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On Bremermann's conjecture for the Silov boundary of pseudoconvex Riemann domains

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Introduction

H. J. Bremermann [6] introduced a Dirichlet problem for plurisubharmonic functions on some bounded pseudoconvex domain and characterized the Silov boundary as the unique closed regular boundary for this Dirichlet problem, and presented the following conjecture.

CONJECTURE. *The Silov boundary of a bounded pseudoconvex domain Ω with C^2-boundary in $\mathbf{C}^n$ coincides with the topological closure of the set of strictly pseudoconvex boundary points of Ω.*

Recently, using Kohn's global regularity theorem [10] for $\bar{\partial}$, M. Hakim-N. Sibony [9] showed that this conjecture is valid when Ω has C^∞-boundary.

In the present paper, we consider this problem when Ω is a Riemann domain over $\mathbf{C}^n$ and we show that the above conjecture is valid when $\bar{\Omega}$ is a holomorphically convex set with respect to some uniform algebras of holomorphic functions on Ω.

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§1. Notation and definitions

A (unramified) *Riemann domain over* $\mathbf{C}^n$ is a Hausdorff space R with a locally homeomorphic mapping $p: R \to \mathbf{C}^n$, called a projection of R. There is a unique complex structure on R such that $p: R \to \mathbf{C}^n$ gives a local coordinate system at each point of R. We shall assume in this paper that R has a countable base of open sets. Let Ω be a relatively compact subdomain of R and $O(\Omega)$ be the algebra of all holomorphic functions on Ω and $O(\bar{\Omega})$ be the algebra of the restrictions to $\bar{\Omega}$ of holomorphic functions on some neighborhood of $\bar{\Omega}$.

Let $A(\bar{\Omega}) = C(\bar{\Omega}) \cap O(\Omega)$, which is a uniform algebra. When A is a closed subalgebra of $A(\bar{\Omega})$ which contains 1, we call A a *uniform subalgebra* of $A(\bar{\Omega})$. Further when A separates points of $\bar{\Omega}$, A is said to be *separating*.

Let S_A be the *spectrum* of A, i.e., the set of non-trivial continuous multiplicative linear functionals on A, which is endowed with the weakest topology making any *Gelfand transform* $\hat{f}$ of $f \in A$ continuous, where $\hat{f}$ is defined by $\hat{f}(\chi) = \chi(f)$ for all $\chi \in S_A$. We denote by $\hat{A}$ the set of all Gelfand transforms of functions in A. If A is separating, there is a natural topological imbedding $\bar{\Omega} \subset S_A$ by the point evaluation mapping $\lambda: \bar{\Omega} \to S_A$ given by $\lambda(z)(f) = f(z)$. When $\bar{\Omega} = S_A$, i.e., λ is bijective, and so homeomorphic, $\bar{\Omega}$ is called an A-convex set or holomorphically convex set with respect to A.

We denote by Γ_A the *Silov boundary* for A. A point $z \in \bar{\Omega}$ is called a *peak point* for A if there is an $f \in A$ such that $f(z)=1$ and $|f|<1$ on $\bar{\Omega} \backslash \{z\}$, and a *local peak point* for A if there is an $f \in A$ such that $f(z)=1$ and $|f|<1$ on $(U \cap \bar{\Omega}) \backslash \{z\}$ for some neighborhood U of z in R. Let M_A be the set of peak points for A. Since $\bar{\Omega}$ is metrizable, by Bishop's Theorem [3], M_A is the *minimal boundary* for A if A is separating (see Gamelin [7]). When Ω has C^2-boundary, a boundary point z of Ω is said to be *strictly pseudoconvex*, if the restriction of the Levi form of the defining function of $\partial\Omega$ to the complex tangent space at z is positive definite. Let $SP(\partial\Omega)$ be the set of strictly pseudoconvex boundary points of Ω.

