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The Riemann-Hurwitz relation, parallel slit covering map, and continuation of an open Riemann surface of finite genus

Dedicated to Prof. M. Ohtsuka on his 60^{th} birthday

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Introduction

Riemann proved, in 1851, the famous mapping theorem which is now named after him. It is the fountainhead of the study of conformal mapping. As its generalization, Schottky [32] suggested in his thesis that every finitely connected plane domain is mapped conformally onto a parallel slit plane. The first complete proof of this fact was due to Cecioni [4] and Hilbert [12]. Hilbert also outlined a proof for the case of infinite connectivity. Courant and Koebe carried out Hilbert's plan and, in fact, they finally showed that an arbitrary planar (=schlichtartig) Riemann surface can be mapped conformally onto a parallel slit plane. The mapping function is furnished by a "Strömungsfunktion" which is derived from a dipole "Strömungspotential" (see [14], p. 454 and p. 484).

In 1950, Nehari [23] first succeeded in generalizing the above result to (the interior of) compact bordered Riemann surfaces. Later Kusunoki [18] proved the same theorem again as an application of his theory of Abelian integrals on open Riemann surfaces. Mori [21] and Mizumoto [20] dealt with the general case — surfaces of finite genus but with infinitely many ideal boundary components.

Every author mentioned above at first constructed a single-valued meromorphic function on the surface which gives rise to a parallel slit (covering) mapping. Such a function is immediately recognized as a natural generalization of a "Strömungsfunktion". For this reason, we shall refer to it as an *S-function.* The existence of a non-constant S-function on an arbitrary surface of finite genus is assured by *e.g.*, the Riemann-Roch theorem (on open surfaces). It is known that every non-constant S-function defines a finite-sheeted covering surface of the extended complex plane $\hat{C}$.

Mizumoto's work as well as Mori's concerning the general case left some important problems open, however. The geometric structure of the covering determined by an S-function has not been fully analyzed. For instance, they asserted nothing about the branch points. In the beginning of this paper we shall show that the total number of branch points is finite. (The finiteness of the number of sheets does not always imply the finiteness of the number of branch points!)

Another problem to be discussed in detail is, as Mizumoto himself noticed, how each ideal boundary component of the given surface is realized by means of an S-function. To deal with this problem, we first establish a formula (see Theorem 1) which yields a relationship among the genus of the surface, the number of sheets, the total order of (interior) branch points, and another quantity which represents how many of the ideal boundary components are, by the mapping in question, not univalently realized (see Theorem 6). Thus, the formula can be recognized as a generalization of the classical Riemann-Hurwitz relation to a (special) case of open surfaces, and seems to be interesting in its own right. It follows from this formula, via a differential-geometric approach similar to Riemann's, that ideal boundary components which are not univalently realized are finite in number (see Theorem 3). In particular, this fact gives an answer to Mizumoto's conjecture in [20] (see Theorem 5). (In this connection we shall refer to Francis [10] and Quine [27], which also study the branching structure of covering surfaces. For the detail see the end of section 7.

Next to these somewhat preliminary considerations comes our main theorem (Theorem 4): *An arbitrary open Riemann surface of finite genus is, by means of a (non-constant) S-function on it, conformally mapped onto a finite-sheeted vertical slit covering surface of the extended plane. All but finitely many exceptional ideal boundary components are realized as univalent vertical slits; the exceptional boundary components are also realized as vertical slits but they are not univalent. What is more, the total area of these slits vanishes.*

Note that the above theorem is a direct generalization of the classical result due to Koebe and Courant. Also note that the theorem is used to obtain a compact continuation of the given open surface, onto which the preassigned S-function extends holomorphically. [This result will be proved elsewhere (see [33]). It reveals a notable, so far unknown property of S-functions: *Every S-function is an algebraic function.*] The continuation above is of measure zero (hence "dense" or "inessential") and is of the same genus. The ideal boundary of the original surface is realized, on the new compact covering surface, as a set of vertical slit with a vanishing total area. The relationship between this result and Open Question 3 in [31] will be also discussed. See section 25.

