HIROSHIMA MATH. J
13 (1983), 39–51

On the asymptotic properties for simple semilinear heat equations

KUSUO KOBAYASHI

(Received May 20, 1982)

§ 1. Introduction

It is known that solutions of Cauchy problems for some semilinear evolution equations may blow up in a finite time (or grow up to infinity as $t \to \infty$) for some initial values. There are several works concerning the asymptotic behavior of the solution of the Cauchy problem for the equation

$$\frac{\partial}{\partial t} u(t, x) = \Delta u(t, x) + g(u(t, x)), \quad t > 0, \; x \in \boldsymbol{R}^N, \tag{1.1}$$

with the initial condition

$$u(0, x) = a(x), \quad x \in \boldsymbol{R}^N. \tag{1.2}$$

The case when $g(\lambda) = \lambda^{1+\alpha}$ $(\alpha > 0)$ has been studied by H. Fujita [1], [2], K. Hayakawa [3] and S. Sugitani [7]. Assume that the initial value $a(x)$ is non-negative bounded continuous. Then these results can be stated as follows;

(i) in case $0 < \alpha N \leq 2$, for any initial value $a(x)$ not vanishing identically, the solution $u(t, x)$ of (1.1) with (1.2) blows up in a finite time, and

(ii) in case $\alpha N > 2$, (a) for sufficiently small initial values $a(x)$ $(\not\equiv 0)$ the solutions $u(t, x)$ of (1.1) with (1.2) converge to 0 uniformly in x as $t \to \infty$, and (b) for sufficiently large initial values $a(x)$ the solutions $u(t, x)$ of (1.1) with (1.2) blow up in a finite time.

For general f, there is a work of K. Kobayashi-T. Sirao-H. Tanaka [5].

Under what condition on the initial value $a(x)$ does the solution $u(t, x)$ of (1.1) with (1.2) converge to 0 as $t \to \infty$ in case $\alpha N > 2$? And, under what condition on $a(x)$ does the solution $u(t, x)$ blow up in a finite time in the same case?

In this paper we shall consider these kinds of problems for the equation (1.1) replacing g by f defined as follows:

$$f(\lambda) = \begin{cases} p\lambda - pq, & \lambda \geq q, \\ 0, & 0 \leq \lambda < q, \end{cases} \tag{1.3}$$

where p and q are positive constants.

For any bounded continuous function $a(x)$ on $\boldsymbol{R}^N$, it is known that the equation (1.1) replacing g by f with (1.2) has a global solution, which is denoted by $u(t, x)$ or $u(t, x; a, f)$, and that the following comparison theorem holds:

$$
\begin{aligned}
(1.4) \qquad & a(x) \leq \tilde{a}(x) \quad \text{and} \quad a(x) \not\equiv \tilde{a}(x) \quad \text{imply that} \\
& u(t, x; a, f) < u(t, x; \tilde{a}, f) \quad \text{for} \quad t > 0.
\end{aligned}
$$

In this paper we assume that the initial value $a(x)$ is non-negative bounded continuous and the dimension N is not less than 3. For these initial values $a(x)$ not vanishing identically the solutions $u(t, x; a, f)$ of the equation (1.1) replacing g by f with (1.2) are positive global solutions, that is, $u(t, x; a, f)>0$ for any $t>0$ and $x \in \boldsymbol{R}^N$. The positive global solution $u(t, x; a, f)$ may grow up to infinity as $t \to \infty$ for some large initial value $a(x)$, that is, for each positive constant M and each compact set K in $\boldsymbol{R}^N$ there exists $T>0$ such that $t>T$ and $x \in K$ imply $u(t, x; a, f)>M$. We seek a sufficient condition on the initial value $a(x)$ under which the solution $u(t, x; a, f)$ grows up to inifinity as $t \to \infty$. For this purpose we seek the stationary solutions of (1.1). The simplest stationary solutions will be radially symmetric ones, that is, solutions $u(x)$ of $\Delta u + f(u) = 0$ that depend only on $|x|$. To find such solutions $u(|x|)$, we set $|x| = r$. Then $u(r)$ satisfies the equation

$$
(1.5) \qquad \begin{cases} \dfrac{d^2u}{dr^2} + \dfrac{N-1}{r}\dfrac{du}{dr} + f(u) = 0, & r>0, \\[2ex] u(0) > 0, \quad \dfrac{du}{dr}(0) = 0. \end{cases}
$$

It is obvious that for each $\ell (0 \leq \ell \leq q)$, $u(r) = \ell$ is a solution of (1.5) with $0 \leq u(0) = \ell \leq q$. In Theorem 2.2 in §2, it is shown that for each $\ell (0 \leq \ell < q)$ there exists a unique solution $u_\ell(r)$ of (1.5) such that $u_\ell(0) > q$, $u_\ell(r) > \ell$ for any $r \geq 0$ and $u_\ell(r) \downarrow \ell$ as $r \to \infty$.

