuation.

THEOREM 5. *Suppose $v>0$ on E and let h be harmonic in $\Omega\backslash E$. If*

$$\limsup_{x\to y, x\in\Omega\backslash E} \{|h(x)|/v(x)\} < +\infty \tag{4}$$

for each y in E and if

$$\liminf_{r\to 0+} \frac{\mathscr{M}(h, y, r)}{\mathscr{M}(v, y, r)} \leq 0 \leq \limsup_{r+0+} \frac{\mathscr{M}(h, y, r)}{\mathscr{M}(v, y, r)}$$

for v-almost all y in E, then h has a harmonic continuation to Ω.

COROLLARY. *Suppose that $v>0$ on E and let h be harmonic in $\Omega\backslash E$. If* (4) *holds for each y in E and if*

$$\lim_{x\to y, x\in\Omega\backslash E} \{h(x)/v(x)\} = 0$$

for v-almost all y in E, then h has a harmonic continuation to Ω.

This corollary improves [6; Theorem 2]. For a good account of the main applications of extension results, we refer to [6].

3. Proof of Lemma 1

Let u be δ-superharmonic in ω and let u_1, u_2, u_3, u_4 be superharmonic functions in ω such that $u=u_1-u_2=u_3-u_4$ wherever the differences are well-defined. Then $u_1+u_4=u_2+u_3$ $q.p.$ in ω. Since superharmonic functions which are equal $q.p.$ are identical, the last written equation holds throughout ω. Hence if μ_j $(j=1, 2, 3, 4)$ is the measure associated to u_j, then $\mu_1+\mu_4=\mu_2+\mu_3$, so that $\mu_1(F)-\mu_2(F)=\mu_3(F)-\mu_4(F)$ whenever $\mu_3(F)+\mu_4(F)<+\infty$ and, in particular, whenever F is a Borel set whose closure is a compact subset of ω.

4. A preliminary result

The following is the key result in our proof of Theorem 1. It is essentially well-known and leads easily to other known results which we give below as corollaries.

LEMMA 3. *Let u be δ-superharmonic in Ω. If $0<r\leq R$ and $\bar{B}(x, R)\subset\Omega$, then*

$$\mathcal{M}(u, x, r) = \mathcal{M}(u, x, R) + p_n\int_r^R t^{1-n}\mu(\bar{B}(x, t))dt. \tag{5}$$

Clearly it is enough to prove the result in the case where u is superharmonic in Ω. In this case we have

$$u(x) = \mathcal{M}(u, x, r) + p_n\int_0^r t^{1-n}\mu(\bar{B}(x, t))dt$$

$$= \mathcal{M}(u, x, R) + p_n\int_0^R t^{1-n}\mu(\bar{B}(x, t))dt$$

(see [3, pp. 126–127]). If $u(x)<+\infty$, (5) follows immediately by subtraction. If $u(x)=+\infty$, then we replace u in $B(x, r)$ by the Poisson integral of the function $u|_{S(x,r)}$. The resulting function u', say, is superharmonic in Ω and $u'(x)<+\infty$. Hence if μ' is the measure associated to u', (5) holds with u replaced by u' and μ replaced by μ'. The equation (5) itself follows from the facts that $u=u'$ in $\Omega\backslash\bar{B}(x, r)$ and $\mu(\bar{B}(x, t))=\mu'(\bar{B}(x, t))$ when $r<t<R$. Although the latter fact is well-known, I know of no convenient reference; it can be proved as follows. Let G^{μ} and $G^{\mu'}$ denote the Green's potentials in $B(x, R)$ of the restrictions of the measures μ and μ' to $\bar{B}(x, t)$. It is easy to see that $G^{\mu}=G^{\mu'}$ in $B(x, R)\backslash\bar{B}(x, t)$. Suppose that $t<\rho<R$. The balayage in $B(x, R)$ of the characteristic function of $B(x, \rho)$ is equal to 1 in $B(x, \rho)$ and is the Green's potential of a measure λ supported

on $S(x, \rho)$. Hence

$$\mu(\bar{B}(x, t)) = \int_{B(x,t)} G^{\lambda} d\mu = \int_{S(x,\rho)} G^{\mu} d\lambda$$
$$= \int_{S(x,\rho)} G^{\mu'} d\lambda = \int_{B(x,t)} G^{\lambda} d\mu' = \mu'(\bar{B}(x, t)).$$

Now define σ on the interval $[0, +\infty)$ by $\sigma(0) = +\infty$ and

$$\sigma(r) = \begin{cases} -\log r & (n = 2,\ r > 0) \\ r^{2-n} & (n \geq 3,\ r > 0), \end{cases}$$

so that, if $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^n$ and if $y \in \mathbf{R}^n$, the function $x \to \sigma(|x-y|)$ is the fundamental superharmonic function of $\mathbf{R}^n$ with pole y.