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I. Preliminaries

1. Let R be an open Riemann surface of finite genus g and ∂R the Kerékjártó-Stoïlow ideal boundary of R. Denote by $\{R_n\}_{n=1}^{\infty}$ a regular exhaustion of R. We may assume, without loss of generality, that R_1 is of genus g. Let ∂R_n be the border of $\bar{R}_n$, the closure of R_n, and $\partial R_n = \sum_{i=1}^{h_n} \beta_n^{(i)}$ the decomposition of ∂R_n into contours. We are particularly interested in the case where $h_n \to \infty$ as $n \to \infty$.

In the present study we shall be concerned with a single-valued non-constant meromorphic function f on R which satisfies any one of the following three conditions:

(A) $\mathrm{Re}(df)$ is a *real distinguished harmonic differential* in the sense of Ahlfors ([2], [30]); in other words, $\mathrm{Re}(df)$ has Γ_{hm}-behavior in Yoshida's sense ([36]).

(K) df is a *canonical semiexact differential* in the sense of Kusunoki ([18]).

(S) $\mathrm{Re} f$ is a *$(Q)L_1$-principal function* on R in the sense of Sario ([2], [29], and [31]), Q being the canonical partition of ∂R.

That these conditions are equivalent to each other can be found in, for example, Rodin-Sario [29], Mori [22] and Yoshida [36]. We could therefore start with any one of conditions (A), (K) and (S). However, it seems more or less convenient for us to use (A).

We generally use the notation of Ahlfors (see [2], Chap. V). *The only exception is that we consider mainly real differentials instead of complex ones.* Thus $\Gamma = \Gamma(R)$ stands for the real Hilbert space of square integrable *real* differentials on R. Similarly $\Gamma_h = \Gamma_h(R)$ stands for the class of square integrable *real* harmonic differentials on R, and $\Gamma_{e0}^1 = \Gamma_{e0}^1(R)$ is the space of exact differentials dh on R such that h is a real C^2-function with compact support. The closure of Γ_{e0}^1 in Γ is denoted by Γ_{e0}. The intersection of Γ_{e0} with the class Γ^1 of all smooth differentials in Γ does not coincide with Γ_{e0}^1: $\Gamma_{e0} \cap \Gamma^1 \supsetneqq \Gamma_{e0}^1$. Denote by Γ_c^1 (resp. Γ_e^1) the subspace of closed (resp. exact) C^1-differentials, and let Γ_c (resp. Γ_e) be the closure of Γ_c^1 (resp. Γ_e^1).

Among many important subclasses of Γ_h we shall later need only the following three:

Γ_{hm}: the space of harmonic measures,

Γ_{he}: the space of harmonic exact differentials,

Γ_{hse}: the space of harmonic semiexact differentials.

It is well known that $\Gamma_{hm} \subset \Gamma_{he} \subset \Gamma_{hse}$ and that $\Gamma_h = \Gamma_{hm} \oplus \Gamma_{hse}^*$. Here $\oplus$ means the direct sum and the asterisk stands for the conjugation.

The following decompositions are of particular importance:

de Rham's decomposition

$$\Gamma = \Gamma_h \oplus \Gamma_{e0} \oplus \Gamma_{e0}^*,$$

Dirichlet's principle

$$\Gamma_c = \Gamma_h \oplus \Gamma_{e0}, \quad \Gamma_c^1 = \Gamma_h \oplus (\Gamma_{e0} \cap \Gamma^1), \quad \Gamma_e = \Gamma_{he} \oplus \Gamma_{e0}.$$

A harmonic differential ω on R whose only singularities are harmonic poles is called *distinguished* if it has the representation

$$\omega = \omega_{hm} + \omega'_{e0}, \; \omega^* = \omega_{hse} + \omega''_{e0}$$

outside a compact subset of R, where $\omega_{hm} \in \Gamma_{hm}$, $\omega_{hse} \in \Gamma_{hse}$ and ω'_{e0}, $\omega''_{e0} \in \Gamma_{e0} \cap \Gamma^1$ (see [2], p. 313). Note that the representation for ω^* trivially holds whenever ω^* is semiexact.