Consequently the non-negative radially symmetric stationary solutions of (1.1) are constatns $\ell (0 \leq \ell \leq q)$ and positive functions $u_\ell(|x|)$ $(0 \leq \ell < q)$. Using these stationary solutions we can obtain the following results for each $0 \leq \ell < q$ (see Theorem 3.1 in §3):

(a) In the case when $\ell \leq a(x) \leq u_\ell(|x|)$ and $\ell \not\equiv a(x) \not\equiv u_\ell(|x|)$, the solution $u(t, x; a, f)$ greater than ℓ converges to ℓ uniformly in x as $t \to \infty$.

(b) In the case when $a(x) \geq u_\ell(|x|)$ and $a(x) \not\equiv u_\ell(|x|)$, the solution $u(t, x; a, f)$ grows up to infinity as $t \to \infty$.

(c) In the case when $a(x) \geq q$ and $a(x) \not\equiv q$, the solution $u(t, x; a, f)$ grows up to infinity as $t \to \infty$.

The author wishes to thank Professor H. Tanaka and Professor M. Mimura for their helpful suggestions and advice.

§2. On positive solutions of $u''+(N-1)u'/r+f(u)=0,\ N\geq 3$

In order to study the existence of stationary solutions of the equation (1.1) replacing g by f defined by (1.3), we shall consider the equation (1.5). If $0\leq u(0)=\ell\leq q$, then $u(r)=\ell$ is the unique solution of (1.5). We shall seek the positive solution of (1.5) with $u(0)>q$. Let $(\cdot)'=d(\cdot)/dr$ and first consider the following equation

$$(2.1)\qquad \begin{cases} u''+(N-1)u'/r+pu-pq=0, & r>0,\\ u(0)=u_0>q, \quad u'(0)=0. \end{cases}$$

Let $J_\nu(r)$ be a standard Bessel function, that is,

$$J_\nu(r)=\textstyle\sum_{k=0}^{\infty}(-1)^k(r/2)^{2k+\nu}/k!\Gamma(\nu+k+1).$$

It is known that the unique solution $u_{(1)}(r)$ of the above equation (2.1) is given by

$$u_{(1)}(r)=(u_0-q)\Gamma(N/2)(\sqrt{p}r/2)^{(2-N)/2}J_{(N-2)/2}(\sqrt{p}r)+q.$$

LEMMA 2.1. *The positive solution $u(r)$ of* (1.5) *with $u(0)>q$, if it exists, is strictly decreasing in r and converges to some non-negative constant ℓ less than q as $r\to\infty$.*

PROOF. First we note that $u(r)$ satisfies the integral equation

$$u(r)=u(0)+\frac{1}{N-2}\int_0^r\{r^{2-N}-s^{2-N}\}s^{N-1}f(u(s))ds.$$

Since $u'(r)=-r^{1-N}\int_0^r s^{N-1}f(u(s))ds<0$, $u(r)$ is strictly decreasing in r. Suppose that $u(r)$ converges to $\ell\geq q$ as $r\to\infty$. Then $u(r)>q$ for $r\geq 0$, and hence $f(u(r))=pu(r)-pq$. Therefore the positive solution $u(r)$ is equal to the solution $u_{(1)}(r)$ of the equation (2.1) with $u_0=u(0)$. On the other hand, $u_{(1)}(r)$ is smaller than q for some $r>0$ since $J_{(N-2)/2}(\sqrt{p}r)$ has simple zeros in $(0,\infty)$. This contradiction completes the proof of Lemma 2.1.

If $u(r)\leq q$, then $f(u(r))=0$. Therefore we next consider the equation $u''+(N-1)u'/r=0$. For $0\leq\ell<q$,

$$u_{(2)}(r)=\ell+mr^{2-N},\ m>0$$

are the positive solutions of the equation

$$(2.2)\qquad \begin{cases} u''+(N-1)u'/r=0, & r>0,\\ \lim_{r\to\infty}u(r)=\ell. \end{cases}$$

Choosing suitable constants u_0 and m, and using a part of $u_{(1)}(r)$ and a part of $u_{(2)}(r)$, we can construct a positive solution $u(r)$ of the equation (1.5) with $u(0)>q$.

THEOREM 2.2. *For each constant ℓ with $0\leq\ell<q$, there exists a positive solution $u(r)$ of* (1.5) *with $u(0)>q$ such that $u(r)$ is strictly decreasing in r and converges to ℓ as $r\to\infty$.*

PROOF. Let r_0 be the smallest positive zero of the Bessel function $J_{(N-2)/2}(\sqrt{p}r)$, that is,

$$r_0 = \min\{r > 0 : u_{(1)}(r) = q\}. \tag{2.3}$$

Putting $m=(q-\ell)r_0^{N-2}$, we have $u_{(1)}(r_0)=q=u_{(2)}(r_0)$. Differentiating $u_{(1)}(r)$ and $u_{(2)}(r)$ yields

$$u'_{(1)}(r_0) = -(u_0-q)\Gamma(N/2)\sqrt{p}(\sqrt{p}r_0/2)^{(2-N)/2}J_{N/2}(\sqrt{p}r_0) < 0,$$

and

$$u'_{(2)}(r_0) = m(2-N)r_0^{1-N} = (2-N)(q-\ell)r_0^{-1} < 0.$$

Therefore, we can choose $u_0>q$ such that $u'_{(1)}(r_0)=u'_{(2)}(r_0)$. That is, putting

