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Supplement to "Compact transformation groups on Z_2-cohomology spheres with orbit of codimension 1"

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§1. Introduction

In the main theorem of the previous paper [1], we have proved the following

(1.1) *Let (G, M) be a smooth action of a compact connected Lie group G on a connected closed smooth manifold M with orbit of codimension* 1. *If M is a Z_2-cohomology sphere, then (G, M) is (essentially) isomorphic to*

(a) *the linear action on the sphere S^n via a representation $G \to SO(n+1)$,*

(b) *the standard action on the Brieskorn manifold $W^{2m-1}(r)$ for odd $r \geqq 1$, given in* [1; Ex. 1.2], *or*

(c) *the action $(SO(4), M)$ with* $\dim M = 7$, *given in* [1; Ex. 1.3], *which exists for each relatively prime integers l_s and m_s $(s=1, 2)$ with*

$$l_s \equiv m_s \equiv 1 \bmod 4, \quad 0 < l_1 - m_1 \equiv 4 \bmod 8, \quad l_2 - m_2 \equiv 0 \bmod 8.$$

The purpose of this supplement is to prove the following (1.2) whose sufficiency is asserted in [1; Ex. 1.3]:

(1.2) *Among the actions $(SO(4), M)$ in* (c) *of* (1.1), *M is a homotopy sphere if and only if $(l_1, m_1, l_2, m_2) = (1, -3, 1, 1)$, and then $M = S^7$ and the action is linear.*

By virtue of (1.2), the following theorem is an immediate consequence of (1.1), because it is well-known that $W^{2m-1}(r)$ in (b) is a homotopy sphere if and only if both m and r are odd (cf. [2; Satz 1]).

THEOREM 1.3. *If M is a homotopy sphere in addition, then (G, M) in* (1.1) *is (essentially) isomorphic to a linear action in* (a) *or the action on $W^{2m-1}(r)$ in* (b) *for odd m and odd $r \geqq 1$.*

We prepare some lemmas on the cohomology of certain coset spaces of $S^3 \times S^3$ in § 2, and prove (1.2) in § 3.

§2. Preliminaries

Let $G=S^3\times S^3$ and consider its subgroups

$$(2.1)\quad \begin{aligned} D &= D^*(8) = \{(p, p);\ p\in D'\} \quad (D' = \{z, zj;\ z\in S^1(\subset C),\ z^4=1\}\subset S^3),\\ S &= S^1(l, m) = \{(z^l, z^m);\ z\in S^1\}, \quad U = U(l, m) = S\cup S(j, j), \end{aligned}$$

given in [1; § 9.7], where l and m are given integers such that

(2.2) l and m are relatively prime and $l\equiv m\equiv 1 \bmod 4$.

Then we have the following lemmas on the integral cohomology (its coefficient Z is omitted throughout this note) of G/D, G/S and G/U.

LEMMA 2.3. (i) *$S^3\times(S^3/D')\approx G/D$ by sending $(p, [q])$ to $[pq, q]$ $(p, q\in S^3)$.*

(ii) *$H^*(G/D)\cong H^*(S^3)\otimes H^*(S^3/D')$ and*

$$H^i(S^3/D')\cong Z \text{ if } i=0, 3, \quad \cong Z_2\oplus Z_2 \text{ if } i=2, \quad \cong 0 \text{ otherwise.}$$

PROOF. (i) The inverse is given by sending $[p, q]$ to $(pq^{-1}, [q])$.

(ii) The first half is a consequence of (i). The second half holds, since S^3/D' is orientable and $H^i(S^3/D')\cong H^i(D')$ for $i=1, 2$ (cf. [3; 12–7]). *q.e.d.*

LEMMA 2.4. (i) *$H^*(G/S)\cong H^*(S^2)\otimes H^*(S^3)$.*

(ii) *Let j be the involution of G/S given by $j([p, q])=[pj, qj]$. Then the induced automorphism j^* of $H^i(G/S)$ is -1 if $i=2$ or 5, and 1 otherwise.*

(iii) *G/U is the orbit space of the free involution j in (ii).*

(iv) *$H^i(G/U)\cong H^i(P_2(R)\times S^3)$ ($P_2(R)$ is the real projective plane).*

(v) *The projection θ: $G/S\to G/U$ induces the isomorphism θ^*: $H^3(G/U)\cong H^3(G/S)$.*

PROOF. (i) We see immediately (i) from the Gysin sequence of the circle bundle s: $G\to G/S$ for the projection s.