DEFINITION. A single-valued meromorphic function f on R is called an *S-function* if $\mathrm{Re}\,(df)$ is distinguished.

The order (*i.e.*, the number of poles counted with their multiplicities) of an S-function f is necessarily finite, which we denote by $\mu = \mu(f)$. We may assume that these μ poles of f are entirely contained in R_1. If this is the case, the above representation holds on $R \setminus \bar{R}_1$ (cf. [36]). The Riemann-Roch theorem (on open Riemann surfaces) tells us that there exists certainly a non-constant S-function, provided we let μ large enough (see section 28).

It turns out that an S-function f on a planar surface with $\mu(f) = 1$ is precisely a *Strömungsfunktion* of Hilbert, Courant and Koebe. This is the reason why we call it an S-function. See section 29 and [33].

2. We summarize here some important properties of S-functions on R, which we shall need later on. Suppose that a non-constant S-function f is fixed once and for all, unless otherwise stated.

PROPOSITION 1 ([18], [21]). *The covering $f: R \to \hat{C}$ is at most μ-sheeted. More precisely, over any point $w \in \hat{C}$, at most μ points of R lie.*

Let $\nu(w)$ denote the number of w-points of f. Here a w-point is supposed to be counted as many times as its multiplicity. The above proposition asserts $\nu(w) \leqq \mu$ for all $w \in \hat{C}$. What is more, we have

PROPOSITION 2 ([19], [21]). *The two-dimensional Lebesgue measure of the set $\{w \in C \mid \nu(w) < \mu\}$ is zero.*

For each $n = 1, 2, \ldots$ we can construct a meromorphic semiexact differential df_n on R_n which has the same singularities as df and has the corresponding representation

$$\mathrm{Re}\,(df_n) = \omega_{hm}^{(n)} + \omega_{e0}^{(n)}, \quad \omega_{hm}^{(n)} \in \Gamma_{hm}(R_n), \quad \omega_{e0}^{(n)} \in \Gamma_{e0}(R_n) \cap \Gamma^1(R_n)$$

on a neighborhood of ∂R_n (see [2]). In general, f_n is not single-valued on R_n. But it is possible to find a single-valued branch of f_n on $R_n \setminus \bar{R}_1$, since $R_n \setminus \bar{R}_1$ is of planar character and df_n is semiexact. Hence it makes sense to speak of the boundary values of f_n modulo additive constants — the image $f_n(\beta_n^{(i)})$ is determined modulo euclidean translations. The following proposition is easily shown.

PROPOSITION 3. $\mathrm{Re} f_n$ *is constant on each contour* $\beta_n^{(i)}$ *of* ∂R_n, *while* $\mathrm{Im} f_n$ *is not constant on* $\beta_n^{(i)}$ $(1 \leqq i \leqq h_n)$. *In other words, (every branch of)* f_n *maps* $\beta_n^{(i)}$ *onto a vertical segment of positive length.*

Another property of df_n which we shall need is:

PROPOSITION 4 ([18], [21]). *If we appropriately normalize the periods of* df_n *along non-dividing cycles, then*

(i) df_n *converges to* df *locally uniformly on* R, *and*

(ii) f_n *(with appropriate additive constant) converges to* f *locally uniformly on* $R \setminus \bar{R}_1$.

In the sequel we shall always assume that df_n and f_n are normalized as in this proposition.

