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Continuous measure representations on harmonic spaces

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In a series of papers ([7], [8], [9], [10], [11]) F-Y. Maeda has developed a theory of Dirichlet integrals on those harmonic spaces X which admit a so-called measure representation.

By definition (see [11], p. 33) a *measure representation* is a homomorphism $\sigma=(\sigma_U)_{U\in\mathfrak{U}}$ of the sheaf $\mathscr{R}=(\mathscr{R}(U))_{U\in\mathfrak{U}}$ into the sheaf $\mathscr{M}=(\mathscr{M}(U))_{U\in\mathfrak{U}}$ of all signed Radon measures, such that

$$\sigma_U(f)\geq 0 \iff f \text{ is superharmonic on } U \quad (f\in\mathscr{R}(U),\ U\in\mathfrak{U}).$$

Here, $\mathscr{R}(U)$ denotes the set of all functions $f\colon U\to\mathscr{R}$, which are locally representable as differences of continuous superharmonic functions, and $\mathfrak{U}$ is the system of all open subsets of X.

In the special case where X is an open subset of $\boldsymbol{R}^n$ and the function 1 as well as the coordinate functionals $\pi_1,\dots,\pi_n$ belong to $\mathscr{R}(X)$, F-Y. Maeda was able (again using the hypothesis of the existence of a measure representation) to associate a differential operator L to the given harmonic space. The coefficients of this operator are measures on X and the following property holds:

$$L(h)=0 \iff h \text{ is harmonic on } U \quad (h\in\mathscr{C}^2(U),\ U\in\mathfrak{U}).$$

This note consists in the proof of the following

THEOREM. *Every harmonic space* (see [1] or [5]) *with a countable base of its topology admits a measure representation.*

Moreover, there exists a measure representation σ with the following *continuity property*: the restriction of σ_U to the space $\mathscr{S}_c(U)$ of all continuous superharmonic functions on $U\in\mathfrak{U}$ is continuous with respect to the topology of local uniform convergence on $\mathscr{S}_c(U)$ and the vague topology on $\mathscr{M}(U)$.

The proof of the existence of σ essentially relies on the results of N. Boboc, Gh. Bucur and A. Cornea concerning the carrier theory in standard H-cones (see [2]). In the first part of this paper we mainly compile those results in [2], which are important for our purposes. In the second and the third part, the existence of a measure representation and its continuity property will be proved.

In general the notations of [2] and [5] are used. In addition, $\mathscr{S}_c(U):=\mathscr{S}(U)\cap\mathscr{C}(U)$ denotes the set of all continuous superharmonic functions on $U\in$

$\mathfrak{U}$, and $\mathscr{P}_c(U) := \mathscr{P}(U) \cap \mathscr{C}(U)$ and $\mathscr{P}_0(U)$ denote respectively the set of all continuous potentials on U and that of all continuous potentials on U with compact superharmonic support. For $p \in \mathscr{P}(U)$ and a bounded Borel measurable function f on U

$$V_p(f) = f \odot p$$

is the specific product of f and p (as defined for example in [5], p. 196). $\mathscr{K}(U)$ denotes the set of all continuous functions on U having compact support.

§ 1. Representation measures for continuous potentials with compact superharmonic support

In this section let $(X, \mathscr{H}^*)$ denote a $\mathfrak{P}$-harmonic space with a countable base of its topology and $1 \in \mathscr{H}^*(X)$. We fix a (positive) Radon measure $\bar{\mu}$ on X such that

$$0 < \int p \, d\bar{\mu} < \infty \qquad \text{for all} \quad p \in \mathscr{P}_0(X),\ p \neq 0.$$

(1.1) REMARKS.

1) A measure $\bar{\mu}$ with the above properties always exists; take for example

$$\bar{\mu} := \sum_{n=1}^{\infty} \frac{1}{2^n} \varepsilon_{x_n},$$

where $\{x_n : n \in \mathbf{N}\}$ is a countable dense subset of X.