$$u_0 = q + (N-2)(q-\ell)r_0^{-1}/\{\Gamma(N/2)\sqrt{p}(\sqrt{p}r_0/2)^{(2-N)/2}J_{N/2}(\sqrt{p}r_0)\} > q,$$

we obtain the positive solution

$$u_\ell(r) = \begin{cases} (N-2)(q-\ell)(r/r_0)^{(2-N)/2}J_{(N-2)/2}(\sqrt{p}r)/\sqrt{p}r_0J_{N/2}(\sqrt{p}r_0) + q, \\ \qquad\qquad 0 < r \leq r_0, \\ \ell + (q-\ell)r_0^{N-2}r^{2-N}, \qquad r > r_0, \end{cases} \tag{2.4}$$

of (1.5) with $u(0)>q$ and $\lim_{r\to\infty} u(r)=\ell$.

EXAMPLE. When $N=3$,

$$u_\ell(r) = \begin{cases} (q-\ell)\sin\sqrt{p}r/\sqrt{p}r + q, & 0 < r\leq p^{-1/2}\pi, \\ \ell + (q-\ell)(\sqrt{p}r)^{-1}\pi, & r > p^{-1/2}\pi. \end{cases}$$

§3. Asymptotic behavior of the solution $u(t, x; a, f)$

In the present section we consider the Cauchy problem for the semilinear heat equation

$$(3.1) \qquad \frac{\partial u}{\partial t} = \Delta u + f(u), \quad t > 0,\ x \in \boldsymbol{R}^N,$$

with the initial condition

$$(3.2) \qquad u(0, x) = a(x), \quad x \in \boldsymbol{R}^N,$$

where $f(\lambda)$ is the function defined by (1.3) for the positive given numbers p and q, $a(x)$ is a non-negative bounded continuous function on $\boldsymbol{R}^N$ and N is greater than or equal to 3. For any non-negative bounded continuous initial value $a(x)$ not vanishing identically, the equation (3.1) with (3.2) has a unique positive global solution $u(t, x; a, f)$. There exist non-negative stationary solutions of (3.1) as was seen in §2, that is, the non-negative bounded radially symmetric solutions of $\Delta u + f(u) = 0$ are non-negative constants $\ell (0 \le \ell \le q)$ and $u_\ell(|x|)\ (0 \le \ell < q)$, where $u_\ell(r)$ is the positive solution of (1.5) with $u(0) > q$ defined by (2.4) and satisfying $\lim_{r\to\infty} u_\ell(r) = \ell$. We denote $u_\ell(|x|)$ by $u_\ell(x)$ for simplicity throughout this section.

THEOREM 3.1. *Let ℓ be a constant with $0 \le \ell < q$ and suppose that $a(x)$ is a non-negative bounded continuous function.*

(i) *If $\ell \le a(x) \le u_\ell(x)$ and $a(x) \not\equiv u_\ell(x)$, then the solution $u(t, x; a, f)$ of* (3.1) *with* (3.2) *converges to ℓ uniformly in x as $t \to \infty$.*

(ii) *If $a(x) \ge u_\ell(x)$ and $a(x) \not\equiv u_\ell(x)$, then the solution $u(t, x; a, f)$ of* (3.1) *with* (3.2) *grows up to infinity as $t \to \infty$.*

When $a(x) \ge \ell$ and $a(x) \not\equiv \ell$, the comparison theorem (1.4) implies that $u(t, x; a, f) > \ell$ for any $t > 0$ and $x \in \boldsymbol{R}^N$. Therefore $v(t, x) = u(t, x; a, f) - \ell$ is the positive solution of the following equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + f_\ell(v), & t > 0, \\ v(0, x) = a(x) - \ell, & \end{cases}$$

where

$$f_\ell(\lambda) = \begin{cases} p\lambda - p(q-\ell), & \lambda \ge q - \ell, \\ 0, & 0 \le \lambda < q - \ell. \end{cases}$$

Hence it is sufficient to show this theorem for any $q > 0$ on the assumption $\ell = 0$.

For proving this theorem we prepare several lemmas. We start with some estimates on $f(\lambda)$ defined by (1.3).

LEMMA 3.2. *For each $\lambda_0 > 0$, there exists a positive number α such that*

$$(3.3) \qquad f(\gamma\lambda) \le \gamma^{1+2\alpha} f(\lambda) \quad \textit{for} \quad 0 \le \lambda \le \lambda_0,\ 0 < \gamma \le 1,$$

and

$$f(\gamma\lambda) \geq \gamma^{1+2\alpha} f(\lambda) \quad \textit{for} \quad 0 \leq \gamma\lambda \leq \lambda_0,\ \gamma \geq 1. \tag{3.4}$$