II. An extension of the Riemann-Hurwitz relation

3. We attempt to generalize the classical Riemann-Hurwitz relation to our case $f: R \to \hat{C}$, the covering which is determined by the (non-constant) S-function f. Since the interior branch points of the covering correspond to the zeros (and poles) of df and df is locally uniformly approximated by df_n (see Proposition 4), we shall first study the zeros of df_n.

Let V_n^0 be the number of zeros *in* R_n of df_n counted with multiplicities. As is well known, df_n is holomorphic on ∂R_n, and it is customary to count the zeros of df_n on ∂R_n as half as the actual number. See Nevanlinna [24]. For our aim this convention is still useful and furthermore, due to Proposition 3, it turns out to be more convenient to consider the quantity

$$W_n = \sum_{i=1}^{h_n} \left\{ \frac{1}{2} (\text{actual number of zeros of } df_n \text{ on } \beta_n^{(i)}) - 1 \right\}.$$

Indeed, on each contour $\beta_n^{(i)}$, there are at least two points at which df_n vanishes (cf. Proposition 3). These two zeros on the contour should be accepted as indispensable ones, and in W_n these zeros are not counted. We can easily see that W_n is a non-negative integer. When $W_n > 0$, df_n has more zero(s) than the minimum number. In such a case, we shall say that df_n has W_n *surplus* zeros on the border ∂R_n.

Suppose f has poles of order μ_i at p_i, $i=1, 2,\dots, r$, where $\mu=\mu(f)=\mu_1+\mu_2+\cdots+\mu_r$. Since f_n has the same singularities as f, the number of poles of df_n is exactly $\sum_{i=1}^{r}(\mu_i+1)=\mu+r$.

We shall first recall the following well known facts:

LEMMA 1. *The degree of a canonical divisor on a closed Riemann surface is equal to the Euler characteristic of the surface.*

LEMMA 1′. *Let Ω be the interior of a compact bordered Riemann surface $\bar{\Omega}$ and ω an analytic differential on Ω whose real part is distinguished. The divisor of ω on $\bar{\Omega}$, denoted by $(\omega)_{\bar{\Omega}}$, shall be the formal symbol $p_1^{m_1}p_2^{m_2}\cdots p_r^{m_r}/q_1^{n_1}q_2^{n_2}\cdots q_s^{n_s}$, where $p_j\in\bar{\Omega}$ (resp. $q_k\in\bar{\Omega}$) is a zero (resp. pole) of ω of order m_j (resp. n_k). (Note that m_j or n_k may be a half-integer if p_j or q_k lies on the border $\partial\Omega$.) Set $\deg(\omega)_{\bar{\Omega}}=\sum_{j=1}^{r}m_j-\sum_{k=1}^{s}n_k$ as in the classical case, and let $\chi(\bar{\Omega})$ denote the Euler characteristic of $\bar{\Omega}$. Then*

$$\deg(\omega)_{\bar{\Omega}} = \chi(\bar{\Omega}).$$

The proof of Lemma 1 can be found in any standard textbook on Riemann surfaces, *e.g.*, [24] and [34]. Lemma 1′ can be easily proved if we extend ω meromorphically across $\partial\Omega$ onto the Schottky double of Ω and apply Lemma 1. For the details, see Nevanlinna [24], p. 133.

From Lemma 1′ and the definition of W_n immediately follows the next lemma.

LEMMA 1″. For the Abelian differential df_n on R_n

$$2g-2+(\mu+r) = V_n^0 + W_n.$$

4. Though a very natural generalization of the classical result (Lemma 1), the formula in Lemma 1′ is meaningless for certain open surfaces. The next proposition (see also Proposition 5′ below) is, on the contrary, valid for an *arbitrary* Riemann surface.