2) Every positive hyperharmonic function h being a limit of an increasing sequence $(p_n)_{n \in N}$ in $\mathscr{P}_0(X)$, we have

$$\int h \, d\bar{\mu} = 0 \iff h = 0.$$

3) A first step towards proving the existence of a measure representation consists in assigning to each potential $p \in \mathscr{P}_0(X)$ in a "reasonable way" a measure μ_p. Obviously a measure μ_p — even for arbitrary potentials p — can be defined by

$$\mu_p(f) := \int f \odot p \, d\bar{\mu} \quad (f \in \mathscr{K}(X)),$$

i.e. the $\bar{\mu}$-integral of the specific product of f and p, provided that this integral exists for all $f \in \mathscr{K}(X)$. First difficulties arise then in proving the following property of measure representations:

$$\mu_{p_1} - \mu_{p_2} \geq 0 \implies p_1 - p_2 \in \mathscr{S}(X);$$

in other words, the injectivity of the map $p \mapsto \mu_p$ on suitable subcones of $\mathscr{P}(X)$.

(1.2) EXAMPLE. Let $X =]-1, +1[$ and $\bar{\mu}$ the Lebesgue measure on X. For every open interval $U \subset X$ let $\mathscr{H}(U)$ denote the vector space of all continuous functions $f: U \to \boldsymbol{R}$ such that

1) f is locally affine on $U \backslash \{0\}$;

2) f is constant on $U \cap]-1, 0]$, provided that $0 \in U$

(see [5], Exercise 3.1.7). The corresponding harmonic structure possesses two non-proportional potentials with the same superharmonic support $\{0\}$, namely

$$p_1: x \longmapsto 1 - |x|, \quad p_2: x \longmapsto 1_{]0,1[}(x)(1-x).$$

Using the above notations we get $\mu_{p_1} = \varepsilon_0$, $\mu_{p_2} = (1/2)\varepsilon_0$, and hence $\mu_{p_1} - \mu_{p_2} \geq 0$, but $p_1 - p_2 \notin \mathscr{S}(X)$.

As demonstrated in [2] such problems cannot occur to representations of continuous potentials.

(1.3) The cone $S := \bar{\mathscr{S}}_+(X)$ of all positive hyperharmonic functions on X which are finite on a dense set, is a standard H-cone of functions on X such that $S \cong S^{**}$.

PROOF: By [3], Théorème 5, S is an H-cone which is canonically isomorphic to its bidual S^{**}. (In the cited theorem "superharmonic" means "hyperharmonic and finite on a dense set". Hence the cone $\mathscr{S}$ of [3], Théorème 5, coincides with our cone S). The proof of this theorem shows that there exists an absolutely continuous resolvent $\mathscr{V}$ such that S coincides with the cone of all $\mathscr{V}$-excessive functions. Hence, by [2], Example 3 on p. 113, S is a standard H-cone.

(1.4) The set $K^* := \{s \in S: \bar{\mu}(s) \leq 1\}$ is a compact metrizable Choquet simplex with respect to the natural topology; K^* is a cap of the cone S ([2], Proposition 4.2.4, remark after Corollary 4.2.5. The fact that $\bar{\mu}$ is a weak unit in S^* follows from (1.1), property 2, and from the remark at the end of p. 96 in [2]). Hence each $s \in S$ such that $\bar{\mu}(s) < \infty$ admits a representation

$$s(x) = \int_{X^*} s'(x)\mu(ds') \quad (x \in X),$$

where μ is a finite positive measure carried by the set X^* of all non-zero extreme points of K^*.

(1.5) There exists a semipolar subset S_X of X and a measurable map Θ_X^* from $E_X := X \backslash S_X$ into X^* such that the following properties hold:

i) The carrier of the function $p_x := \Theta_X^*(x)$, $\operatorname{carr}_X p_x := \{y \in X: \hat{R}_{p_x}^{X \setminus U} \neq p_x$ for every neighbourhood U of $y\}$, is the one-point set $\{x\}$, $x \in E_X$. (If p_x is superharmonic, then $\operatorname{carr}_X p_x$ coincides with the usual superharmonic support, as defined for example in [1], p. 163.)

ii) To each $p \in \mathscr{P}_0(X)$ there exists a unique finite Borel measure μ_p on E_X such that

$$p(y) = \int_{E_X} \Theta_X^*(z)(y)\mu_p(dz) \qquad \text{for all} \quad y \in X.$$

