Hiroshima Math. J.
13 (1983), 393–399

The Bergman kernel function for symmetric Siegel domains of type III

Dedicated to Professor K. Murata for his 60th birthday

Toru INOUE

(Received December 14, 1982)

It is known (Wolf-Korányi [7]) that every hermitian symmetric space of noncompact type has a standard realization as a Siegel domain of type III. In this note we give an explicit formula for the Bergman kernel function of such a symmetric Siegel domain.

The general definition of Siegel domain of type III was given by Pyatetskii-Shapiro [4] as follows. Let U, V and W be complex vector spaces. Let $U_{\boldsymbol{R}}$ be a real form of U, Ω an open convex cone in $U_{\boldsymbol{R}}$, and B a bounded domain in W. Given any $w \in B$, let Φ_w be a semi-hermitian form of $V \times V$ to U, i.e., $\Phi_w = \Phi_w^h + \Phi_w^b$ where Φ_w^h is hermitian relative to the complex conjugation of U over $U_{\boldsymbol{R}}$ and Φ_w^b is symmetric $\boldsymbol{C}$-bilinear. Then the domain

$$\{(u, v, w) \in U \oplus V \oplus W; \operatorname{Im} u - \operatorname{Re} \Phi_w(v, v) \in \Omega, w \in B\}$$

is called a Siegel domain of type III. Siegel domains of type II are degenerate special case $W=0$, i.e., $B=(0)$, $\Phi_0^b=0$ and Φ_0^h is positive definite relative to Ω.

For Siegel domains of type II (not necessarily symmetric nor homogeneous), an explicit formula for the Bergman kernel was given by Gindikin [1, Theorem 5.4] in terms of a certain integral over the dual cone of Ω (see also Korányi [3, Proposition 5.3]).

Every hermitian symmetric space of noncompact type can be written as G/K, where G is a connected semi-simple linear Lie group and K is a maximal compact subgroup of G. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. We denote the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ by $\mathfrak{g}_{\boldsymbol{C}}$, $\mathfrak{k}_{\boldsymbol{C}}$, $\mathfrak{p}_{\boldsymbol{C}}$, respectively. Then $\mathfrak{p}_{\boldsymbol{C}}$ is decomposed into the direct sum of two complex subalgebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$, which are $(\pm i)$-eigenspaces of the complex structure of $\mathfrak{p}$, respectively, and are abelian subalegbras of $\mathfrak{g}_{\boldsymbol{C}}$ normalized by $\mathfrak{k}_{\boldsymbol{C}}$.

Let $G_{\boldsymbol{C}}$ be the complexification of G and let $P^{\pm}$, $K_{\boldsymbol{C}}$ be the connected subgroups of $G_{\boldsymbol{C}}$ corresponding to $\mathfrak{p}^{\pm}$, $\mathfrak{k}_{\boldsymbol{C}}$, respectively. It is known that the map $\mathfrak{p}^+ \times K_{\boldsymbol{C}} \times \mathfrak{p}^- \to G_{\boldsymbol{C}}$, given by $(X^+, k, X^-) \to \exp X^+ \cdot k \cdot \exp X^-$, is a holomorphic diffeomorphism onto a dense open subset $P^+K_{\boldsymbol{C}}P^-$ of $G_{\boldsymbol{C}}$, which contains G. Therefore, every element $g \in P^+K_{\boldsymbol{C}}P^- \subset G_{\boldsymbol{C}}$ can be written in a unique way as

(1) $$g = \pi_+(g) \cdot \pi_0(g) \cdot \pi_-(g), \quad \pi_0(g) \in K_{\boldsymbol{C}}, \pi_\pm(g) \in P^\pm.$$