PROOF. Since (3.4) follows from (3.3), we shall show (3.3). When $\gamma\lambda \leq q$, (3.3) holds for any $\alpha > 0$ since $f(\gamma\lambda) = 0$. Therefore it is sufficient to show (3.3) for $q/\gamma < \lambda \leq \lambda_0$ and $0 < \gamma < 1$. For $q < \gamma\lambda < \lambda$, we write

$$\frac{f(\gamma\lambda)}{f(\lambda)} = \frac{p\gamma\lambda - pq}{p\lambda - pq} = \frac{\gamma(\lambda - q) - q(1-\gamma)}{\lambda - q} = \gamma\left\{1 - \frac{q}{\lambda - q}\left(\frac{1}{\gamma} - 1\right)\right\}. \tag{3.5}$$

Putting $\beta = 1/\gamma - 1 > 0$ and $\lambda' = q/(\lambda - q) > 0$ in (3.5), we have

$$\frac{\log\{f(\gamma\lambda)/f(\lambda)\}}{\log\gamma} = \frac{-\log(1+\beta) + \log(1-\lambda'\beta)}{-\log(1+\beta)} = 1 - \frac{\log(1-\lambda'\beta)}{\log(1+\beta)}.$$

Because $\log(1-\lambda'\beta) \leq -\lambda' \log(1+\beta)$, we have for $\lambda \leq \lambda_0$

$$\frac{\log\{f(\gamma\lambda)/f(\lambda)\}}{\log\gamma} \geq 1 + \lambda' = 1 + \frac{q}{\lambda - q} > 1 + \frac{q}{\lambda_0 - q} \equiv 1 + 2\alpha > 1,$$

and hence

$$\log\{f(\gamma\lambda)/f(\lambda)\} \leq (1+2\alpha)\log\gamma \quad \text{for} \quad q/\gamma < \lambda \leq \lambda_0,\ 0 < \gamma < 1,$$

which completes the proof of Lemma 3.2.

The heat equation (3.1) with (3.2) is transformed into the integral equation

$$u(t, x) = H_t a(x) + \int_0^t H_{t-s} f(u(s, \cdot))(x) ds, \tag{3.6}$$

where

$$H_t a(x) = \int_{\boldsymbol{R}^N} H(t, x, y) a(y) dy,$$

$$H(t, x, y) = (4\pi t)^{-N/2} \exp(-|x-y|^2/4t).$$

We consider a class of monotone radially symmetric functions as follows:

$$\mathscr{A} = \{a \in C(\boldsymbol{R}^N) : a(x) \geq 0,\ \not\equiv 0,\ a(x) \geq a(y) \quad \text{for} \quad |x| \leq |y|\}.$$

It is obvious that $u_0(x)$ belongs to $\mathscr{A}$. Since f is non-decreasing, the following lemma holds (see Lemma 3.2 in [5]).

LEMMA 3.3. *If* $a(x) \in \mathscr{A}$, *then* $H_t a(x) \in \mathscr{A}$ *and* $u(t, x; a, f) \in \mathscr{A}$ *for each* $t \geq 0$.

Let $a(x)$ be an initial value satisfying the assumption in (i) of Theorem 3.1, that is, a bounded continuous function with $0 \leq a(x) \leq u_0(x)$ and $a(x) \not\equiv u_0(x)$.

Then, applying the comparison theorem (1.4) to the equation (3.1) with initial values $a(x)$ and $u_0(x)$, we have

$$0 \leq u(t, x; a, f) < u(t, x; u_0, f) = u_0(x) \qquad \text{for} \quad t > 0.$$

Since $u_0(x) \in \mathscr{A}$, for each $t_* > 0$ there exists a function $a^*(x) \in \mathscr{A}$ such that

$$0 \leq u(t_*, x; a, f) \equiv a_*(x) \leq a^*(x) < u_0(x),$$

for any $x \in \boldsymbol{R}^N$. Applying the comparison theorem (1.4) to the equation (3.1) with initial values $a_*(x)$, $a^*(x)$ and $u_0(x)$, we have

$$u(t, x; a_*, f) \leq u(t, x; a^*, f) < u(t, x; u_0, f),$$

and hence

$$u(t+t_*, x; a, f) \leq u(t, x; a^*, f) < u_0(x),$$

for any $t > 0$ and $x \in \boldsymbol{R}^N$. Therefore it is sufficient to prove (i) of Theorem 3.1 for each initial value $a(x) \in \mathscr{A}$ satisfying $0 \leq a(x) < u_0(x)$. Similarly, we have only to prove (ii) of Theorem 3.1 for each initial value $a(x) \in \mathscr{A}$ satisfying $a(x) > u_0(x)$.

We shall estimate the solution $u(t, x; a, f)$ of (3.1) with (3.2) using the stationary solution $u_0(x)$.

LEMMA 3.4. (i) *If* $0 \leq a(x) < u_0(x)$ *for any* $x \in \boldsymbol{R}^N$, *then for each* $r \geq r_0$ *there exists a positive number* $\gamma < 1$ *such that*

$$u(t, x; a, f) \leq \gamma u_0(x)$$

whenever $t \geq 0$ *and* $|x| \leq r$, *where* r_0 *is the positive constant given by* (2.3).

(ii) *If* $a(x) > u_0(x)$ *for any* $x \in \boldsymbol{R}^N$, *then for each* $r \geq r_0$ *there exists a positive number* $\gamma > 1$ *such that*

$$u(t, x; a, f) \geq \gamma u_0(x)$$

whenever $t \geq 0$ *and* $|x| \leq r$.