§2. Steinness of A-convex set

E. Bishop [4] and H. Rossi [13] proved that the spectrum $S_{O(\Omega)}$ of the Fréchet algebra $O(\Omega)$ with the topology of uniform convergence on compact subsets of a Riemann domain Ω over $\mathbf{C}^n$ (abbreviated to c.o. topology) can be given a structure of a Stein Riemann domain over $\mathbf{C}^n$ and is precisely the envelope of holomorphy of Ω.

We state this result as follows (see [2], [8] I–G).

THEOREM 1 (Bishop, Rossi). *Let Ω be a connected Riemann domain over $\mathbf{C}^n$ with a projection $p=(p_1,\dots, p_n)$ and $S_{O(\Omega)}$ be the spectrum of the Fréchet algebra $O(\Omega)$ with c.o. topology. Then $S_{O(\Omega)}$ can be endowed with a structure of a Stein Riemann domain over $\mathbf{C}^n$ with a projection $\hat{p}=(\hat{p}_1,\dots, \hat{p}_n)$ satisfying the following conditions* (*a*), (*b*), (*c*), (*d*):

(*a*) *The complex structure of $S_{O(\Omega)}$ is compatible with the natural topology, i.e., the weak $*$ topology of $S_{O(\Omega)}$.*

(*b*) *The point evaluation mapping $\lambda: \Omega \to S_{O(\Omega)}$ is locally biholomorphic and it is into biholomorphic when $O(\Omega)$ is separating.*

(*c*) *The Gelfand transformation gives a ring isomorphism from $O(\Omega)$ to $O(S_{O(\Omega)})$; that is, $O(S_{O(\Omega)}) = O(\Omega)^\wedge$.*

(*d*) *If Ω' is a Riemann domain over $\mathbf{C}^n$ and a holomorphic mapping ψ: $\Omega \to \Omega'$ gives a canonical ring isomorphism $\psi^*: O(\Omega') \to O(\Omega)$ defined by $\psi^*(f) = f \circ \psi$, then there exists a holomorphic mapping $h: \Omega' \to S_{O(\Omega)}$ such that $\lambda = h \circ \psi$.*

F.T. Birtel [2] used this theorem to show the following proposition (see [2] p. 44).

PROPOSITION 2 (Birtel). *Let K be a compact set in $\mathbf{C}^n$ with $\mathring{K} \neq \emptyset$ and A be a uniform subalgebra of $A(K)$ which contains the coordinate functions $z_1, \dots, z_n$. If K is A-convex, then $\mathring{K}$ is a Stein domain.*

We extend this proposition as follows.

THEOREM 3. *Let R be a Riemann domain over $\mathbf{C}^n$ with a projection $p=(p_1,\dots, p_n)$ and Ω be a relatively compact subdomain of R and suppose $(\bar{\Omega})^\circ = \Omega$. Let A be a uniform subalgebra of $A(\bar{\Omega})$ which contains $p_1|_{\bar{\Omega}},\dots, p_n|_{\bar{\Omega}}$. If $\bar{\Omega}$ is A-convex, then Ω is a Stein Riemann domain over $\mathbf{C}^n$.*

PROOF. It suffices to prove that any connected component of Ω is a Stein Riemann domain over $\mathbf{C}^n$, so we shall suppose Ω is connected.

If Ω is not a Stein Riemann domain, then by Theorem 1 the point evaluation mapping λ is not surjective. Hence there is a point χ^0 of the boundary $\partial(\lambda(\Omega))$ of $\lambda(\Omega)$ in $S_{O(\Omega)}$. Let $\{\chi^n\}_{n\in N} \subset \lambda(\Omega)$ be a sequence which converges to χ^0. By compactness of $\bar{\Omega}$, the sequence $\{z^n = \lambda^{-1}(\chi^n)\}_{n\in N}$ has a cluster point $z^0 \in \partial\Omega$. Then $\hat{p}(\chi^0) = p(z^0)$ in $\mathbf{C}^n$. Since $\hat{p}$ and p are locally biholomorphic mappings, there are neighborhoods V of z^0 in R and U of χ^0 in $S_{O(\Omega)}$ respectively such that the mapping $\hat{p}^{-1} \circ p$: $V \to U$ is biholomorphic.