PROPOSITION 5. *The total number V^0 of the zeros of df on R does not exceed $2g-2+(\mu+r)$.*

PROOF. Set $\mu_0=2g-2+(\mu+r)$ and suppose, contrary to the assertion, that V^0 were greater than μ_0. Take arbitrary μ' $(>\mu_0)$ of these zeros and choose m so large that R_m contains all of these μ' zeros. Then by a theorem of Hurwitz, there exists an integer n_1 $(\geqq m)$ such that df_n $(n>n_1)$ would have the same number μ' of zeros in R_m. This contradicts, however, the fact that the zeros of df_n in R_n is exactly $V_n^0=2g-2+(\mu+r)-W_n\leqq\mu_0$. q.e.d.

REMARK. As a matter of fact, we have proved a stronger inequality: $V^0 \leqq V_n^0$ *for every sufficiently large n.*

Proposition 5 generalizes the classical Lemma 1 as follows:

PROPOSITION 5′. *If ω is an Abelian differential on R whose real part is distinguished, then* $\deg(\omega) \leqq 2g-2$.

We cannot generally assert the equality in the above proposition; see Proposition 15 in section 21. Roughly speaking, a strict inequality is brought about by ghost branch points on the ideal boundary. We shall later analyze these ghost branch points in greater detail. See Theorems 1 and 6.

5. In order to continue the study of the relationship of the number of zeros of df with the genus g of R, we prepare several lemmas.

The next lemma is known as Riemann's *Umlaufsatz* or the *theorem of turning tangents.*

LEMMA 2. *If C is a positively oriented smooth Jordan curve in the plane, then*

$$\frac{1}{2\pi}\int_C d\arg dz = 1.$$

For the proof see Riemann [28], pp. 128–129 (Werke, pp. 113–114) or Osgood [25], pp. 369–372. Cf. also [5] and [13]. (Riemann himself used Lemma 2 to prove the classical Riemann-Hurwitz relation.)

Now let Ω be a planar Riemann surface bounded by N analytic Jordan curves $C_0, C_1, \ldots, C_{N-1}$ $(1 \leqq N < \infty)$, which are positively oriented with respect to Ω; namely, $\partial\Omega = C_0 + C_1 + \cdots + C_{N-1}$. Let ψ be a holomorphic function on the bordered surface $\bar{\Omega} = \Omega \cup \partial\Omega$. Denote by $Z(d\psi, \bar{\Omega})$ the number of zeros on $\bar{\Omega}$ of $d\psi$ counted with multiplicities. As usual, we count the zeros on the boundary as half as the actual number. We shall also use the notation $Z(d\psi, \Omega)$ and $Z(d\psi, \partial\Omega)$ whose meaning will be self-explanatory. Obviously, $Z(d\psi, \bar{\Omega})$ is finite and $Z(d\psi, \bar{\Omega}) = Z(d\psi, \Omega) + Z(d\psi, \partial\Omega)$.

Let q_j $(0 \leqq j \leqq n, 0 \leqq n \leqq 2Z(d\psi, \partial\Omega))$ be the points on $\partial\Omega$ at which $d\psi = 0$. Suppose that $\partial\Omega \setminus \{q_1, q_2, \ldots, q_n\} = \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m \cup \alpha_{m+1}^* \cup \alpha_{m+2}^* \cup \cdots \cup \alpha_M^*$ is the decomposition of $\partial\Omega \setminus \{q_1, q_2, \ldots, q_n\}$ into connected components, where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are open arcs and $\alpha_{m+1}^*, \alpha_{m+2}^*, \ldots, \alpha_M^*$ are Jordan curves. To each α_i $(1 \leqq i \leqq m)$ we add its two endpoints q_{j_1} and q_{j_2} $(0 \leqq j_1, j_2 \leqq n, j_1 \neq j_2)$ to form a 1-simplex, which we denote by α_i^*. Of course, we orient these α_s' just as C_s'. We consider a (singular) 1-chain (in smooth category)

$$\alpha^* = \sum_{i=1}^{M} \alpha_i^*$$

and define

$$\kappa_\psi^*(\partial\Omega) = \frac{1}{2\pi}\int_{\alpha^*} d\arg d\psi = \frac{1}{2\pi}\sum_{i=1}^{M}\int_{\psi(\alpha_i^*)} d\arg dw.$$