COROLLARY 1. *Let u be superharmonic in Ω and suppose that $x \in \Omega$. Then $\mu(\{x\}) = 0$ if and only if $\mathscr{M}(u, x, r) = o(\sigma(r))$ as $r \to 0+$. In particular, if $u(x) < +\infty$ (so that $\mathscr{M}(u, x, r) = O(1)$ as $r \to 0+$), then $\mu(\{x\}) = 0$.*

A simple proof of the particular case has been given by Kuran [5].
To prove this corollary, suppose that $\bar{B}(x, R) \subset \Omega$, so that, by (5),

$$\mathscr{M}(u, x, r) = p_n \int_r^R t^{1-n} \mu(\bar{B}(x, t)) dt + O(1) \qquad (r \longrightarrow 0+). \tag{6}$$

Since $\mu(\bar{B}(x, t))$ is increasing on $(0, R]$, it is easy to see that the integral in (6) is $o(\sigma(r))$ as $r \to 0+$ if and only if $\mu(\bar{B}(x, t)) \to 0$ as $t \to 0+$. Since $\mu(\{x\}) = \lim_{t \to 0+} \mu(\bar{B}(x, t))$, the result follows.

COROLLARY 2. *If u is non-negative and superharmonic in $\mathbf{R}^n$ $(n \geq 3)$, then*

$$\mu(\bar{B}(x, r)) \leq r^{n-2} \mathscr{M}(u, x, r).$$

From (5), we have

$$\mathscr{M}(u, x, r) \geq (n-2) \int_r^R t^{1-n} \mu(\bar{B}(x, t)) dt,$$

for each number $R > r$. Hence

$$\mathscr{M}(u, x, r) \geq \mu(\bar{B}(x, r))(n-2) \int_r^{\infty} t^{1-n} dt$$
$$= r^{2-n} \mu(\bar{B}(x, r)).$$

Corollary 2 has been proved by Kuran [5; Theorem 4] who also gives the analogue for a disc in $\mathbf{R}^2$.

5. Proof of Theorem 1

It is enough to show that the last inequality in Theorem 1 holds, for the first inequality will then follow by working with $-u$ instead of u.

If the last expression in (1) is $+\infty$, the required inequality is trivial. Suppose now that this expression has the value λ and that $\lambda<\Lambda<+\infty$. Let R be a positive number such that

$$\mu(\bar{B}(x, r)) < \Lambda\nu(\bar{B}(x, r))$$

whenever $0<r\leq R$. By Lemma 3, if $0<r\leq R$, then

$$\begin{aligned}\mathscr{M}(u, x, r) &= p_n\int_r^R t^{1-n}\mu(\bar{B}(x, t))dt + O(1) \qquad (r\longrightarrow 0+)\\ &\leq \Lambda p_n\int_r^R t^{1-n}\,\nu(\bar{B}(x, t))dt + O(1)\\ &= \Lambda\mathscr{M}(v, x, r) + O(1).\end{aligned}$$

Since

$$\lim_{r\to 0+}\mathscr{M}(v, x, r) = v(x) = +\infty,$$

we obtain

$$\limsup_{r\to 0+}\frac{\mathscr{M}(u, x, r)}{\mathscr{M}(v, x, r)}\leq\Lambda,$$

and the theorem follows.

6. Proof of Theorem 2

Consider first the case of Theorem 2 in which $f(0)=0$. Let R be such that $\bar{B}(x, R)\subset\Omega$ and $0<R\leq\alpha$. Put $g(0)=0$, $g(t)=t^{1-n}f'(t)$ $(0<t<R)$, $g(t)=0$ $(t\geq R)$, and define a measure ν on Ω by writing $d\nu(y)=g(|x-y|)dy$. Then, if $0<r<R$,

$$\begin{aligned}\nu(\bar{B}(x, r)) &= \int_{B(x,r)}|x-y|^{1-n}f'(|x-y|)dy\\ &= s_n\int_0^r f'(t)dt = s_n f(r),\end{aligned}$$

and, by Lemma 3, if v is the Green's potential in Ω associated to ν, then

$$\begin{aligned}\mathscr{M}(v, x, r) &= s_n p_n\int_r^R t^{1-n}f(t)dt + O(1)\\ &= s_n\hat{f}(r) + O(1).\end{aligned}$$

Since

$$v(x) = \lim_{r\to 0+} \mathscr{M}(v, x, r) = s_n p_n \int_0^R t^{1-n} f(t) dt = +\infty,$$

the result now follows from Theorem 1.