(ii) Put $T=S^1\times S^1(\supset S)$, and let j' and j'' be the free involutions of S^3 and S^3/S^1 given by $j'(p)=pj$ and $j''([p])=[pj]$, respectively. Then we have the commutative diagrams

$$\begin{tikzcd} H^2(G/S) \arrow[r, "j^*"] & H^2(G/S) \\ H^2(G/T) \arrow[u, "v^*"] \arrow[r, "(j''\times j'')^*"] & H^2(G/T), \arrow[u, "v^*"] \end{tikzcd} \qquad \begin{tikzcd} H^3(G/S) \arrow[r, "j^*"] \arrow[d, "s^*"] & H^3(G/S) \arrow[d, "s^*"] \\ H^3(G) \arrow[r, "(j'\times j')^*"] & H^3(G), \end{tikzcd}$$

where v is the projection. In these diagrams, we see that v^* is epimorphic and s^*

is monomorphic by the Gysin sequence of the circle bundles $v: G/S \to G/T$ and $s: G \to G/S$, respectively. Furthermore, $(j'' \times j'')^* = -1$ and $(j' \times j')^* = 1$ because j'' reverses the orientations and j' preserves them. Thus we see (ii).

(iii) The definition (2.1) shows (iii).

(iv), (v) By (iii) and [3; 12–2, Th. 2], there is a spectral sequence $\{E^r_{i,j}, d^r\}$ such that $E^2_{i,j} \cong H^i(Z_2; H^j(G/S))$ and E^∞ is the associated graded group of $H^*(G/U)$. By (i), (ii) and [3; 3–7], we have

$$(*) \quad E^2_{i,j} \cong \begin{cases} H^3(G/S) \cong Z & \text{if } i = 0 \text{ and } j = 3, \\ Z_2 & \text{if } i \text{ is odd} > 0 \text{ and } j = 2, 5, \text{ or } i \text{ is even} > 0 \text{ and } j = 0, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that $H^1(G/U) = 0$ and $H^2(G/U) \cong Z_2$. On the other hand, $H^5(G/U) \cong Z_2$ and $H^i(G/U) = 0$ $(i \geqq 6)$ because G/U is a non-orientable 5-manifold.

We now show that

$(**)$ the differential $d^3: E^3_{1,2}\ (= E^2_{1,2} \cong Z_2) \longrightarrow E^3_{4,0}\ (= E^2_{4,0} \cong Z_2)$ is isomorphic.

Assume the contrary. Then $E^4_{4,0} = E^3_{4,0}$, and $(*)$ implies that $H^4(G/U) \cong Z_2$ or $H^3(G/U) \cong Z \oplus Z_2$ according as $d^4: E^4_{0,3}\ (= E^2_{0,3}) \to E^4_{4,0}$ is trivial or non-trivial. Hence $H^4(G/U; Z_2) \cong Z_2 \oplus Z_2$ (since $H^5(G/U) \cong Z_2$) or $H^3(G/U; Z_2) \supset Z_2 \oplus Z_2$ by the universal coefficient theorem. This contradicts that $H^i(G/U; Z_2) \cong Z_2$ for $0 \leqq i \leqq 5$ ([1; Lemma 9.7.1 (i)]). Thus $(**)$ holds.

By $(*)$ and $(**)$, we see that $H^4(G/U) = E^4_{4,0} = 0$ and $H^3(G/U) = E^\infty_{0,3} = E^2_{0,3} \cong Z$. Thus (iv) holds. Furthermore (v) holds, because θ^* is the composition of $H^3(G/U) = E^2_{0,3} \cong H^3(G/S)$. *q.e.d.*

LEMMA 2.5. *Consider the commutative diagram*

$$\begin{array}{ccccc} S^3 & \xrightarrow{i_t} & G(=S^3 \times S^3) & \xleftarrow{\Delta} & S^3 \\ \| & & \downarrow d & & \downarrow d' \\ S^3 & \xrightarrow{i'_t} & G/D & \xleftarrow{\Delta'} & S^3/D', \end{array}$$

where i_t $(t=1, 2)$ is the inclusion into the t-th factor, Δ is the diagonal map, d and d' are the projections, $i'_t = d i_t$ and $\Delta'([p]) = [p, p]$. Further consider the projections $s: G \to G/S$ and $u: G/D \to G/U$. Then the homomorphisms induced from these maps on H^3 satisfy

$$(2.6) \qquad i_1^* s^*(\delta) = m^2 v, \quad i_2^* s^*(\delta) = -l^2 v, \quad \Delta^* s^*(\delta) = (m^2 - l^2) v,$$

$$(2.7) \qquad i_1'^* u^*(\delta') = m^2 v, \quad \Delta'^* u^*(\delta') = ((m^2 - l^2)/8) v',$$

for some generators $\delta \in H^3(G/S)$, $\delta' \in H^3(G/U)$, $v \in H^3(S^3)$ and $v' \in H^3(S^3/D')$ of the infinite cyclic groups.