Furthermore, the map ζ: $P^+K_{\boldsymbol{C}}P^- \to \mathfrak{p}^+$, given by

(2) $$\zeta(g) = \log \pi_+(g),$$

induces a holomorphic diffeomorphism of G/K onto $\zeta(G) = D$, and D is a bounded domain in $\mathfrak{p}^+$.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{t}_{\boldsymbol{C}}$, the complexification of $\mathfrak{t}$, is a Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{C}}$. For each root α of $\mathfrak{g}_{\boldsymbol{C}}$ relative to $\mathfrak{t}_{\boldsymbol{C}}$, let $\mathfrak{g}^\alpha$ be the corresponding root space, and let $H_\alpha \in i\mathfrak{t} \cap [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ be the unique element such that $\alpha(H_\alpha) = 2$. We choose a linear order in the dual of the real vector space $i\mathfrak{t}$ such that $\mathfrak{p}^+$ is spanned by root spaces for noncompact positive roots. For each noncompact root α, we choose a root vector $E_\alpha \in \mathfrak{g}^\alpha$ such that $[E_\alpha, E_{-\alpha}] = H_\alpha$ and $\bar{E}_\alpha = E_{-\alpha}$ where the bar denotes the complex conjugation of $\mathfrak{g}_{\boldsymbol{C}}$ with respect to $\mathfrak{g}$.

Let Ψ be a maximal set of strongly orthogonal noncompact positive roots. Following Wolf-Korányi [7], we define, for every subset $\Gamma \subset \Psi$, the partial Cayley transform by

$$c_\Gamma = \prod_{\alpha \in \Gamma} \exp \frac{\pi}{4} i(E_\alpha + E_{-\alpha}).$$

Then $c_\Gamma G \subset P^+K_{\boldsymbol{C}}P^-$ and we can define

(3) $$S_\Gamma = \zeta(c_\Gamma G) \subset \mathfrak{p}^+$$

where ζ is as in (2). In [7] it is shown that S_Γ is a Siegel domain of type III and in the special case $\Gamma = \Psi$, S_Ψ is a Siegel domain of type II. For our purpose, however, the precise description of S_Γ as a Siegel domain of type III is not needed.

For $(g, z) \in G_{\boldsymbol{C}} \times \mathfrak{p}^+$, such that $g \exp z \in P^+K_{\boldsymbol{C}}P^-$, we define $g \cdot z \in \mathfrak{p}^+$ by

$$g \cdot z = \log(\pi_+(g \exp z))$$

where π_+ is as in (1). Then the map $z \to g \cdot z$ is holomorphic and S_Γ is the partial Cayley transform of the bounded symmetric domain D, i.e.,

$$S_\Gamma = c_\Gamma \cdot D.$$

Let $\mathscr{K}_\Gamma(z, w)$ be the Bergman kernel function of S_Γ and let χ: $K_{\boldsymbol{C}} \to \boldsymbol{C}^\times$ be the holomorphic character of $K_{\boldsymbol{C}}$ defined by

$$\chi(k) = \det(\mathrm{Ad}(k)|_{\mathfrak{p}^+}), \quad k \in K_{\boldsymbol{C}}.$$

Then we have

PROPOSITION. *The Bergman kernel function of S_Γ is given by*

$$\mathscr{K}_\Gamma(z, w) = \mathrm{vol}\,(D)^{-1}\chi(\pi_0(\exp(-\bar{w})c_\Gamma^{-2}\exp z))$$

where vol(D) *is the Euclidean volume of the bounded symmetric domain* D, π_0 *is as in* (1), *and* $w \to \bar{w}$ *denotes the complex conjugation of* $\mathfrak{g}_C$ *with respect to* $\mathfrak{g}$.

PROOF. Let $\mathscr{K}_D(z, w)$ be the Bergman kernel of the bounded symmetric domain D. Then it is known ([2], [5]) that

$$\mathscr{K}_D(z, w) = \mathrm{vol}\,(D)^{-1}\chi(\pi_0(\exp(-\bar{w})\exp z)). \tag{4}$$

For $z \in S_\Gamma$, let $j(c_\Gamma^{-1}, z)$ denote the complex Jacobian (determinant) of the holomorphic map $z \to c_\Gamma^{-1} \cdot z$ at the point z. Then by [5, Lemma 5.3, p. 65],

$$j(c_\Gamma^{-1}, z) = \chi(\pi_0(c_\Gamma^{-1}\exp z)).$$

On the other hand, since $D = c_\Gamma^{-1} \cdot S_\Gamma$, the general theory of the Bergman kernel implies that, for $z, w \in S_\Gamma$,

$$\mathscr{K}_\Gamma(z, w) = j(c_\Gamma^{-1}, z)\mathscr{K}_D(c_\Gamma^{-1}\cdot z, c_\Gamma^{-1}\cdot w)\overline{j(c_\Gamma^{-1}, w)} \tag{5}$$