PROOF. Let X' denote the set of all non-zero extreme points of the cap $K := \{\mu \in S^*: \mu(1) \leq 1\}$ (S^* denotes the dual H-cone, see [2]). X can be viewed as a subspace of X' via the embedding $x \mapsto \varepsilon_x$. By [2], p. 194, there exists a Borel measurable subset E of X', a Borel measurable subset E^* of X^* and a bijection $\Theta^*: E \to E^*$ such that

(1) The sets $X' \backslash E$ and $X^* \backslash E^*$ both are semipolar;
(2) $\text{carr}_{\overline{X'}}\, \Theta^*(x) = \{x\}$ for all $x \in E$, where $\overline{X'}$ denotes the closure of X' in K;
(3) Both Θ^* and its inverse $\Theta: E^* \to E$ are Borel measurable.

The notion of "carr" is introduced in [2], §3.4. It is easy to see that the usual superharmonic support $S(p)$ of $p \in \mathscr{P}_0(X)$ coincides with $\text{carr}_{\overline{X'}}\, p$ and that

$$\text{carr}_{\overline{X'}}\, \Theta^*(x) = \text{carr}_X\, \Theta^*(x) \qquad \text{for} \quad x \in E \cap X.$$

By (1.4) each $p \in \mathscr{P}_0(X)$ is representable by a measure μ_p' on X^*. Since p is universally continuous (see [2], p. 97/98) the semipolar set $X^* \backslash E^*$ has μ_p'-measure zero; i.e. μ_p' is carried by E^* ([2], p. 197). If μ_p denotes the image measure of μ_p' under the Borel measurable bijection $\Theta: E^* \to E$, then

$$p(y) = \int_E \Theta^*(z)(y)\, \mu_p(dz) \qquad \text{for all} \quad y \in X.$$

Since μ_p is carried by the compact set $\text{carr}_{\overline{X'}}\, p = S(p) \subset X$, we can take

$$E_X := E \cap X$$

(as a subset of $X' \backslash E$ the set $S_X := X \backslash E$ is semipolar) and

$$\Theta_X^* := \Theta^* \mid E_X.$$

(1.6) REMARKS.

1) The measure μ_p introduced in (1.5) for $p \in \mathscr{P}_0(X)$ and regarded as a Radon measure on X coincides with the measure defined in (1.1.3), since

$$p(\cdot) = \int \Theta_X^*(z)(\cdot)\, \mu_p(dz), \quad f \odot p(\cdot) = \int \Theta_X^*(z)(\cdot) f(z)\, \mu_p(dz),$$

and hence

$$\bar{\mu}(f \odot p) = \iint \Theta_X^*(z)(y)\, \bar{\mu}(dy) f(z)\, \mu_p(dz) = \int f(z)\, \mu_p(dz)$$

for every $f \in \mathscr{K}_+(X)$.

2) The map

$$p \longmapsto \mu_p, \quad \text{where} \quad \mu_p(f) = \bar{\mu}(f \odot p), \quad f \in \mathscr{K}_+(X),$$

is one-to-one on $\mathscr{P}_0(X)$, since each $p \in \mathscr{P}_0(X)$ is representable as

$$p(\cdot) = \int \Theta_X^*(z)(\cdot)\,\mu_p(dz).$$

The injectivity assertion still remains true even in the case when $1 \notin \mathscr{H}^*(X)$.

For later purposes we provide the following preparative

(1.7) LEMMA. *Let $U \subset X$ be open. For $p, p' \in \mathscr{P}_0(X)$ the following properties are equivalent*:

1) *$p - p'$ is harmonic on U*;
2) $1_U \odot p = 1_U \odot p'$;
3) *the representation measures μ_p and $\mu_{p'}$ coincide on U.*

PROOF. The representation measure μ_{p_U} of $p_U := 1_U \odot p$ is the measure $1_U \mu_p$. Hence 2) and 3) are equivalent. The equality $1_U \odot p = 1_U \odot p'$ implies $p - p' = 1_{X \setminus U} \odot p - 1_{X \setminus U} \odot p'$, and hence $(p - p')|_U \in \mathscr{H}(U)$ (property 1).

Suppose now conversely that property 1) holds. The potential $1_U \odot p$ is the specific restriction of p with respect to U; hence it depends only on the potential part of the superharmonic function $p|_U$ (see [1], pp. 153–157). Analogously $1_U \odot p'$ only depends on the potential part of $p'|_U$. By condition 1) these two potential parts coincide. Hence $1_U \odot p = 1_U \odot p'$.