If $f(0)\neq 0$, put $\nu=f(0)\delta_x$, where δ_x is the Dirac measure concentrated at x, and let v be given by $v(y)=f(0)\sigma(|x-y|)$. Then $\nu(\bar{B}(x, r))=f(0)$ for each positive r and $\mathscr{M}(v, x, r)=\sigma(r)f(0)\sim\hat{f}(r)$ as $r\to 0+$. Hence, the result again follows from Theorem 1.

To prove the Corollary, take $\alpha=1$ and $f(t)=t^q$ in Theorem 2. Then

$$\hat{f}(r) = \begin{cases} \dfrac{n-2}{n-q-2} r^{q+2-n}(1+o(1)) & (0 \leq q < n-2) \\ p_n \log(1/r) & (q = n-2). \end{cases}$$

7. Proof of Theorem 3

This proof is borrowed from Watson [7].

By the Corollary of Theorem 2, $S_\beta \subseteq S'_\beta$, where

$$S'_\beta = \{x\in\Omega\colon \limsup_{r\to 0+} r^{\beta+2-n}\mu(\bar{B}(x, r)) = +\infty\}.$$

Now

$$\limsup_{r\to 0+} r^{\beta+2-n}\mu(\bar{B}(x, r)) \leq \limsup_{r\to 0+} r^{\beta+2-n}\mu(J(x, r)),$$

where $J(x, r)$ is the closed cube of centre x and side $2r$ with edges parallel to the coordinate axes. Hence

$$\limsup_{r\to 0+} r^{\beta+2-n}\mu(\bar{B}(x, r))$$
$$\leq (2\sqrt{n})^{n-\beta-2} \lim_{\varepsilon\to 0+} [\sup_J \{\mu(J)(d(J))^{\beta+2-n}\colon x\in J,\ d(J) < \varepsilon\}],$$

where J is any non-degenerate n-dimensional interval and $d(J)$ is the diameter of J. Hence $S_\beta \subseteq Z$, where Z is the set of points x in Ω for which the last written limit is $+\infty$. By a result of Rogers and Taylor [7, Lemma 4], $m_{n-2-\beta}(Z)=0$, and therefore $m_{n-2-\beta}(S_\beta)=0$.

If, now, $0<\gamma'<\beta\leq n-2$, then $T_\beta\subseteq S_{\gamma'}$. Hence $m_{n-2-\gamma'}(T_\beta)=0$ for all such γ', so that $m_\gamma(T_\beta)=0$ whenever $\gamma>n-2-\beta$.

8. Proofs of Theorems 4 and 5

We start by quoting two results which we shall need in the proof of Theorem 4.

THEOREM A. *Suppose that $v=+\infty$ on E and that u is superharmonic in $\Omega\backslash E$. If*

$$\liminf_{x\to y, x\in\Omega\backslash E}\{u(x)/v(x)\}\geq 0$$

for all $y\in E$, then u has a superharmonic extension to Ω.

THEOREM B. *Let μ and ν be measures on a ball $B(z, R)$ such that $\nu(B(y, r))>0$ whenever $B(y, r)\subseteq B(z, R)$. If*

$$\limsup_{r\to 0+}\frac{\mu(\bar{B}(x, r))}{\nu(\bar{B}(x, r))}$$

is greater than $-\infty$ for all $x\in B(z, R)$ and is non-negative for ν-almost all $x\in B(z, R)$, then μ is a non-negative measure.

Theorem A is due to Kuran [6, Theorem 1]. Notice that the Corollary of Theorem 4 is an improvement of Theorem A.

Theorem B is due to Watson [9; Theorem 1]. Its proof depends on a result of Besicovitch [1; Theorem 3].