In what follows we write c instead of c_Γ. We also write e^z for $\exp z$. Let σ denote the complex conjugation of G_C with respect to G, and let ι denote the anti-automorphism of G_C defined by $\iota(g) = g^{-1}$. Then

$$\begin{aligned}
\exp(\overline{-c^{-1}\cdot w}) &= \iota\sigma\pi_+(c^{-1}e^w) \\
&= \iota\sigma[c^{-1}e^w \cdot \iota\pi_-(c^{-1}e^w)\cdot \iota\pi_0(c^{-1}e^w)] \\
&= \sigma\pi_0(c^{-1}e^w)\cdot\sigma\pi_-(c^{-1}e^w)\cdot\iota\sigma(c^{-1}e^w), \\
\exp(c^{-1}\cdot z) &= \pi_+(c^{-1}e^z) \\
&= c^{-1}e^z\cdot\iota\pi_-(c^{-1}e^z)\cdot\iota\pi_0(c^{-1}e^z).
\end{aligned}$$

Since K_C normalizes $P^\pm$, and since $\sigma\pi_-(c^{-1}e^w) \in P^+$, $\sigma\pi_0(c^{-1}e^w)$, $\iota\pi_0(c^{-1}e^z) \in K_C$, $\iota\pi_-(c^{-1}e^z) \in P^-$, it follows from (4) that

$$\begin{aligned}
&\mathscr{K}_D(c^{-1}\cdot z, c^{-1}\cdot w) \\
&\quad = \mathrm{vol}\,(D)^{-1}\chi(\pi_0(\exp(\overline{-c^{-1}\cdot w})\exp(c^{-1}\cdot z))) \\
&\quad = \mathrm{vol}\,(D)^{-1}\chi(\sigma\pi_0(c^{-1}e^w))\chi(\pi_0(\iota\sigma(c^{-1}e^w)\cdot c^{-1}e^z))\chi(\iota\pi_0(c^{-1}e^z)).
\end{aligned}$$

On the other hand

$$\begin{aligned}
j(c^{-1}, z) &= \chi(\pi_0(c^{-1}e^z)), \\
\overline{j(c^{-1}, w)} &= \overline{\chi(\pi_0(c^{-1}e^w))} = \chi(\iota\sigma\pi_0(c^{-1}e^w)).
\end{aligned}$$

Hence (5) implies that

$$\mathscr{K}_\Gamma(z, w) = \mathrm{vol}(D)^{-1}\chi(\pi_0(\iota\sigma(c^{-1}e^w)\cdot c^{-1}e^z)).$$

Here $\iota\sigma(c^{-1}e^w)=\exp(-\bar{w})\cdot\iota\sigma(c^{-1})=\exp(-\bar{w})\cdot c^{-1}$, since $\sigma(c)=c^{-1}$, and the proposition follows.

We illustrate our result by the following example.

EXAMPLE. Let $G/K=SU(p, q)/S(U(p)\times U(q))$ $(p\geq q\geq 1)$. We have $G_{\boldsymbol{C}}=SL(p+q, \boldsymbol{C})$ and

$$K_{\boldsymbol{C}} = \left\{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix};\ a\in GL(p, \boldsymbol{C}),\ d\in GL(q, \boldsymbol{C}),\ \det(a)\det(d) = 1\right\}.$$

If we write $(p+q)\times(p+q)$ complex matrices in block form

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} (a \text{ is } p\times p,\ b \text{ is } p\times q,\ c \text{ is } q\times p,\ d \text{ is } q\times q),$$

then

$$\mathfrak{k}_{\boldsymbol{C}} = \left\{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix};\ \mathrm{trace}\,(a) + \mathrm{trace}\,(d) = 0\right\},\quad \mathfrak{p}_{\boldsymbol{C}} = \left\{\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}\right\};$$

furthermore, we can put

$$\mathfrak{p}^+ = \left\{\begin{pmatrix} 0 & b \\ 0 & 0\end{pmatrix}\right\},\quad \mathfrak{p}^- = \left\{\begin{pmatrix} 0 & 0 \\ c & 0\end{pmatrix}\right\}$$

and hence

$$P^+ = \left\{\begin{pmatrix} 1_p & b \\ 0 & 1_q\end{pmatrix}\right\},\quad P^- = \left\{\begin{pmatrix} 1_p & 0 \\ c & 1_q\end{pmatrix}\right\}.$$