(1.8) COROLLARY. *Suppose that the restriction of $\mu_p - \mu_{p'}$ to U is a positive measure. Then there exists a superharmonic function s on U such that*

$$p = p' + s \qquad on \quad U.$$

PROOF. Let $\mu := 1_U(\mu_p - \mu_{p'}) \geq 0$. Then

$$q(\cdot) := \int \Theta_X^*(z)(\cdot)\,\mu(dz)$$

defines a positive hyperharmonic function q on X. The equality

$$q + 1_U \odot p' = 1_U \odot p$$

shows that $q \in \mathscr{P}_0(X)$. An application of (1.7), 3)$\Rightarrow$1), to the potentials p and $q + p'$ then finishes the proof.

§ 2. The existence of a measure representation

For an open subset U of a harmonic space X let $\mathscr{R}(U)$ denote the set of all functions $f: U \to \boldsymbol{R}$ which are locally representable as differences of continuous superharmonic functions:

For each $x \in U$ there exists an open neighbourhood V_x and $f_1, f_2 \in \mathscr{S}_c(V_x)$ such that $f = f_1 - f_2$ on V_x.

If X is a $\mathfrak{P}$-harmonic space, then (according to the extension theorem, see [1], p. 159, or [5], p. 46)

$s \in \mathscr{R}(U) \Leftrightarrow s$ is locally representable as a difference of globally defind continuous potentials with compact support.

(2.1) THEOREM. *Every harmonic space with a countable base admits a measure representation.*

PROOF. First step: Suppose first that $(X, \mathscr{H}^*)$ is a $\mathfrak{P}$-harmonic space. Dividing the sheaf $\mathscr{H}^*$ by a strictly positive continuous superharmonic function, we can assume without loss of generality that $1 \in \mathscr{H}^*(X)$ (see [11], p. 33).

Let $f \in \mathscr{R}(U)$, $U \in \mathfrak{U}$. If $f = p - p'$ on some open subset $V \subset U$ with $p, p' \in \mathscr{P}_0(X)$, then we define the restriction of the measure $\sigma_U(f)$ to V by $(\mu_p - \mu_{p'})|_V$. Then:

1) $\sigma_U(f)$ is well-defined: Suppose that $f = p_1 - p_1'$ on some open set $V_1 \subset U$, $p_1, p_1' \in \mathscr{P}_0(X)$. Then

$$p + p_1' = p' + p_1 \quad \text{on} \quad V \cap V_1,$$

and hence by (1.7)

$$\mu_p + \mu_{p_1'} = \mu_{p'} + \mu_{p_1}, \quad \mu_p - \mu_{p'} = \mu_{p_1} - \mu_{p_1'} \qquad \text{on} \quad V \cap V_1.$$

2) $\sigma_U(f)$ belongs to $\mathscr{M}(U)$, since the measures μ_p, $p \in \mathscr{P}_0(X)$, are finite Radon measures.

3) $\sigma = (\sigma_U)_{U \in \mathfrak{U}}$ is a measure representation: Obviously σ is a homomorphism of the sheaf $\mathscr{R}$ into the sheaf $\mathscr{M}$. By (1.8) $\sigma_U(f)$ is positive iff f is superharmonic on $U \in \mathfrak{U}$.

Second step: Let $(X, \mathscr{H}^*)$ be a harmonic space with a countable base of its topology. Then there exists a locally finite covering $(U_i)_{i \in I}$ of X consisting of $\mathfrak{P}$-sets and a subordinate continuous partition (φ_i) of the function 1. By the first step each of the harmonic spaces $(U_i, \mathscr{H}^* |_{U_i})$ admits a measure representation σ^i. Obviously

$$\sigma := \sum_{i \in I} \varphi_i \sigma^i$$

(i.e. $\sigma_U(f)(g)=\sum_{i\in I}\sigma^i_{U\cap U_i}(f)(\varphi_i g)$, $g\in\mathscr{K}(U)$, $f\in\mathscr{R}(U)$, $U\in\mathfrak{U}$) is a measure representation for $(X, \mathscr{H}^*)$. ┘

(2.2) EXAMPLES.

