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Lie algebras in which the join of any set of subideals is a subideal

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Introduction

In the recent paper [2], the author investigated several classes of Lie algebras in which the join of any pair of subideals (resp. weak subideals) is always a subideal (resp. a weak subideal). The purpose of this paper is to present further resutls concerning the class of Lie algebras in which the join of any set of subideals (resp. weak subideals) is always a subideal (resp. a weak subideal).

In Section 1 we shall characterize the class $\mathfrak{L}^{\infty}$ (resp. $\overline{\mathfrak{L}}^{\infty}$) of Lie algebras in which the join of any set of subideals (resp. weak subideals) is always a subideal (resp. a weak subideal), that is, $\mathfrak{L} \cap (\mathfrak{L}^{\infty}$ Max-si$) = \mathfrak{L}^{\infty}$ and $\overline{\mathfrak{L}} \cap (\overline{\mathfrak{L}}^{\infty}$ Max-wsi$) = \overline{\mathfrak{L}}^{\infty}$ (Theorem 1). In Section 2 we shall show that over fields of characteristic zero, $\mathfrak{E} \cap (\mathfrak{L}^{\infty}$ Max-si$) \leq \mathfrak{L}^{\infty}$ (Theorem 7). As concrete subclasses of $\mathfrak{L}^{\infty}$ of Lie algebras over a field of characteristic zero, we shall present $\mathfrak{N}$ Max-si and $\mathfrak{E} \cap$ (($\mathfrak{N}$ Max-si) Max-si) (Theorem 8). In Section 3 we shall show that over fields of characteristic zero, $(\mathfrak{F} \cap \mathfrak{N})$ (Min-si $\cap$ Max-si) $\leq \mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}$ (Theorem 10).

Throughout the paper we employ the notations and terminology in [1] and [2], and all Lie algebras are over a field of arbitrary characteristic unless otherwise specified.

1.

Tôgô [4] introduced the class Min-wsi of Lie algebras satisfying the minimal condition for weak subideals. We analogously introduce the class Max-wsi of Lie algebras satisfying the maximal condition for weak subideals. On the other hand, we introduced the class $\mathfrak{L}$ (resp. $\mathfrak{L}^{\infty}$) [2] of Lie algebras in which the join of any pair (resp. any set) of subideals is always a subideal. We similarly introduced the class $\overline{\mathfrak{L}}$ (resp. $\overline{\mathfrak{L}}^{\infty}$) for the case of weak subideals. We shall first show the following

Theorem 1. (1) $\mathfrak{L} \cap (\mathfrak{L}^{\infty}$ Max-si$) = \mathfrak{L}^{\infty}$.
(2) $\overline{\mathfrak{L}} \cap (\overline{\mathfrak{L}}^{\infty}$ Max-wsi$) = \overline{\mathfrak{L}}^{\infty}$.

Proof. Here we only prove (1), since (2) is proved similarly. Clearly we have $\mathfrak{L}^{\infty} \leq \mathfrak{L} \cap (\mathfrak{L}^{\infty}$ Max-si$)$. Conversely, let $L \in \mathfrak{L} \cap (\mathfrak{L}^{\infty}$ Max-si$)$ and let $\{H_{\alpha}:$

$\alpha<\rho\}$ be any ascending chain of subideals of L, where ρ is an ordinal. We put $J=\bigcup_{\alpha<\rho} H_\alpha$ and show that J si L. Since $L \in \mathfrak{L}^\infty$ Max-si, L has an $\mathfrak{L}^\infty$-ideal I such that $L/I \in$ Max-si. $\{(H_\alpha+I)/I: \alpha<\rho\}$ is an ascending chain of subideals of L/I. Since $L/I \in$ Max-si, we can find an ordinal $\mu<\rho$ such that $(H_\alpha+I)/I \leq (H_\mu+I)/I$ for all $\alpha<\rho$. Then we have $J+I=H_\mu+I$. By modular law

$$J = J \cap (H_\mu+I) = H_\mu + (J \cap I) = \langle H_\mu, J \cap I \rangle .$$

Since $H_\alpha \cap I$ si $I \in \mathfrak{L}^\infty$ for all $\alpha<\rho$, we have $J \cap I$ si L. Therefore J si L as $L \in \mathfrak{L}$. By using [2, Lemma 3.3], we obtain $L \in \mathfrak{L}^\infty$.