Suppose now that the hypotheses of Theorem 4 are satisfied. It is enough to prove that if $\bar{B}(z, R)\subset\Omega$, then u has a superharmonic extension to $B(z, R)$. Now there exists a positive superharmonic function v_1 in $B(z, 2R)$, with associated measure ν_1 such that $v_1=+\infty$ on $E\cap B(z, 2R)$ and $\nu_1(E)=0$. (To construct such a function v_1, take w to be a positive superharmonic function in $B(z, 2R)$ such that $w=+\infty$ on $E\cap B(z, 2R)$ and put $v=\sum_{m=1}^{\infty} m^{-2}\min(w, m)$.) In order to be able to apply Theorems A and B we put $\Omega'=B(z, 2R)\cap\Omega$ and work with the function v^*, defined in Ω' by

$$v^*(x) = v(x) + v_1(x) - |x|^2,$$

instead of v. The following properties of v^* and its associated measure ν^* are easily verified: (i) v^* is superharmonic in Ω', (ii) $\nu^*(B(y, r))>0$ whenever $B(y, r)\subseteq\Omega'$, (iii) $\nu^*=\nu$ on $\Omega'\cap E$, (iv) $v^*=+\infty$ on $\Omega'\cap E$, (v) condition (2) is satisfied with v^* replacing v for each y in $\bar{B}(z, R)\cap E$, (vi) condition (3) is satisfied with v^* replacing v for ν-almost all y in $\bar{B}(z, R)\cap E$. Define a function Φ on $\bar{B}(z, R)\cap E$ by

$$\Phi(y) = \liminf_{x\to y, x\in\Omega\backslash E}\{u(x)/v^*(x)\}.$$

Clearly Φ is lower semi-continuous on E. Also $\Phi>-\infty$ on E. Hence Φ is bounded below on the compact set $\bar{B}(z, R)\cap E$. Let κ be a non-positive lower bound of Φ on $\bar{B}(z, R)\cap E$. Then

$$\liminf_{x\to y, x\in\Omega\backslash E}\{(u(x)-\kappa v^*(x))/v^*(x)\}\geq 0 \qquad (y\in B(z, R)\cap E).$$

Applying Theorem A to the superharmonic function $u-\kappa v^*$ in $B(z, R) \cap E$, we find that $u-\kappa v^*$ has a superharmonic extension to $B(z, R)$. Hence u has a δ-superharmonic extension, $\bar{u}$ say, to $B(z, R)$. Let $\bar{\mu}$ be the measure on $B(z, R)$ associated to $\bar{u}$. By Theorem 1,

$$\limsup_{r\to 0+} \frac{\bar{\mu}(\bar{B}(y, r))}{v^*(\bar{B}(y, r))} \geq \limsup_{r\to 0+} \frac{\mathscr{M}(\bar{u}, y, r)}{\mathscr{M}(v^*, y, r)}$$
$$= \limsup_{r\to 0+} \frac{\mathscr{M}(u, y, r)}{\mathscr{M}(v^*, y, r)} \geq 0$$

for ν-almost all $y \in B(z, R) \cap E$. Also, by Lemma 2,

$$\limsup_{r\to 0+} \frac{\bar{\mu}(\bar{B}(y, r))}{v^*(\bar{B}(y, r))} \geq \liminf_{r\to 0+} \frac{\mathscr{M}(u, y, r)}{\mathscr{M}(v^*, y, r)}$$
$$\geq \liminf_{x\to y, x\in\Omega\setminus E} \{u(x)/v^*(x)\} > -\infty$$

for each $y \in B(z, R) \cap E$. Finally, since $\bar{\mu}=\mu$ in $B(z, r)\setminus E$ and μ is a non-negative measure, we have

$$\limsup_{r\to 0+} \frac{\bar{\mu}(\bar{B}(y, r))}{v^*(\bar{B}(y, r))} \geq 0$$

for each y in $B(z, R)\setminus E$. Hence $\bar{\mu}$ and v^* satisfy the hypotheses of Theorem B and therefore $\bar{\mu}$ is non-negative. It follows easily that $\bar{u}$ is superharmonic in $B(z, R)$, and the proof is complete.

The Corollary is an immediate consequence of the theorem and Lemma 2.

If the hypotheses of Theorem 5 are satisfied then Theorem 4 is applicable to both h and $-h$, so that h has a superharmonic extension h_1 and a subharmonic extension h_2 to Ω; but since E is polar, $h_1=h_2$ everywhere in Ω, so that they are harmonic.

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