1) Let $X=]-1, +1[$, endowed with the harmonic structure of the solutions of the equation $u''=0$, and $\bar{\mu}$ the restriction of the Lebesgue measure on X. For every potential $p\in\mathscr{P}(X)$ the measure μ_p satisfies the condition

$$p(\cdot) = \int G(\cdot, y)\mu_p(dy),$$

where

$$G(x, y) = \min\left(\frac{1+x}{1+y}, \frac{1-x}{1-y}\right), \quad x, y\in X,$$

denotes the Green function (normed by $\bar{\mu}(G(\cdot, y))=1$ for $y\in X$). For $f\in\mathscr{R}(U)$, $U\in\mathfrak{U}$, we get

$$\sigma_U(f) = -\frac{1-x^2}{2}f'' \quad \text{(in the distribution sense)}.$$

2) Let $(X, \mathscr{H}^*)$ be the harmonic space considered in (1.2), and $\bar{\mu}$ the restriction of the Lebesgue measure to $X=]-1, +1[$. Then

$$\sigma_U(f) = -\varphi f'' + f'_-(0)\varepsilon_0.$$

Here, $f'_-(0)$ denotes the left derivative of f at 0, ε_0 is the Dirac measure at 0, f'' denotes the second derivative in the distribution sense of f on $X\setminus\{0\}$ and $\varphi\colon X\to\mathbf{R}$ is defined by

$$\varphi(y) = \begin{cases} \left(1-\frac{y}{2}\right)(1+y), & y<0 \\ \frac{y}{2}(1-y), & y\geq 0. \end{cases}$$

(A similar measure representation was considered by F-Y. Maeda in [11], Example 3.3).

(2.3) Without going into details we remark:

1) Let $(X, \mathscr{H}^*)$ be an abelian harmonic group with a countable base. Starting from a translation invariant compatible family of strict continuous potentials (see [13], VI) a translation invariant measure representation can be constructed.

2) In [4], A. Boukricha and W. Hansen study perturbations of harmonic spaces. These perturbations can be characterized with the help of measure representations: Let $(X, \mathscr{H})$ be a Bauer space with a countable base and let σ be a measure representation. For a sheaf $\mathscr{H}'$ of continuous functions on X the

following conditions are equivalent:

(1) $\mathscr{H}'$ is obtained by a perturbation of $\mathscr{H}$.

(2) $\mathscr{H}'=[\sigma+\mu=0]$ (i.e. $\mathscr{H}'(U)=\{f\in\mathscr{R}(U)\colon \sigma_U(f)+f\mu|_U=0\}$, $U\in\mathfrak{U}$), where μ is the image with respect to σ of a compatible family ([4], p. 78) of continuous potentials (p_U) (i.e. $\mu|_U=\sigma_U(p_U)$), and $\mathscr{R}$ denotes the sheaf of local differences of continuous superharmonic functions with respect to the given harmonic structure $\mathscr{H}$.

§3. Continuity of the measure representation

In §2 we showed that there exists a measure representation σ on a $\mathfrak{P}$-harmonic space $(X, \mathscr{H}^*)$ such that

$$\sigma_X(p_0)(f) = \bar{\mu}(f\odot p_0) \qquad \text{for every} \quad p_0\in\mathscr{P}_0,\, f\in\mathscr{K}_+(X).$$

For the study of continuity properties of this measure representation we need the following preparations.

(3.1) LEMMA (Hansen). *Let $(X, \mathscr{H}^*)$ be a $\mathfrak{P}$-harmonic space with a countable base. For every $p\in\mathscr{P}_0(U)$, $U\in\mathfrak{U}$, there exists a unique $\tilde{p}\in\mathscr{P}_0(X)$ such that*

$$R_{\tilde{p}}^{X\setminus U} + p = \tilde{p} \text{ on } U \quad \text{and} \quad S(p) = S(\tilde{p}).$$

The extension map $p\mapsto\tilde{p}$ is increasing.

For the proof see [6].