In [2] we introduced the class $\mathfrak{L}^*$ of Lie algebras as the largest ɪ-closed subclass of $\mathfrak{L}$ having the following property: If H, K si L and $J=\langle H, K\rangle \in \mathfrak{L}^*$, then J si L. We similarly introduced the class $\bar{\mathfrak{L}}^*$ for the case of weak subideals. Then we obtain

COROLLARY 2. (1) $\mathfrak{N}(\mathfrak{L}^* \cap \text{Max-si}) \leq \mathfrak{L}^\infty$.
(2) $\mathfrak{N}(\bar{\mathfrak{L}}^* \cap \text{Max-wsi}) \leq \bar{\mathfrak{L}}^\infty$.

PROOF. Evidently $\mathfrak{N} \leq \mathfrak{L}^\infty$. Since $\mathfrak{N}\mathfrak{L}^* \leq \mathfrak{L}$ ([2, Theorem 2.6(2)]), we have (1) by Theorem 1. (2) is proved similarly.

In [2] we defined the class $\mathfrak{A}_1$ of Lie algebras L such that either $L \in \mathfrak{A}$, or $L \in \mathfrak{A}^2$ with $\dim(L/L^2)=1$. We also defined the class $\mathfrak{M}_1$ (resp. $\bar{\mathfrak{M}}_1$) of Lie algebras in which every subideal (resp. weak subideal) is an ideal. Then we have some concrete subclasses of $\mathfrak{L}^\infty$ (and $\bar{\mathfrak{L}}^\infty$) in the following

COROLLARY 3. *Let $\mathfrak{X}$ be a class of Lie algebras.*
(1) *If* $\mathfrak{X} = \mathfrak{A}_1 \cap \mathfrak{F}$ *or* $\mathfrak{X} = \mathfrak{M}_1 \cap \text{Max-}\lhd$, *then* $\mathfrak{N}\mathfrak{X} \leq \mathfrak{L}^\infty$.
(2) *If* $\mathfrak{X} = \mathfrak{A}_1 \cap \mathfrak{F}$ *or* $\mathfrak{X} = \bar{\mathfrak{M}}_1 \cap \text{Max-}\lhd$, *then* $\mathfrak{N}\mathfrak{X} \leq \bar{\mathfrak{L}}^\infty$.

PROOF. By [2, Theorems 2.7 and 2.10] we have $\mathfrak{A}_1 \leq \mathfrak{L}^* \cap \bar{\mathfrak{L}}^*$, $\mathfrak{M}_1 \leq \mathfrak{L}^*$ and $\bar{\mathfrak{M}}_1 \leq \bar{\mathfrak{L}}^*$. It is clear that $\mathfrak{M}_1 \cap \text{Max-}\lhd = \mathfrak{M}_1 \cap \text{Max-si}$ and $\bar{\mathfrak{M}}_1 \cap \text{Max-}\lhd = \bar{\mathfrak{M}}_1 \cap \text{Max-wsi}$. Thus the results follow from Corollary 2.

REMARK. We can show that

$$\mathfrak{A}_1 \cap \mathfrak{F} = \mathfrak{A}_1 \cap \text{Max-si} = \mathfrak{A}_1 \cap \text{Max-wsi}.$$

In fact, let $L \in \mathfrak{A}_1 \backslash \mathfrak{A}$. Then $L \in \mathfrak{A}^2$ and $\dim(L/L^2)=1$. Since every weak subideal of L is a subideal of L, $L \in$ Max-si if and only if $L \in$ Max-wsi. Suppose that $L \in$ Max-si. Since every subspace of L^2 is a 2-step subideal of L, we have $L^2 \in \mathfrak{F}$, so that $L \in \mathfrak{F}$.