Proof. Take the subgroup $L=S^3\times S^1(\supset S)$ of G. Then $S^3/Z_m\approx L/S$ by the map induced by i_1, because $i_1^{-1}(S)=Z_m$ by (2.1). Thus we have the fibering $S^3/Z_m \xrightarrow{i_1} G/S\to G/L$ with $G/L\approx S^3/S^1\approx S^2$, and its Wang exact sequence is in the commutative diagram

$$\begin{array}{ccccccc} H^3(G/S)(\cong Z) & \xrightarrow{i_1^*} & H^3(S^3/Z_m)(\cong Z) & \longrightarrow & H^2(S^3/Z_m)\ (\cong Z_m) & \longrightarrow & 0 \\ \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle g^*} & & & & \\ H^3(G) & \xrightarrow{i_1^*} & H^3(S^3)(\cong Z) & & & & \end{array}$$

($H^4(G/S)=0$ by Lemma 2.4 (i)), where $g\colon S^3\to S^3/Z_m$ is the projection of the m-fold covering. Therefore

$$i_1^*s^*(\delta) = g^*i_1^*(\delta) = m^2v \quad \text{for some generators } \delta\in H^3(G/S) \text{ and } v\in H^3(S^3).$$

By interchanging the factors of $G=S^3\times S^3$ in the above proof, we have

$$(*) \qquad i_2^*s^*(\delta) = \varepsilon l^2v\ (\varepsilon=\pm 1), \quad \text{and hence} \quad \Delta^*s^*(\delta) = (m^2+\varepsilon l^2)v.$$

Now put $n=|l-m|$. Then we can define a map $\Delta_0\colon S^3/Z_n\to G/S$ by $\Delta_0([p])=[\Delta(p)]$, because $\Delta^{-1}(S)=Z_n$ by (2.1). Therefore we have the commutative diagram

$$\begin{array}{ccc} H^3(G/S) & \xrightarrow{\Delta_0^*} & H^3(S^3/Z_n) \\ \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle h^*} \\ H^3(G) & \xrightarrow{\Delta^*} & H^3(S^3), \end{array}$$

where h is the projection of the n-fold covering. By this diagram, the last equality in $(*)$ implies that $m^2+\varepsilon l^2$ is a multiple of $n=|l-m|$. On the other hand, the assumption (2.2) implies that $l-m\equiv 0$ and $l^2+m^2\equiv 2 \bmod 4$. Therefore $\varepsilon=-1$, and (2.6) is proved.

Set $\delta'=\theta^{*-1}(\delta)\in H^3(G/U)$, where θ^* is isomorphic by Lemma 2.4 (v). Since $ui_1'=udi_1=\theta si_1$, the first equality in (2.7) follows from the one in (2.6). Since $u\Delta'd'=ud\Delta=\theta s\Delta$, the last equality in (2.6) implies $d'^*\Delta'^*u^*(\delta')=(m^2-l^2)v$. This implies the second equality in (2.7), because $d'\colon S^3\to S^3/D'$ is an 8-fold covering. *q.e.d.*

§3. Proof of (1.2)

Let l_s and m_s $(s=1, 2)$ be given integers such that

(3.1) $\quad l_s$ and m_s are relatively prime and $l_s\equiv m_s\equiv 1 \bmod 4$ $(s=1, 2)$,

and by using the subgroups in (2.1), set

$$(3.2)\quad \begin{aligned} &G = S^3 \times S^3, \quad K_s = U(l_s, m_s)(\supset K_s^\circ = S^1(l_s, m_s)) \qquad \text{for} \quad s = 1, 2, \\ &K_2' = \beta^{-1}K_2\beta \quad (\beta = (\beta', \beta'), \beta' = (1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})/2), \quad K = D^*(8). \end{aligned}$$

Then [1; Ex. 1.3, Prop. 9.4.2 (o), § 9.7, (3.2–6)] shows the following

(3.3) *The simply connected closed 7-manifold M in* (1.1) (c) *is given by*

$$(3.4)\qquad M = X_1 \cup X_2', \quad X_1 \cap X_2' = G/K,$$

where X_1 and X_2' are the mapping cones of the projections f_1: $G/K \to G/K_1$ and f_2': $G/K \to G/K_2'$, respectively.

Now we can prove (1.2) by the following

PROPOSITION 3.5. *Let $A=(a_{s,t})$ be the 2×2 matrix given by*

$$(3.6)\qquad a_{s,1} = (-1)^{s+1}m_s^2, \quad a_{s,2} = (-1)^{s+1}(m_s^2 - l_s^2)/8 \qquad (s = 1, 2).$$

Then the integral cohomology of M in (3.4) *satisfies the following* (i) *and* (ii):

(i) $H^i(M) \cong H^i(S^7)$ *if* $i \neq 3, 4$.