Now we define a mapping $\tilde{\lambda}$: $\Omega \cup V \to \lambda(\Omega) \cup U$ by

$$\tilde{\lambda} = \begin{cases} \lambda & \text{on } \Omega \\ \hat{p}^{-1} \circ p & \text{on } V \end{cases}$$

Since $p = \hat{p} \circ \lambda$ on Ω, $\tilde{\lambda}$ is well defined and a biholomorphic mapping. We set $\tilde{f} = \hat{f} \circ \tilde{\lambda}$ for $f \in A(\bar{\Omega})$. Then $\tilde{f}$ is a unique holomorphic continuation of f to $\Omega \cup V$ by Theorem 1 and clearly $\tilde{f}$ extends continuously to $\bar{\Omega}$ so that $\tilde{f} = f$ on $\bar{\Omega}$, i.e., $\tilde{f} \in A(\bar{\Omega}) \cap O(\Omega \cup V)$.

Since $(\bar{\Omega})^\circ = \Omega$, there is a point $w \in V_0 \backslash \bar{\Omega}$ for any neighborhood V_0 of z^0 with $V_0 \subset V$. Consider the mapping ϕ: $A \to \mathbf{C}$ defined by $\phi(f) = \tilde{f}(w)$. Then ϕ is a multiplicative linear functional on A. By Theorem 1, we have $f(\Omega) = \hat{f}(S_{O(\Omega)})$ for all $f \in A(\bar{\Omega})$, since if $\hat{f}(S_{O(\Omega)}) \backslash f(\Omega)$ is not empty the function $1/(f-c)$ for some $c \in \hat{f}(S_{O(\Omega)}) \backslash f(\Omega)$ is holomorphic on Ω but $(1/(f-c))^\wedge = 1/(\hat{f}-c)$ has poles on $S_{O(\Omega)}$. It follows that $|\tilde{f}(w)| \leq \|f\|_{\bar{\Omega}}$, so that $\phi \in S_A$. Hence by the hypothesis that $\bar{\Omega}$ is A-convex, there is a $z \in \bar{\Omega}$ such that $\phi = \lambda(z)$. Thus, if we show that there is a function $g \in A$ such that $\tilde{g}$ separates the points z and w for some V_0 and $w \in V_0 \backslash \bar{\Omega}$, then we obtain a contradiction.

Now Ω is a relatively compact subdomain of an unramified Riemann domain R, so $\bar{\Omega}$ is finitely sheeted. Let D_ε be a ball in $\mathbf{C}^n$ with center $p(z^0)$ and radius $\varepsilon > 0$. Since $\bar{\Omega}$ is finitely sheeted, we can take ε_0 so small that each component of $p^{-1}(D_{\varepsilon_0}) \cap \bar{\Omega}$ contains a unique point of $p^{-1}(p(z^0)) \cap \bar{\Omega}$. We denote the points of $(p^{-1}(p(z^0)) \cap \bar{\Omega}) \backslash \{z^0\}$ by $z^1, \ldots, z^m$. Since A separates points of $\bar{\Omega}$, there are functions $g_j \in A$, $j=1, \ldots, m$, such that $g_j(z^0) \neq g_j(z^j)$ for $j=1, \ldots, m$. By continuity of $\tilde{g}_j$, there are ε_j, $j=1, \ldots, m$ and connected neighborhoods $V_j(z^0)$ of z^0 in R and $U_j(z^j)$ of z^j in R for $j=1, \ldots, m$ such that $V_j(z^0) \subset V$, $p(V_j(z^0)) = p(U_j(z^j)) = D_{\varepsilon_j}$ and $\tilde{g}_j(\tau) \neq \tilde{g}_j(\zeta)$ for any $\tau \in V_j(z^0)$ and any $\zeta \in U_j(z^j) \cap \bar{\Omega}$. Let $\varepsilon' = \mathrm{Min}\,\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m\}$ and let V_0 be the connected neighborhood of z^0 in R such that $p(V_0) = D_{\varepsilon'}$. Then clearly $V_0 \subset V$, and any point w of V_0 is separated from each point of $(p^{-1}(p(V_0)) \cap \bar{\Omega}) \backslash V_0$ by some $\tilde{g}_j$ $(j=1, \ldots, m)$ and each point of $\bar{\Omega} \backslash p^{-1}(p(w))$ by some p_i $(i=1, \ldots, n)$.