LEMMA 3. *Let Ω, C_i $(0\leqq i\leqq N-1)$ and ψ be as above. Then*

$$Z(d\psi, \bar{\Omega}) = \kappa_\psi^*(\partial\Omega) + (N-2).$$

For the proof Ω may be assumed to be a plane domain, since the conclusion of the lemma is invariant under any conformal mapping of $\bar{\Omega}$. Then use Lemma 2 and modify the classical proof of the Riemann-Hurwitz relation (cf. [25] or [28]). Details are omitted.

REMARKS. (1) The image curve $\psi(\partial\Omega)$ has corners at $\psi(q_j)$, $1\leqq j\leqq n$. The integral

$$\kappa_\psi^*(\partial\Omega) = \frac{1}{2\pi}\int_{\alpha^*} d\arg d\psi$$

does not take account of the change of tangent at these corners. Thus it does not coincide with the (ordinary) rotation index (see Chern [5], p. 21 or Hopf [13], p. 61) of the piecewise smooth chain $\psi(\partial\Omega)$. Compare Nevanlinna [24], p. 123, too.

(2) It is also possible to have a similar formula for meromorphic functions on $\bar{\Omega}$. Cf. [25], p. 372.

The following lemma is a simple corollary of Lemma 3.

LEMMA 4. *Let Ω, C_i $(0\leqq i\leqq N-1)$ be as before and suppose φ is a holomorphic function on $\bar{\Omega}$ such that* (i) *$d\varphi\neq 0$ on C_0, and* (ii) *$\varphi(C_i)$ is a straight line segment* (*of positive length*) *for each $i=1, 2,\ldots, N-1$. Then*

$$Z(d\varphi, \bar{\Omega}) = \frac{1}{2\pi}\int_{C_0} d\arg d\varphi + (N-2).$$

6. We continue studying the zeros of df. To this end, take an integer k such that R_k contains all the zeros of df (see Proposition 5). Let n $(>k)$ be large enough so that $df_n\neq 0$ on ∂R_k. $R_n\setminus\bar{R}_k$ consists of h_k planar surfaces $R_{n,k}^{(i)}$, $i=1, 2,\ldots, h_k$. Let $N_{n,k}^{(i)}$ be the connectivity of $R_{n,k}^{(i)}$. It is easy to see that

$$\sum_{i=1}^{h_k}(N_{n,k}^{(i)} - 1) = h_n.$$

To each $R_{n,k}^{(i)}$ and the restriction $f_n\,|\,R_{n,k}^{(i)}$, $1\leqq i\leqq h_k$, we apply Lemma 4, and obtain

$$Z(df, R_{n,k}^{(i)}\cup\partial R_{n,k}^{(i)}) = -\frac{1}{2\pi}\int_{\beta_k^{(i)}} d\arg df_n + (N_{n,k}^{(i)} - 2),\ i = 1, 2,\ldots, h_k.$$

Summing up these h_k identities, we have

$$Z(df_n, \bar{R}_n \setminus R_k) = -\frac{1}{2\pi}\int_{\partial R_k} d\arg df_n + h_n - h_k.$$

Since, on the other hand,

$$Z(df_n, \bar{R}_n \setminus R_k) = Z(df_n, R_n \setminus \bar{R}_k) + W_n + h_n,$$

we have proved the following

LEMMA 5. *For every sufficiently large n and k $(n>k)$,*

$$\frac{1}{2\pi}\int_{\partial R_k} d\arg df_n = -Z(df_n, R_n \setminus \bar{R}_k) - W_n - h_k.$$