PROOF. First, we note that the solution $u(t, x; a, f) = u(t, x)$ of (3.6) and the stationary solution $u_0(x)$ can be constructed by iteration as follows. Putting

$$\begin{cases} u^{(0)}(t, x) = H_t a(x), \\ u^{(n+1)}(t, x) = H_t a(x) + \int_0^t ds \int_{\boldsymbol{R}^N} H(t-s, x, y) f(u^{(n)}(s, y)) dy, \; n = 0, 1, 2, \ldots, \end{cases}$$

$$\begin{cases} u_0^{(0)}(t, x) = H_t u_0(x), \\ u_0^{(n+1)}(t, x) = H_t u_0(x) + \int_0^t ds \int_{\boldsymbol{R}^N} H(t-s, x, y) f(u_0^{(n)}(s, y)) dy, \; n = 0, 1, 2, \ldots, \end{cases}$$

we have $u^{(n)}(t, x) \uparrow u(t, x)$ and $u_0^{(n)}(t, x) \uparrow u_0(x)$ as $n \to \infty$. Let α be a positive number defined in Lemma 3.2 for $\lambda_0 = 2\max_{x \in \boldsymbol{R}^N} u_0(x) = 2u_0(0)$ and let β be such that $0 < \beta < 2\alpha/(1+2\alpha)$. For arbitrarily fixed $r \geq r_0$, we choose a constant q' so that $0 < q' < \min_{|x| \leq r} u_0(x)$. Recall that $u_0(x) = q$ for $|x| = r_0$ and note $q' < q$. Furthermore we define

$$(3.7) \quad \tau = \inf\{t\colon H_t u_0(x) \leq \beta q' \quad \text{for any} \quad x \in \boldsymbol{R}^N\} = \inf\{t\colon H_t u_0(0) \leq \beta q'\}.$$

Now we proceed with the proof of (i). Define a positive number

$$(3.8) \qquad \gamma = \max_{0 \leq t \leq \tau, |x| \leq r} (H_t a(x)/H_t u_0(x)) \vee \gamma_0 < 1,$$

where $a \vee b = \max(a, b)$ and γ_0 is the solution in (0, 1) of

$$\varphi(\gamma) \equiv (1-\beta)\gamma^{1+2\alpha} - \gamma + \beta = 0.$$

Step 1 is to prove that

$$(3.9) \qquad u^{(n)}(t, x) \leq \gamma u_0^{(n)}(t, x), \qquad n = 0, 1, 2, \ldots,$$

provided $|x| \leq r$ and $u_0^{(n)}(t, x) > q'$. We shall show (3.9) by induction. Noting that $H_t u_0(x) \in \mathscr{A}$ and using the definition (3.7) of τ, we have for $t \geq \tau$

$$H_t u_0(x) \leq H_\tau u_0(0) = \beta q' < q' \qquad \text{for any} \quad x \in \boldsymbol{R}^N.$$

Therefore, assuming that $u_0^{(0)}(t, x) = H_t u_0(x) > q'$, we get $0 \leq t \leq \tau$, and hence by the definition (3.8) of γ we have (3.9) for $n = 0$.

Next we shall show that (3.9) holds for $n+1$ under the hypothesis that (3.9) holds for n. Since $u^{(n)}(s, y) \leq u(s, y) < u_0(y) < q$ for $|y| > r \geq r_0$, $f(u^{(n)}(s, y)) = 0$ for $|y| > r$ and hence

$$(3.10) \qquad u^{(n+1)}(t, x) = H_t a(x) + \int_0^t ds \int_{|y| \leq r} H(t-s, x, y) f(u^{(n)}(s, y)) dy.$$

In the case when $|y| \leq r$ and $u_0^{(n)}(s, y) > q'$, using the induction hypothesis, and next using (3.3) of Lemma 3.2 with $\lambda = u_0^{(n)}(s, y) \leq u_0(y) \leq u_0(0) < \lambda_0$, we have

$$f(u^{(n)}(s, y)) \leq f(\gamma u_0^{(n)}(s, y)) \leq \gamma^{1+2\alpha} f(u_0^{(n)}(s, y)).$$

While in the case when $|y| \leq r$ and $u_0^{(n)}(s, y) \leq q'$, noting $u^{(n)}(s, y) \leq u_0^{(n)}(s, y) \leq q' < q$, we have

$$f(u^{(n)}(s, y)) = 0 = \gamma^{1+2\alpha} f(u_0^{(n)}(s, y)).$$

Therefore we get from (3.10)

$$(3.11) \qquad u^{(n+1)}(t, x) \leq H_t a(x) + \gamma^{1+2\alpha} \int_0^t ds \int_{|y| \leq r} H(t-s, x, y) f(u_0^{(n)}(s, y)) dy$$

$$\leq H_t a(x) + \gamma^{1+2\alpha} \int_0^t ds \int_{\mathbf{R}^N} H(t-s, x, y) f(u_0^{(n)}(s, y)) dy.$$