Hence for any point $w \in V_0 \backslash \bar{\Omega}$ there is a function $g \in A$ such that $\tilde{g}$ separates the points $z \in \bar{\Omega}$ and w. Q.E.D.

§3. Silov boundary of A-convex set

R. F. Basener [1] showed that a peak point for $A(\bar{\Omega})$ is a limit of strictly pseudoconvex boundary points. We can state this result as follows.

PROPOSITION 4 (Basener). *Let Ω be a relatively compact subdomain with C^2-boundary of a Riemann domain R over $\mathbf{C}^n$. Let A be a subset of $A(\bar{\Omega})$. Then, we have $M_A \subset \overline{SP(\partial\Omega)}$.*

We shall need the following lemma.

LEMMA 5. *Let Ω be as in Proposition 4. Let A be a uniform subalgebra of $A(\bar{\Omega})$ which contains $O(\bar{\Omega})$. Then any strictly pseudoconvex boundary point of Ω is a local peak point for A.*

PROOF. Let $z \in SP(\partial\Omega)$. Since a projection $p: R \to \mathbf{C}^n$ is locally biholomorphic, (U, p) is a coordinate neighborhood of z for some small neighborhood U of z in R. Then by H. Rossi [12] (the proof of Theorem 5.6), there are a suitable regular affine transformation L and a polynomial F such that $(F \circ L \circ p)(z) = 0$ and $\mathrm{Re} F < 0$ on $(L \circ p)(U \cap \bar{\Omega}) \backslash (L \circ p)(z)$. If we set $f = \exp(F \circ L \circ p)$, then by the above, f is a holomorphic function on R such that $f(z) = 1$ and $|f| < 1$ on $(U \cap \bar{\Omega}) \backslash \{z\}$. Hence z is a local peak point for A. Q.E.D.

Now we state and prove our main theorem.

THEOREM 6. *Let Ω be a relatively compact subdomain with C^2-boundary of a Riemann domain R over $\mathbf{C}^n$. Let A be a uniform subalgebra of $A(\bar{\Omega})$ which contains $O(\bar{\Omega})$. If $\bar{\Omega}$ is A-convex, then $\Gamma_{A'} = \overline{SP(\partial\Omega)}$ for any uniform subalgebra A' such that $A \subset A' \subset A(\bar{\Omega})$, especially $\Gamma_{A(\bar{\Omega})} = \overline{SP(\partial\Omega)}$.*

PROOF. $A \subset A' \subset A(\bar{\Omega})$ implies $M_A \subset M_{A'} \subset M_{A(\bar{\Omega})}$. Since $\bar{\Omega}$ is metrizable, by Bishop's Theorem [3], we have $\Gamma_{A'}=\overline{M}_{A'}$. Since $M_{A'}\subset\overline{SP(\partial\Omega)}$ by Proposition 4, it suffices to prove that $SP(\partial\Omega)\subset M_A$.

For $z\in SP(\partial\Omega)$, by Lemma 5 there is an $f\in A$ such that $f(z)=1$ and $|f|<1$ on $(U\cap\bar{\Omega})\backslash\{z\}$ for some neighborhood U of z in R. By the assumption that $\bar{\Omega}$ is A-convex, the point evaluation mapping λ is a homeomorphism. Since $f=\hat{f}\circ\lambda$ on $\bar{\Omega}$, we have $\hat{f}(\lambda(z))=1$ and $|\hat{f}|<1$ on $\lambda(U\cap\bar{\Omega})\backslash\{\lambda(z)\}$, i.e., $\lambda(z)$ is a local peak point for $\hat{A}$ on S_A. Then by the local peak point theorem of Rossi [11] (see Gamelin [7]), $\lambda(z)$ is a peak point for $\hat{A}$. It follows that z is a peak point for A. Hence $z\in M_A$. Q.E.D.