(ii) *If $i=3, 4$, then the rank of $H^i(M)$ is equal to $2-\operatorname{rank} A$. Furthermore, if $\det A \neq 0$, then $H^3(M)=0$ and $H^4(M)$ is a finite group of order* $|\det A| = |l_1^2m_2^2 - l_2^2m_1^2|/8$.

PROOF OF (1.2) BY PROPOSITION 3.5. If M is a homotopy sphere, then Proposition 3.5 shows that

$$(l_1m_2 - l_2m_1)(l_1m_2 + l_2m_1) = \pm 8.$$

Since $l_1m_2 \equiv l_2m_1 \equiv 1 \mod 4$ by (3.1), this implies that $l_1m_2 - l_2m_1 = \pm 4$ and $l_1m_2 + l_2m_1 = \pm 2$, and hence $(l_1m_2, l_2m_1) = (-3, 1)$ or $(1, -3)$. Therefore $(l_1, m_1, l_2, m_2) = (1, -3, 1, 1)$ by (3.1) and the assumption $l_1 > m_1$ in (1.1) (c), and (1.2) is proved. *q.e.d.*

REMARK 3.7. Proposition 3.5 implies also the fact in [1; p. 613] that M is a Z_2-cohomology sphere if and only if $(l_1 - m_1 + l_2 - m_2)/4$ is odd, because $l_1^2m_2^2 - l_2^2m_1^2 \equiv 2(l_1 - m_1 + l_2 - m_2) \mod 16$ by (3.1).

PROOF OF PROPOSITION 3.5. Since M is a simply connected 7-manifold, (i) holds for $i=0$, 1, 6, 7. Consider the Mayer-Vietoris exact sequence of (M, X_1, X_2') in (3.4). Then, by noticing that X_1 and X_2' are homotopy equivalent to G/K_1 and $G/K_2' \approx G/K_2$ respectively, and by using Lemmas 2.3 (ii) and 2.4 (iv), we see (i) for $i=5$, and hence for $i=2$; and furthermore we have the exact sequence

$$(3.7)\quad 0 \to H^3(M) \to H^3(G/K_1) \oplus H^3(G/K_2') \xrightarrow{f_1^* - f_2'^*} H^3(G/K) \to H^4(M) \to 0.$$

In this sequence, we see that $H^3(G/K_s) \cong H^3(G/K_2') \cong Z$ $(s=1, 2)$, $H^3(G/K) \cong Z \oplus Z$ and

$$(3.8) \qquad f_s(\delta_s) = m_s^2 v + ((m_s^2 - l_s^2)/8)v'$$

for some generators $\delta_s \in H^3(G/K_s)$ and $v, v' \in H^3(G/K)$,

by (3.2), Lemmas 2.4 (iv), 2.3 and (2.7), (f_s: $G/K \to G/K_s$ is the projection).

On the other hand, we have the commutative diagram

$$\begin{array}{ccccc} G/K_2 & \xleftarrow{f_2} & G/K & \approx & S^3 \times (S^3/D') \\ \downarrow{\scriptstyle c_\beta} & & \downarrow{\scriptstyle c_\beta} & & \downarrow{\scriptstyle c_{\beta'} \times c_{\beta'}} \\ G/K_2' & \xleftarrow{f_2'} & G/K & \approx & S^3 \times (S^3/D'), \end{array}$$

where $c_\beta([x]) = [\beta^{-1}x\beta]$ $(x \in G)$, $c_{\beta'}(p) = \beta'^{-1}p\beta'$, $c_{\beta'}([p]) = [\beta'^{-1}p\beta']$ $(p \in S^3)$, and the homeomorphism is the one given in Lemma 2.3 (i). It is easy to see that the two $c_{\beta'}$ preserve the orientations. Thus this diagram shows that $c_\beta^* = 1$: $H^3(G/K) \to H^3(G/K)$ and

$$(3.9) \qquad f_2'^*(\delta_2') = m_2^2 v + ((m_2^2 - l_2^2)/8)v'$$

for a generator $\delta_2' = c_\beta^{*-1}(\delta_2) \in H^3(G/K_2')(\cong Z)$,

by (3.8) for $s=2$.

Now (3.8) and (3.9) show that the homomorphism $f_1^* - f_2'^*$: $Z \oplus Z \to Z \oplus Z$ in (3.7) is represented by the matrix $A = (a_{s,t})$ given by (3.6). Thus we see (ii) by the exact sequence (3.7), and the proof of the proposition is completed. *q.e.d.*

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