Let us assume that $|x| \leq r$ and $u_0^{(n+1)}(t, x) > q'$. In case $0 \leq t \leq \tau$, noting the definition (3.8) of γ, we get from (3.11)

$$\begin{aligned} u^{(n+1)}(t, x) &\leq \gamma H_t u_0(x) + \gamma^{1+2\alpha} \int_0^t ds \int_{\mathbf{R}^N} H(t-s, x, y) f(u_0^{(n)}(s, y)) dy \\ &\leq \gamma u_0^{(n+1)}(t, x). \end{aligned}$$

Noting that $H_t a(x) \leq H_t u_0(x)$, we get from (3.11)

$$\begin{aligned} (3.12) \quad & u^{(n+1)}(t, x) - \gamma u_0^{(n+1)}(t, x) \\ &\leq (1-\gamma) H_t u_0(x) + (\gamma^{1+2\alpha} - \gamma) \int_0^t ds \int_{\mathbf{R}^N} H(t-s, x, y) f(u_0^{(n)}(s, y)) dy \\ &\leq (1-\gamma) H_t u_0(x) + (\gamma^{1+2\alpha} - \gamma)(q' - H_t u_0(x)) \\ &= (1-\gamma^{1+2\alpha}) H_t u_0(x) + (\gamma^{1+2\alpha} - \gamma) q', \end{aligned}$$

where we have used the assumption $u_0^{(n+1)}(t, x) > q'$, that is,

$$\int_0^t ds \int_{\mathbf{R}^N} H(t-s, x, y) f(u_0^{(n)}(s, y)) dy > q' - H_t u_0(x)$$

in deriving the last inequality of the above. In case $t > \tau$, noting the definition (3.7) of τ, we get from (3.12)

$$u^{(n+1)}(t, x) - \gamma u_0^{(n+1)}(t, x) \leq (\beta - \beta\gamma^{1+2\alpha} + \gamma^{1+2\alpha} - \gamma) q' = \varphi(\gamma) q' \leq 0,$$

because $\varphi(\gamma) \leq 0$ for $\gamma_0 \leq \gamma < 1$, and hence we obtain (3.9).

Step 2. Here the proof of (i) will be completed as follows. Let a positive number T be fixed. Since $\min_{|x| \leq r} u_0(x) > q'$ and $u_0^{(n)}(t, x)$ converges to $u_0(x)$ uniformly in $(t, x) \in [0, T] \times \mathbf{R}^N$ as $n \to \infty$, there exists a positive integer M such that $n \geq M$ implies that

$$u_0^{(n)}(t, x) > q' \qquad \text{for } 0 \leq t \leq T \text{ and } |x| \leq r.$$

Therefore, by Step 1, $n \geq M$ implies that

$$u^{(n)}(t, x) \leq \gamma u_0^{(n)}(t, x) \qquad \text{for } 0 \leq t \leq T \text{ and } |x| \leq r,$$

and hence

$$u(t, x) \leq \gamma u_0(x) \qquad \text{for } 0 \leq t \leq T \text{ and } |x| \leq r,$$

which establishes the proof of (i) since T is arbitrary.

(ii) can be proved along similar lines to (i). Put

(3.13) $$\gamma \equiv \min_{0\le t\le \tau, |x|\le r} (H_t a(x)/H_t u_0(x)) \wedge 2 > 1,$$

where $a \wedge b = \min(a, b)$. We shall prove

(3.14) $$u^{(n)}(t, x) \ge \gamma u_0^{(n)}(t, x), \qquad n = 0, 1, 2,\dots,$$

provided $|x| \le r$ and $u_0^{(n)}(t, x) > q'$, which yields (ii) similarly to (i). When $n=0$, (3.14) follows from the definition (3.13) of γ. Suppose that (3.14) holds for n. In the case when $|y| \le r$ and $u_0^{(n)}(s, y) > q'$, using the induction hypothesis and next using (3.4) of Lemma 3.2 with $\lambda = u_0^{(n)}(s, y) \le (2/\gamma) u_0(0) = \lambda_0/\gamma$, we get

(3.15) $$f(u^{(n)}(s, y)) \ge \gamma^{1+2\alpha} f(u_0^{(n)}(s, y)).$$

While in the case when $|y| > r \ge r_0$ or $u_0^{(n)}(s, y) \le q' < q$, (3.15) also holds since the right hand side of (3.15) vanishes. Therefore we have

(3.16) $$u^{(n+1)}(t, x) \ge H_t a(x) + \gamma^{1+2\alpha} \int_0^t ds \int_{\boldsymbol{R}^N} H(t-s, x, y) f(u_0^{(n)}(s, y)) dy.$$

Assume that $|x| \le r$ and $u_0^{(n+1)}(t, x) > q'$. In case $0 \le t \le \tau$, it follows from (3.16) that

$$u^{(n+1)}(t, x) \ge \gamma u_0^{(n+1)}(t, x),$$

using the definition (3.13) of γ. While in case $t > \tau$, using the definition (3.7) of τ, we have from (3.16)

$$u^{(n+1)}(t, x) - \gamma u_0^{(n+1)}(t, x) \ge (1-\gamma^{1+2\alpha}) H_t u_0(x) + (\gamma^{1+2\alpha} - \gamma) q' \ge \varphi(\gamma) q' > 0,$$

since $\varphi(\gamma) > 0$ for $\gamma > 1$. Thus the proof of Lemma 3.4 is completed.