2.

In this section we shall consider the class $\mathfrak{L}^{\infty}$ over fields of characteristic zero.

We begin with

LEMMA 4. *If $\mathfrak{X}$ is an* ɪ*-closed coalescent class, then $\mathfrak{X}\leq\mathfrak{L}^{*}$ and therefore $\mathfrak{N}\mathfrak{X}\leq\mathfrak{L}$.*

PROOF. Let H, K si L and let $J=\langle H, K\rangle$. First assume that $L\in\mathfrak{X}$. Since $\mathrm{I}\mathfrak{X}=\mathfrak{X}$, we have $H, K\in\mathfrak{X}$. By the coalescency of $\mathfrak{X}$ we have J si L. Hence $L\in\mathfrak{L}$ and therefore $\mathrm{I}\mathfrak{X}=\mathfrak{X}\leq\mathfrak{L}$. Next assume that $J\in\mathfrak{X}$. Since H, K si J, we have $H, K\in\mathrm{I}\mathfrak{X}=\mathfrak{X}$ and so J si L by the coalescency of $\mathfrak{X}$. Therefore we obtain $\mathfrak{X}\leq\mathfrak{L}^{*}$. Thus $\mathfrak{N}\mathfrak{X}\leq\mathfrak{L}$ by [2, Theorem 2.6].

[1, Theorem 3.3.3] states that over fields of characteristic zero, the following classes are all coalescent:

$$\text{Min-si, Max-si, } \mathfrak{G}^{1}.$$

Therefore by Lemma 4 we have

COROLLARY 5. *Over fields of characteristic zero, the following classes are contained in $\mathfrak{L}$:*

$$\mathfrak{N}\,\text{Min-si}, \mathfrak{N}\,\text{Max-si}, \mathfrak{N}\mathfrak{G}^{1}.$$

For a Lie algebra L, the set of left Engel elements of L is denoted by $\mathfrak{e}(L)$. It is clear that $L\in\mathfrak{E}$ if and only if $L=\mathfrak{e}(L)$.

LEMMA 6. *Let L be a Lie algebra over a field of characteristic zero and let $\mathscr{S}=\{H\colon H \text{ si } L \text{ and } H\subseteq\mathfrak{e}(L)\}$. Assume that $\mathfrak{e}(L)\leq L$. If $\mathscr{S}$ is closed under the formation of unions of ascending chains, then the join of any subset of $\mathscr{S}$ always belongs to $\mathscr{S}$.*

If particularly $L\in\mathfrak{E}$, then $L\in\mathfrak{L}^{\infty}$ if and only if $\mathscr{S}=\{H\colon H \text{ si } L\}$ is closed under the formation of unions of ascending chains.

PROOF. It suffices to prove the first half, since the latter half is immediately deduced from the first half. First we shall show that $\mathscr{S}$ is closed under the join of any pair in it. Let $H, K\in\mathscr{S}$ and let $J=\langle H, K\rangle$. Since $\mathfrak{e}(L)\leq L$, we have $J\leq\mathfrak{e}(L)$. So we have to show that J si L. To do this we use induction on $s=\mathrm{si}(L\colon H)$. It is trivial for $s\leq 1$. Let $s>1$ and suppose that the result is true for $\leq s-1$. Since $K\leq\mathfrak{e}(L)$, for each $x\in K$ $\mathrm{ad}_L x$ is a nil derivation of L and so $\exp(\mathrm{ad}_L x)$ is well-defined. Then by [1, Lemma 1.4.9] we have

$$\langle H^K \rangle = \langle H^e \colon e \in \exp(\mathrm{ad}_L K) \rangle .$$