Every element $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in P^+K_{\boldsymbol{C}}P^-$ can be written uniquely as

$$g = \begin{pmatrix} 1_p & bd^{-1} \\ 0 & 1_q\end{pmatrix}\begin{pmatrix} a-bd^{-1}c & 0 \\ 0 & d\end{pmatrix}\begin{pmatrix} 1_p & 0 \\ d^{-1}c & 1_q\end{pmatrix}. \tag{6}$$

Therefore, $\zeta(g)=\begin{pmatrix} 0 & bd^{-1} \\ 0 & 0\end{pmatrix}$ ($\zeta(g)$ is as in (2)) and it follows (cf. Wolf [6]) that

$$D = \zeta(G) = \left\{\begin{pmatrix} 0 & z \\ 0 & 0\end{pmatrix}\in\mathfrak{p}^+;\ 1_q - z^*z > 0\right\}$$

where z^* is the conjugate transpose of z and ">" means "is positive definite".

For $1\leq k\leq q$, let

$$c_k = \frac{1}{\sqrt{2}}\begin{pmatrix} 1_k & 0 & i1_k & 0 \\ 0 & \sqrt{2}\,1_{p-k} & 0 & 0 \\ i1_k & 0 & 1_k & 0 \\ 0 & 0 & 0 & \sqrt{2}\,1_{q-k} \end{pmatrix}$$

These elements c_k will play the role of partial Cayley transforms c_Γ. As in (3) we put

$$S_k = \zeta(c_k G) \subset \mathfrak{p}^+.$$

Then the domain S_k can be described as follows (cf. Pyatetskii-Shapiro [4]). We identify $\mathfrak{p}^+$ with the space $M_{p,q}(\boldsymbol{C})$ of $p\times q$ complex matrices and write $z\in M_{p,q}(\boldsymbol{C})$ in the form

$$z = \begin{pmatrix} u & v_2 \\ v_1 & w \end{pmatrix} \quad \begin{pmatrix} u \text{ is } k\times k,\ v_1 \text{ is } (p-k)\times k \\ v_2 \text{ is } k\times(q-k),\ w \text{ is } (p-k)\times(q-k) \end{pmatrix}.$$

Corresponding to the decomposition

$$\begin{pmatrix} u & v_2 \\ v_1 & w \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & v_2 \\ v_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix},$$

we have the direct sum decomposition

$$M_{p,q}(\boldsymbol{C}) = U \oplus V \oplus W,$$

where $U=M_{k,k}(\boldsymbol{C})$, $V=M_{p-k,k}(\boldsymbol{C})\oplus M_{k,q-k}(\boldsymbol{C})$ and $W=M_{p-k,q-k}(\boldsymbol{C})$. For a real form $U_{\boldsymbol{R}}$ of U, we take $U_{\boldsymbol{R}}=\{u\in U;\ u^*=u\}$. Let

$$B = \{w\in W;\ 1_{q-k} - w^*w > 0\},$$

and for each $w\in B$, we put $[w]=(1_{q-k}-w^*w)^{-1}$. For $w\in B$ and $v=\begin{pmatrix} 0 & v_2 \\ v_1 & 0 \end{pmatrix}$, $\tilde{v}=\begin{pmatrix} 0 & \tilde{v}_2 \\ \tilde{v}_1 & 0 \end{pmatrix}\in V$ we define

$$\Phi_w(v, \tilde{v}) = 2(\tilde{v}_1^*(1_{p-k}-ww^*)^{-1}v_1 + v_2[w]\tilde{v}_2^*) + i(v_2[w]w^*\tilde{v}_1 + \tilde{v}_2[w]w^*v_1).$$

Then Φ_w: $V\times V\to U$ is a semi-hermitian form and we have

$$S_k = \left\{\begin{pmatrix} u & v_2 \\ v_1 & w \end{pmatrix}\in M_{p,q}(\boldsymbol{C});\ \mathrm{Im}\ u - \mathrm{Re}\ \Phi_w(v, v) > 0,\ w\in B\right\}$$

where $v=\begin{pmatrix} 0 & v_2 \\ v_1 & 0 \end{pmatrix}$. Therefore, S_k is a Siegel domain of type III.