(3.2) LEMMA. *Suppose that $(s_n)_{n\in N}$ is a sequence in $\mathscr{S}_{c+}(U)$ converging locally uniformly to some $s_0\in\mathscr{S}_c(U)$ and let K be a compact subset of U. Then there exists a sequence $(\tilde{p}_n)_{n\in N}$ in $\mathscr{P}_0(X)$ such that*

1) *$(\tilde{p}_n)_{n\in N}$ converges locally uniformly to some $\tilde{p}_0\in\mathscr{P}_0(X)$ and*

$$\bar{\mu}(\tilde{p}_0) = \lim_{n\to\infty}\bar{\mu}(\tilde{p}_n),$$

2) *$\tilde{p}_n - s_n$ is harmonic on some neighbourhood of K, $n\geq 0$,*

3) *$\sigma_X(\tilde{p}_n)|_K = \sigma_U(s_n)|_K$, $n\geq 0$.*

PROOF. We choose a function $\varphi\in\mathscr{K}(U)$ such that $0\leq\varphi\leq 1$, $\varphi=1$ on some neighbourhood of K, and apply (3.1) to the potentials $p_n := {}^U R_{\varphi s_n}$ (${}^U R$ denotes the reduced function with respect to U). Obviously the extended potentials $\tilde{p}_n$ satisfy condition 2); property 3) is an immediate consequence of 2) and (1.7).

For the proof of 1) let $p'\in\mathscr{P}_0(U)$ such that $p'\geq 1$ on the compact set $L := \operatorname{supp}(\varphi)$. Then for any $\varepsilon>0$ there exists $N_\varepsilon\in N$ such that

$$s_n \leq s_0 + \varepsilon, \quad s_0 \leq s_n + \varepsilon \qquad \text{on} \quad L,$$

for every $n \geq N_\varepsilon$. Hence

$$\varphi s_n \leq \varphi s_0 + \varepsilon p', \quad \varphi s_0 \leq \varphi s_n + \varepsilon p',$$
$${}^U R_{\varphi s_n} \leq {}^U R_{\varphi s_0} + \varepsilon p', \quad {}^U R_{\varphi s_0} \leq {}^U R_{\varphi s_n} + \varepsilon p',$$

i.e. $p_n \leq p_0 + \varepsilon p'$, $p_0 \leq p_n + \varepsilon p'$ for every $n \geq N_\varepsilon$. The monotonicity and additivity of the extension map then imply

$$\tilde{p}_n \leq \tilde{p}_0 + \varepsilon \tilde{p}', \quad \tilde{p}_0 \leq \tilde{p}_n + \varepsilon \tilde{p}' \qquad \text{for} \quad n \geq N_\varepsilon.$$

Consequently the sequence $(\tilde{p}_n)_{n \in \mathbf{N}}$ converges locally uniformly to $\tilde{p}_0$. By Lebesgue's dominated convergence theorem

$$\bar{\mu}(\tilde{p}_0) = \lim_{n \to \infty} \bar{\mu}(\tilde{p}_n). \qquad \lrcorner$$

(3.3) Lemma. *Let $(\tilde{p}_n)_{n \in \mathbf{N}}$ be a sequence in $\mathscr{P}_0(X)$ converging locally uniformly to $\tilde{p}_0 \in \mathscr{P}_0(X)$ such that $\bar{\mu}(\tilde{p}_0) = \lim_{n \to \infty} \bar{\mu}(\tilde{p}_n)$. Then the sequence of measures $(\sigma_X(\tilde{p}_n))_{n \in \mathbf{N}}$ converges vaguely to $\sigma_X(\tilde{p}_0)$.*

Proof. The locally uniformly convergent sequence $(\tilde{p}_n)_{n \in \mathbf{N}}$ satisfies the assumptions of [12], (4.4) and (4.5) (applied to $Y = X$) concerning continuity properties of the specific multiplication. Consequently the sequence $(f \odot \tilde{p}_n)_{n \in \mathbf{N}}$ converges to $f \odot \tilde{p}_0$ with respect to the natural topology of the standard H-cone $S = \bar{\mathscr{S}}_+(X)$ for every bounded continuous function $f\colon X \to \mathbf{R}_+$. From

$$\liminf_{n \to \infty} \bar{\mu}(f \odot \tilde{p}_n) \geq \bar{\mu}(f \odot \tilde{p}_0),$$
$$\liminf_{n \to \infty} \bar{\mu}((1 - f) \odot \tilde{p}_n) \geq \bar{\mu}((1 - f) \odot \tilde{p}_0), f \in \mathscr{C}(X), 0 \leq f \leq 1,$$

and

$$\lim_{n \to \infty} \bar{\mu}(\tilde{p}_n) = \bar{\mu}(\tilde{p}_0),$$

we conclude

$$\lim_{n \to \infty} \sigma_X(\tilde{p}_n)(f) = \lim_{n \to \infty} \bar{\mu}(f \odot \tilde{p}_n) = \bar{\mu}(f \odot \tilde{p}_0) = \sigma_X(\tilde{p}_0)(f). \qquad \lrcorner$$