Let the elements of $\exp(\mathrm{ad}_L K)$ be well-ordered, so that $\exp(\mathrm{ad}_L K) = \{e_\alpha \colon \alpha < \rho\}$ for some ordinal ρ. Then we can define the ascending chain $\{H_\alpha \colon \alpha \leq \rho\}$ of subalgebras of L as follows:

$$H_0 = 0,\ H_\alpha = \langle H^{e_\beta} \colon \beta < \alpha \rangle \qquad (0 < \alpha \leq \rho).$$

By transfinite induction on α we show that $H_\alpha \in \mathscr{S}$ for any $\alpha \leq \rho$. It is trivial for $\alpha = 0$. Let $0 < \alpha \leq \rho$ and suppose that $H_\beta \in \mathscr{S}$ for all $\beta < \alpha$. If α is a limit ordinal, then $H_\alpha = \bigcup_{\beta < \alpha} H_\beta$. Since $\{H_\beta \colon \beta < \alpha\}$ is an ascending chain in $\mathscr{S}$, we have $H_\alpha \in \mathscr{S}$ by our assumption. Suppose that α is not a limit ordinal. Then we have $H_\alpha = \langle H_{\alpha-1}, H^{e_{\alpha-1}} \rangle$ and $H_{\alpha-1} \in \mathscr{S}$ by the second induction hypothesis. Since $H_{\alpha-1} \leq H^L$, we have $H_{\alpha-1}$ si H^L. Obviously $\mathfrak{e}(H^L) = H^L \cap \mathfrak{e}(L) \leq H^L$ and $H_{\alpha-1} \leq \mathfrak{e}(H^L)$. Since $H \lhd^s L$, we have $H \lhd^{s-1} H^L \lhd L$ and so $H^{e_{\alpha-1}} \lhd^{s-1} (H^L)^{e_{\alpha-1}} = H^L$. On the other hand, we have $H^{e_{\alpha-1}} \leq \mathfrak{e}(L)^{e_{\alpha-1}} = \mathfrak{e}(L)$. It follows that $H^{e_{\alpha-1}} \leq H^L \cap \mathfrak{e}(L) = \mathfrak{e}(H^L)$. Since $\{M \colon M$ si H^L and $M \leq \mathfrak{e}(H^L)\}$ is closed under the formation of unions of ascending chains, by the first induction hypothesis we have

$$H_\alpha \text{ si } H^L \quad \text{and} \quad H_\alpha \leq \mathfrak{e}(H^L).$$

Therefore we obtain $H_\alpha \in \mathscr{S}$. This completes the second induction. In particular, $\langle H^K \rangle = H_\rho \in \mathscr{S}$. By [2, Lemma 2.4] we have J si L.

Now let $\{K_\alpha \colon \alpha \in A\}$ be any subset of $\mathscr{S}$. Let the elements of A be well-ordered, so that $A = \{\alpha \colon \alpha < \lambda\}$ for some ordinal λ. Then we can define the ascending chain $\{J_\alpha \colon \alpha \leq \lambda\}$ of subalgebras of L as follows:

$$J_0 = 0, \quad J_\alpha = \langle K_\beta \colon \beta < \alpha \rangle \quad (0 < \alpha \leq \lambda).$$

By using transfinite induction on α, we can show as in the proof of [2, Lemma 3.3] that $J_\alpha \in \mathscr{S}$ for any $\alpha \leq \lambda$. In particular, we have $\langle K_\alpha \colon \alpha < \lambda \rangle = J_\lambda \in \mathscr{S}$. This completes the proof.

We now set about showing the main results of this section.