LEMMA 3.5. (i) *If $0 \le a(x) < u_0(x)$ for any $x \in \boldsymbol{R}^N$, then for each $r \ge r_0$ there exist positive numbers $\alpha > 0$, $0 < \gamma < 1$ and an increasing sequence $\{t_n\}$ with $t_0 = 0$ such that*

(3.17) $$u(t, x; a, f) \le \gamma^{(1+\alpha)^n} u_0(x)$$

for $t \ge t_n$, $|x| \le r$ and $n = 0, 1, 2,\dots$.

(ii) *If $a(x) > u_0(x)$ for any $x \in \boldsymbol{R}^N$, then for each $r \ge r_0$ and $M' > \sup_{x\in\boldsymbol{R}^N} a(x)$ there exist positive numbers $\alpha > 0$, $\gamma > 1$ and an increasing sequence $\{t_n\}$ with $t_0 = 0$ such that*

(3.18) $$u(t, x; a, f) \ge \gamma^{(1+\alpha)^n} u_0(x)$$

for $t \ge t_n$ and $|x| \le r$, provided

(3.19) $$\gamma^{(1+\alpha)^{n-1}} u_0(0) \le M'.$$

PROOF. Put $u(t, x) = u(t, x; a, f)$ and let $r \ge r_0$ be fixed.

(i) Let γ be a positive number less than 1 defined in Lemma 3.4 for r, and let α be a positive number defined in Lemma 3.2 for $\lambda_0=u_0(0)$. We shall prove (3.17) by induction. When $n=0$, (3.17) holds by (i) of Lemma 3.4. Suppose (3.17) is true for n. Let $t>t_n$. Since $u(t, x; a, f)=u(t-t_n, x; u(t_n, \cdot), f)$, noting that $u(t_n+s, y)<u_0(y)<q$ namely $f(u(t_n+s, y))=0$ for $|y|>r$, we have

$$(3.20)\quad u(t, x) = H_{t-t_n}u(t_n, x) + \int_0^{t-t_n} ds \int_{\mathbf{R}^N} H(t-t_n-s, x, y)f(u(t_n+s, y))dy$$

$$= H_{t-t_n}u(t_n, x) + \int_0^{t-t_n} ds \int_{|y|\le r} H(t-t_n-s, x, y)f(u(t_n+s, y))dy.$$

Using the induction hypothesis and next applying (3.3) of Lemma 3.2 to $f(\tilde{\gamma}\lambda)$ with

$$0 < \tilde{\gamma} = \gamma^{(1+\alpha)^n} < 1 \quad \text{and} \quad \lambda = u_0(x) \le u_0(0) = \lambda_0,$$

we have

$$f(u(t_n+s, y)) \le f(\gamma^{(1+\alpha)^n}u_0(y)) \le \gamma^{(1+\alpha)^n(1+2\alpha)}f(u_0(y))$$

for $|y|\le r$. Therefore, noting $u(t_n, x)<u_0(x)$ for $x\in\mathbf{R}^N$, we have from (3.20)

$$(3.21)\quad u(t, x) \le H_{t-t_n}u_0(x)$$

$$+ \gamma^{(1+\alpha)^n(1+2\alpha)}\int_0^{t-t_n} ds \int_{|y|\le r} H(t-t_n-s, x, y)f(u_0(y))dy.$$

Since $u_0(y)<q$ namely $f(u_0(y))=0$ for $|y|>r\ge r_0$, noting that $u_0(x)$ is the solution of the integral equation (3.6) replacing $a(x)$ by $u_0(x)$, we have

$$\int_0^{t-t_n} ds \int_{|y|\le r} H(t-t_n-s, x, y)f(u_0(y))dy$$

$$= \int_0^{t-t_n} ds \int_{\mathbf{R}^N} H(t-t_n-s, x, y)f(u_0(y))dy = u_0(x) - H_{t-t_n}u_0(x).$$

Combining this with (3.21) yields

$$(3.22)\quad u(t, x) \le H_{t-t_n}u_0(x) + \gamma^{(1+\alpha)^n(1+2\alpha)}\{u_0(x) - H_{t-t_n}u_0(x)\}$$

$$= \gamma^{(1+\alpha)^{n+1}}u_0(x)$$

$$+ \{1-\gamma^{(1+\alpha)^n(1+2\alpha)}\}H_{t-t_n}u_0(x) + \gamma^{(1+\alpha)^{n+1}}\{\gamma^{(1+\alpha)^n\alpha}-1\}u_0(x)$$

$$\equiv \gamma^{(1+\alpha)^{n+1}}u_0(x) + I_n(t, x).$$

Since $H_{t-t_n}u_0(x)$ converges to 0 uniformly in x as $t\to\infty$, there exists a positive number t_{n+1} greater than t_n such that $I_n(t, x)<0$ for any $t\ge t_{n+1}$ and $|x|\le r$, and hence by (3.22)

$$u(t, x) \le \gamma^{(1+\alpha)^{n+1}}u_0(x) \qquad \text{for} \quad t \ge t_{n+1}, |x| \le r,$$

which completes the proof of (i).