(3.4) Corollary. *Let $(s_n)_{n \in \mathbf{N}}$ be a sequence in $\mathscr{S}_c(U)$ converging locally uniformly to $s \in \mathscr{S}_c(U)$. Then the sequence $(\sigma_U(s_n))_{n \in \mathbf{N}}$ of Radon measures converges vaguely to $\sigma_U(s)$.*

Proof. Let $K \subset U$ be a compact set. After adding a fixed positive superharmonic function $s' \in \mathscr{S}_c(U)$ we may assume that $s_n \geq 0$ on some fixed neighbourhood U' of K. The assertion follows now from (3.2) and (3.3), applied to the sequence $(s_n|_{U'})_{n \in \mathbf{N}}$ and the fact that $\sigma_U(s_n)|_K = \sigma_{U'}(s_n)|_K$. $\lrcorner$

(3.5) Theorem. *Let $(X, \mathscr{H}^*)$ be a harmonic space with a countable base of its topology and σ the measure representation on X given in* (2.1). *Then for* *every* $U \in \mathfrak{U}$

$$\sigma_U : \mathscr{S}_c(U) \longrightarrow \mathscr{M}_+(U)$$

is continuous with respect to the topology of locally uniform convergence on $\mathscr{S}_c(U)$ *and the vague topology on* $\mathscr{M}_+(U)$.

PROOF. The measure representation σ given in (2.1) is of the form

$$\sigma = \sum_{i \in I} \varphi_i \sigma^i,$$

where each σ^i denotes a "local" measure representation on some $\mathfrak{P}$-set U_i, the family $(U_i)_{i\in I}$ being a locally finite covering of X and $(\varphi_i)_{i\in I}$ a subordinate partition of the constant function one.

To each σ^i the results of (3.4) can be applied. Let now $(s_n)_{n\in \boldsymbol{N}}$ be a sequence in $\mathscr{S}_c(U)$ converging locally uniformly to some $s \in \mathscr{S}_c(U)$, and let $f \in \mathscr{K}_+(U)$. Then there exists a finite subset $J \subset I$ such that

$$\overline{U}_i \cap \operatorname{supp}(f) = \emptyset \qquad \text{for all} \quad i \in I \backslash J.$$

For $i \in J$ the function $\varphi_i f$ is continuous with compact support; hence by (3.4)

$$\begin{aligned}\sigma_U(s)(f) &= \textstyle\sum_{i\in J} \sigma^i_{U\cap U_i}(s)(\varphi_i f) \\ &= \lim_{n\to\infty} \textstyle\sum_{i\in J} \sigma^i_{U\cap U_i}(s_n)(\varphi_i f) = \lim_{n\to\infty} \sigma_U(s_n)(f).\end{aligned}$$ ┘

The following example shows that in general σ is not continuous on $\mathscr{R}$.

(3.6) EXAMPLE. Let $(X, \mathscr{H}^*)$ be the harmonic space of the solutions of the equation $u''=0$ on $X =]-1, +1[$ and σ the measure representation defined by

$$\sigma_U(f) := -f'' \quad \text{(in the distribution sense, } f \in \mathscr{R}(U)).$$

For each $n \in \boldsymbol{N}$, $x \in]-1, +1[$, let

$$p_n(x) := n(1-|x|), \quad q_n(x) := \min\left(p_n(x),\, n - \frac{1}{n}\right).$$

Then $s_n := p_n - q_n \in \mathscr{R}(X)$. Since $0 \leq s_n \leq 1/n$, the sequence $(s_n)_{n\in\boldsymbol{N}}$ converges uniformly to 0, but the sequence of measures $(\sigma_X(s_n))_{n\in\boldsymbol{N}}$ is not vaguely convergent.

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