THEOREM 7. *Over fields of characteristic zero,*

$$\mathfrak{E} \cap (\mathfrak{L}^\infty \text{ Max-si}) \leq \mathfrak{L}^\infty.$$

PROOF. Let $L \in \mathfrak{E} \cap (\mathfrak{L}^\infty$ Max-si) and let $\{H_\alpha \colon \alpha < \rho\}$ be any ascending chain of subideals of L, where ρ is an ordinal. Put $J = \bigcup_{\alpha < \rho} H_\alpha$. Since $L \in \mathfrak{L}^\infty$ Max-si, L has an $\mathfrak{L}^\infty$-ideal I such that $L/I \in$ Max-si. Then there exists an ordinal $\mu < \rho$ such that $(J+I)/I = (H_\mu + I)/I$. Hence $J = J \cap (H_\mu + I) = H_\mu + (J \cap I)$. Since $I \in \mathfrak{L}^\infty$, we have $J \cap I$ si I and so $J \cap I$ si L. Therefore we have J si L by [1, Lemma

2.1.4]. Thus $\{H: H \operatorname{si} L\}$ is closed under the formation of unions of ascending chains. We can now use Lemma 6 to conclude that $L \in \mathfrak{L}^{\infty}$.

THEOREM 8. *Over fields of characteristic zero, the following classes are contained in* $\mathfrak{L}^{\infty}$:

$$\mathfrak{N}\,\text{Max-si}, \quad \mathfrak{E} \cap ((\mathfrak{N}\,\text{Max-si})\,\text{Max-si}).$$

PROOF. By Corollary 5 we have $\mathfrak{N}\,\text{Max-si} \leq \mathfrak{L}$. Hence we obtain

$$\mathfrak{N}\,\text{Max-si} \leq \mathfrak{L} \cap (\mathfrak{L}^{\infty}\,\text{Max-si}) = \mathfrak{L}^{\infty}$$

by Theorem 1 (1). It follows from Theorem 7 that

$$\mathfrak{E} \cap ((\mathfrak{N}\,\text{Max-si})\,\text{Max-si}) \leq \mathfrak{L}^{\infty}.$$

REMARK. It is well known ([1, Lemma 3.1.1]) that over fields of characteristic $p>0$, $\mathfrak{N}\,\text{Max-si} \nleq \mathfrak{L}^{\infty}$.

3.

In this section we shall consider Lie algebras L in which the set of all subideals of L is a complete sublattice of the lattice of all subalgebras of L, that is, $L \in \mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}$, where $\mathfrak{L}_{\infty}$ is the class of Lie algebras in which the intersection of any set of subideals is always a subideal ([2]).

We need the following lemma.

LEMMA 9. Min-si $\leq \mathfrak{L}_{\infty}$.

PROOF. Let $L \in$ Min-si and let $H \leq L$. Then $\{H^{L,n}: n<\omega\}$ is a descending chain of subideals of L. Since $L \in$ Min-si, we have $H^{L,n}=H^{L,n+1}$ for some $n<\omega$. Owing to [2, Proposition 3.1 (3)], we obtain $L \in \mathfrak{L}_{\infty}$.

For Lie algebras over a field of characteristic zero, we can slightly generalize [2, Theorem 3.5 (2)] in the following

THEOREM 10. *Over fields of characteristic zero,*

$$(\mathfrak{F} \cap \mathfrak{N})\,(\text{Min-si} \cap \text{Max-si}) \leq \mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}.$$

PROOF. Using [1, Proposition 8.5.1], we have

$$(\mathfrak{F} \cap \mathfrak{N})\,(\text{Min-si} \cap \text{Max-si}) = \text{Min-si} \cap (\mathfrak{N}\,\text{Max-si}).$$

By Theorem 8 and Lemma 9 we obtain

$$\text{Min-si} \cap (\mathfrak{N}\,\text{Max-si}) \leq \mathfrak{L}^{\infty} \cap \mathfrak{L}_{\infty}.$$

The following example shows that $\mathfrak{A}\mathfrak{F}_1 \nleq \mathfrak{L}_\infty$ over any field. Therefore it seems that we can not extend $\mathfrak{N}$ to subclasses of $\mathfrak{L}_\infty$ as the form $\mathfrak{N}\mathfrak{X}$ for any well-known class $\mathfrak{X}$.