(ii) can be proved by induction in a similar way to (i). Let $M' > \sup_{x \in \boldsymbol{R}^N} a(x)$ be fixed. Let γ be a number greater than 1 defined in (ii) of Lemma 3.4 for r and let α be a positive number defined in Lemma 3.2 for $\lambda_0 = M'$. Since (3.18) is valid for $n=0$, we shall prove (3.18) for $n+1$ under the hypothesis that (3.18) holds for n. Assume $\gamma^{(1+\alpha)^n} u_0(0) \leq M'$. Then $\gamma^{(1+\alpha)^{n-1}} u_0(0) \leq M'$, and hence using the induction hypothesis we have

$$f(u(t_n+s, y)) \geq f(\gamma^{(1+\alpha)^n} u_0(y)) \geq \gamma^{(1+\alpha)^n(1+2\alpha)} f(u_0(y))$$

for $s \geq 0$ and $|y| \leq r$, where we have applied (3.4) of Lemma 3.2 to $f(\gamma^{(1+\alpha)^n} u_0(y))$ since $\gamma^{(1+\alpha)^n} u_0(y) \leq \gamma^{(1+\alpha)^n} u_0(0) \leq M' = \lambda_0$ and $\gamma^{(1+\alpha)^n} > 1$. Consequently we similarly obtain

$$\begin{aligned} u(t, x) &\geq H_{t-t_n} u_0(x) + \gamma^{(1+\alpha)^n(1+2\alpha)} \int_0^{t-t_n} ds \int_{\boldsymbol{R}^N} H(t-t_n-s, x, y) f(u_0(y)) dy \\ &= \gamma^{(1+\alpha)^{n+1}} u_0(x) \\ &\quad + \{1 - \gamma^{(1+\alpha)^n(1+2\alpha)}\} H_{t-t_n} u_0(x) + \gamma^{(1+\alpha)^{n+1}} \{\gamma^{(1+\alpha)^n \alpha} - 1\} u_0(x), \end{aligned}$$

and hence we can prove (ii) in a similar way to (i).

PROOF OF THEOREM 3.1. (i) By the remark we made just before Lemma 3.4 we may assume that $a(x) \in \mathscr{A}$ and $0 \leq a(x) < u_0(x)$. Since $u(t, x; a, f) \in \mathscr{A}$ by Lemma 3.3, (i) of Lemma 3.5 implies that there exist positive numbers $\alpha > 0$, $0 < \gamma < 1$ and a sequence $\{t_n\}$ such that

$$u(t, x; a, f) \leq u(t, 0; a, f) \leq \gamma^{(1+\alpha)^n} u_0(0)$$

for $t \geq t_n$, $x \in \boldsymbol{R}^N$ and $n=0, 1, 2, \ldots$, which completes the proof of (i).

(ii) We may similarly assume that $a(x) > u_0(x)$. For any constants $r \geq r_0$ and $M > \sup_{x \in \boldsymbol{R}^N} a(x)$, let $|x_r| = r$ and $M' = \{u_0(0)/u_0(x_r)\} M$. Then the assumption (3.19) is equivalent to the assumption $\gamma^{(1+\alpha)^{n-1}} u_0(x_r) \leq M$. Therefore, it follows from (ii) of Lemma 3.5 that there exist $\alpha > 0$, $\gamma > 1$ and a sequence $\{t_n\}$ such that

$$u(t, x; a, f) \geq \gamma^{(1+\alpha)^n} u_0(x) \geq \gamma^{(1+\alpha)^n} u_0(x_r)$$

for $t \geq t_n$ and $|x| \leq |x_r| = r$, provided $\gamma^{(1+\alpha)^{n-1}} u_0(x_r) \leq M$, which implies (ii). Thus the proof of Theorem 3.1 is completed.

References

[1] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), 109–124.

[2] H. Fujita, On some nonexistence and nonuniqueness theorems for nonlinear parabolic

equations, Proc. Symp. Pure Math., AMS **18** (1969), 105–113.

[3] K. Hayakawa, On nonexistence of global solutions of some semilinear parabolic differential equations, Proc. Japan Acad. **49** (1973), 503–505.

[4] K. Kobayashi, On some semilinear evolution equations with time-lag, Hiroshima Math. J. **10** (1980), 189–227.

[5] K. Kobayashi, T. Sirao and H. Tanaka, On the growing up problem for semilinear heat equations, J. Math. Soc. Japan **29** (1977), 407–424.

[6] M. Nagasawa and T. Sirao, Probabilistic treatment of the blowing up of solutions for a nonlinear integral equation, Trans. Amer. Math. Soc. **139** (1969), 301–310.

[7] S. Sugitani, On nonexistence of global solutions for some nonlinear integral equations, Osaka J. Math. **12** (1975), 45–51.

[8] G. N. Watson, A treatise on the theory of Bessel functions, 2 nd edition, Cambridge U. P., 1958.

Department of Mathematics,
College of Liberal Arts,
Toyama University