§4. Remarks

(1) Let Ω be a relatively compact subdomain of a Riemann domain R over $\mathbf{C}^n$. Let $H(\bar{\Omega})$ be the topological closure of $O(\bar{\Omega})$ in $C(\bar{\Omega})$. Then $H(\bar{\Omega})$ is a uniform subalgebra of $A(\bar{\Omega})$. $\bar{\Omega}$ is called an S_δ-set when $\bar{\Omega}=\bigcap_{n=1}^{\infty}\Omega_n$ with Stein domains Ω_n in R. H. Rossi [12] showed the following results.

PROPOSITION 7 (Rossi). *If Ω has C^2-boundary and $\bar{\Omega}$ is an S_δ-set, then $\Gamma_{H(\bar{\Omega})}=\overline{SP(\partial\Omega)}$.*

PROPOSITION 8 (Rossi). *If $\bar{\Omega}$ is an S_δ-set, then $\bar{\Omega}$ is $H(\bar{\Omega})$-convex.*

From Proposition 8, Proposition 7 is an immediate consequence of Theorem 6. On the other hand, J. E. Björk [5] presented an example of a bounded pseudoconvex domain Ω in $\mathbf{C}^n$ such that $\bar{\Omega}$ is $H(\bar{\Omega})$-convex but not an S_δ-set. Hence our Theorem 6 can be a proper extension of Proposition 7.

(2) In case Ω has C^∞-boundary, applying Kohn's result [9] we can extend the result of M. Hakim-N. Sibony [8] as follows.

PROPOSITION 9. *Let Ω be a relatively compact pseudoconvex subdomain with C^∞-boundary of a Riemann domain R over $\mathbf{C}^n$ and suppose $A(\bar{\Omega})$ is separating. Then we have $\Gamma_{A(\bar{\Omega})}=\overline{SP(\partial\Omega)}$.*

References

[1] R. F. Basener, *Peak points, barriers and pseudoconvex boundary points,* Proc. Amer. Math. Soc. **65** (1977), 89–92.
[2] F. T. Birtel, *Algebras of holomorphic functions,* Lecture notes, Tulane University, 1972.
[3] E. Bishop, *A minimal boundary for function algebras,* Pacific. J. Math. **9** (1959), 624–642.
[4] E. Bishop, *Holomorphic completion, analytic continuation, and the interpolation of semi norms,* Ann. of Math. **78** (1963), 468–500.
[5] J. E. Björk, *Holomorphic convexity and analytic structures in Banach algebras,* Ark. Math. **9** (1971), 39–54.
[6] H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and*

pseudoconvex domain. Characterization of Silov boundaries, Trans. Amer. Math. Soc. **91** (1959), 246–276.

[7] T. W. Gamelin, *Uniform algebras,* Prentice-Hall, Englewood Cliffs, N. J., 1969.

[8] R. Gunning and H. Rossi, *Analytic functions of several complex variables,* Prentice-Hall, Englewood Cliffs, N. J., 1965.

[9] M. Hakim and N. Sibony, *Frontière de Silov et spectre de $A(\bar{D})$ pour des domaines faiblement pseudoconvexes,* C. R. Acad. Sci. Paris **281**, Série A (1975), 959–962.

[10] J. J. Kohn, *Global regularity for $\bar{\partial}$ on weakly pseudoconvex manifolds,* Trans. Amer. Math. Soc. **181** (1973), 273–292.

[11] H. Rossi, *The local maximum modulus principle,* Ann. of Math. **72** (1960), 1–11.

[12] H. Rossi, *Holomorphically convex sets in several complex variables,* Ann. of Math. **74** (1961), 470–493.

[13] H. Rossi, *On envelopes of holomorphy,* Comm. Pure. Appl. Math. **16** (1963), 9–17.

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