EXAMPLE. We here consider a well-known example. Let A be an abelian Lie algebra with basis $\{a_0, a_1, \dots\}$. Define $x \in \mathrm{Der}(A)$ by $a_n x = a_{n+1}$ $(n \geq 0)$. We construct the split extension $L = A \dotplus \langle x \rangle$ of A by $\langle x \rangle$. Then clearly $L \in \mathfrak{A}\mathfrak{F}_1$. For each $n < \omega$ we put $H_n = \langle x, a_n, a_{n+1}, \dots \rangle$. It is easy to see that

$$L = H_0 \rhd H_1 \rhd \cdots \quad \text{and} \quad \bigcap_{n<\omega} H_n = \langle x \rangle .$$

But we have $I_L(\langle x \rangle) = \langle x \rangle$, so that $L \notin \mathfrak{L}_\infty$.

In [2] we introduced the class $\mathfrak{M}_2$ of Lie algebras in which every subideal is 2-step. By [2, Theorem 2.10] $\mathfrak{A}_1$ is a subclass of $\mathfrak{M}_2$. Furthermore, the Lie algebra constructed in [2, Example 4.3] belongs to $\mathfrak{M}_2$. The following result shows that in any $\mathfrak{M}_2$-algebra the set of all subideals is a complete sublattice of the lattice of all subalgebras.

PROPOSITION 11. $\mathfrak{M}_2 \leq \mathfrak{L}^\infty \cap \mathfrak{L}_\infty$.

PROOF. Owing to [1, Proposition 2.1.10], we have $\mathfrak{M}_2 \leq \mathfrak{L}$. Therefore we obtain the result by [2, Theorem 3.4 (2)].

The following proposition gives us a special type of $\mathfrak{M}_2$-algebras.

PROPOSITION 12. *Let S be a simple Lie algebra and let V be an S-module such that $VS = V$. We regard V as an abelian Lie algebra and construct the split extension $L = V \dotplus S$ of V by S. Then*

$$\{H\colon H \,\mathrm{si}\, L\} = \{H\colon H \leq V \text{ or } H = L\} = \{H\colon H \lhd^2 L\}.$$

PROOF. Since V is an abelian ideal of L, clearly we have

$$\{H\colon H \leq V \text{ or } H = L\} \subseteq \{H\colon H \lhd^2 L\} \subseteq \{H\colon H \,\mathrm{si}\, L\}.$$

Conversely, let $H \,\mathrm{si}\, L$. Then $(H+V)/V \,\mathrm{si}\, L/V \cong S$. Since S is simple, we have $H + V = V$ or L. So we may suppose that $H + V = L$. We can find an integer $n \geq 0$ such that $[L,_n H] \subseteq H$. Since $S \equiv H \bmod V$, we have

$$[V,_n S] = [V,_n H] \subseteq H.$$

On the other hand, $[V,_n S] = V$ by our assumption. It follows that $V \leq H$. Therefore we have $H = L$. This completes the proof.

As a special case of Proposition 12, we can easily obtain [2, Lemma 2.9].

Finally we shall use Proposition 12 to present an example of an $\mathfrak{M}_2$-algebra.

EXAMPLE. Let V be an infinite-dimensional vector space over any field. Let T be the set of endomorphisms of V of trace zero (in the sense of [3, p. 306]). Then it is known ([3, Lemma 4.1]) that T is an infinite-dimensional simple ideal of Der(V), where we regard V as an abelian Lie algebra. It is not hard to show that V is a T-module satisfying $VT=V$. We construct the split extension $L=V\dotplus T$ of V by T. Then we have $L\in\mathfrak{M}_2\leq\mathfrak{L}^{\infty}\cap\mathfrak{L}_{\infty}$ by Propositions 11 and 12.

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