We turn to the Bergman kernel of S_k. Let $z'=\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in \mathfrak{p}^+$ and $w'=\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \in \mathfrak{p}^+$. If $g=\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in K_{\boldsymbol{C}}$, then

$$\chi(g) \equiv \det(\mathrm{Ad}(g)|_{\mathfrak{p}^+}) = (\det a)^q (\det d)^{-p} = (\det d)^{-(p+q)}. \tag{7}$$

Thus if we write

$$\exp(-\bar{w}')c_k^{-2} \exp z' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (\in P^+ K_{\boldsymbol{C}} P^-),$$

(6), (7) and the proposition imply that the Bergman kernel $\mathscr{K}_k$ of S_k is given by

$$\mathscr{K}_k(z', w') = \mathrm{vol}(D)^{-1} (\det d)^{-(p+q)}.$$

To compute d we write $z=\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}$, $w=\begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix}$ where z_1 and w_1 are $k \times k$, z_4 and w_4 are $(p-k)\times(q-k)$, and the sizes of the other rectangular blocks are determined accordingly. Then noting that $\exp(-\bar{w}')=\begin{pmatrix} 1_p & 0 \\ -w^* & 1_q \end{pmatrix}$ ($^-$ denotes the conjugation of $\mathfrak{sl}(p+q, \boldsymbol{C})$ with respect to $\mathfrak{su}(p, q)$), a simple computation shows that

$$d = \begin{pmatrix} i(w_1^* - z_1) - w_2^* z_2 & -iz_3 - w_2^* z_4 \\ iw_3^* - w_4^* z_2 & 1_{q-k} - w_4^* z_4 \end{pmatrix}.$$

Therefore, under the identification $\mathfrak{p}^+ = M_{p,q}(\boldsymbol{C})$, we have for $z=\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}$, $w=\begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix} \in S_k$

$$\mathscr{K}_k(z, w) = \mathrm{vol}\,(\mathrm{D})^{-1} \det \begin{pmatrix} i(w_1^* - z_1) - w_2^* z_2 & -iz_3 - w_2^* z_4 \\ iw_3^* - w_4^* z_2 & 1_{q-k} - w_4^* z_4 \end{pmatrix}^{-(p+q)}$$

In the special case $k=q$,

$$S_q = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in M_{p,q}(\boldsymbol{C});\ \begin{matrix} u \text{ is } q \times q \text{ and } v \text{ is } (p-q) \times q, \\ \frac{1}{2i}(u-u^*) - v^* v > 0 \end{matrix} \right\}$$

and S_q is a Siegel domain of type II. In this case we have for $z=\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $w=\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in S_q$

$$\mathscr{K}_q(z, w) = \mathrm{vol}\,(D)^{-1} \det(i(w_1^* - z_1) - w_2^* z_2)^{-(p+q)}.$$

References

[1] S. G. Gindikin: Analysis in homogeneous domains, Russian Math. Surveys **19** (1964), 1–89.
[2] T. Inoue: Unitary representations and kernel functions associated with boundaries of a bounded symmetric domain, Hiroshima Math. J. **10** (1980), 75–140.
[3] A. Korányi: The Poisson integral for generalized half-planes and bounded symmetric domains, Ann. of Math. **82** (1965), 332–350.
[4] I. I. Pyatetskii-Shapiro: Automorphic Functions and the Geometry of Classical Domains, Gordon and Breach, New York, 1969.
[5] I. Satake: Algebraic Structures of Symmetric Domains, Publ. Math. Soc. Japan 14, Iwanami, Tokyo and Princeton Univ. Press, 1980.
[6] J. A. Wolf: Fine structure of hermitian symmetric spaces, in "Symmetric Spaces", (ed. W. Boothby and G. Weiss), Marcel Dekker, New York, 1972, 271–357.
[7] J. A. Wolf and A. Korányi: Generalized Cayley transformations of bounded symmetric domains, Amer. J. Math. **87** (1965), 899–939.

Department of Mathematics,
Faculty of Science,
Yamaguchi University,
Yamaguchi 753, *Japan*