Next, we consider the single-valued meromorphic function

$$F_n := df_n/df$$

on $\bar{R}_n$, which is holomorphic at the poles of f. We shall compute $\frac{1}{2\pi}\int_{\partial R_k} d\arg F_n$ in two different ways. First, applying the argument principle to $F_n \mid R_k$, we have

$$\frac{1}{2\pi}\int_{\partial R_k} d\arg F_n = Z(df_n, R_k) - V^0.$$

(Observe that F_n has neither poles nor zeros on ∂R_k.) On the other hand, we have by Lemma 5

$$\frac{1}{2\pi}\int_{\partial R_k} d\arg F_n = -Z(df_n, R_n \setminus \bar{R}_k) - W_n - h_k - \frac{1}{2\pi}\int_{\partial R_k} d\arg df.$$

We have thus proved:

LEMMA 6. *For every sufficiently large n and k $(n>k)$*

$$V^0 - \left(\frac{1}{2\pi}\int_{\partial R_k} d\arg df + h_k\right) = V_n^0 + W_n.$$

Lemma 6, together with Lemma 1″, yields

COROLLARY 1.

$$W = \lim_{k\to\infty}\left(-\frac{1}{2\pi}\int_{\partial R_k} d\arg df - h_k\right)$$

exists and is a non-negative integer.

It can be easily seen that W is defined independently of the particular choice of exhaustion $\{R_k\}_{k=1}^{\infty}$.

COROLLARY 2.

$$W + V^0 = W_n + V_n^0$$

for every sufficiently large n.

7. Set $V = V^0 + (\mu - r)$. Then V is exactly the total order of the (interior) branch points of the covering $f: R \to \hat{C}$. It will be seen later that W is exactly equal to the total order of the boundary branch points. For the present, however, we shall content ourselves with the following weaker form of the Riemann-Hurwitz relation, which immediately follows from Lemma 1″ and Corollaries to Lemma 6.

THEOREM 1. *Let f be a non-constant S-function on the open Riemann surface R of genus g, μ the order of f, and V the total order of the (interior) branch points of the covering $f: R \to \hat{C}$. Then*

$$g = 1 - \mu + \frac{V}{2} + \frac{W}{2}$$

with

$$W = \lim_{k \to \infty} \left(- \frac{1}{2\pi} \int_{\partial R_k} d \arg df - h_k \right),$$

where $\{R_k\}_{k=1}^{\infty}$ is a regular exhaustion of R and h_k is the number of boundary contours of R_k.

REMARKS. (1) As for the classical Riemann-Hurwitz relation, see, for example, [3] and [25]. In [25] one can find the proof due to Riemann himself, which uses Lemma 2.

(2) We shall later see that W and V faithfully describe the branching feature of the covering f: $R \to \hat{C}$. That W can be interpreted as the total order of the boundary branch points will be shown in a forthcoming paper [33]. See the end of section 10 and also Theorem 6.

In connection with Theorem 1 we refer to some recent relevant works. In [10] Francis studies a combinatorial covering properties of (compact) bordered surfaces which is analogous to a classical result of Hurwitz. Quine [27] is concerned with a generalization of the classical Riemann-Hurwitz relation to compact bordered surfaces. Ezell and Marx [9] is a further generalization of [10]. As for other related works, see the references of these papers.

Their methods are, like Titus' discussion about Loewner's problem, more topological than ours, but they also investigate the branching structure in detail. It should be noted, however, that they always confine themselves to *compact bordered surfaces*. Also they do *not* consider any branch points on the boundary.

We are concerned here with an *arbitrary open surfaces* (of finite genus) and also *permit boundary branch points*. We have succeeded in dealing with these general cases because of the very favorable boundary behavior of S-functions. Compare Ahlfors' theory of covering surfaces [1], too.

III. Univalence of the S-function near the ideal boundary

We shall now study the valence of f near ∂R. Our aim here is to show that f is univalent near almost all (=except for finitely many) ideal boundary components of R. See Theorems 2 and 3 below.

8. Let C be an oriented analytic Jordan curve on R and ψ a single-valued holomorphic function on (a neighborhood of) C such that $d\psi$ has no zeros on C. Then we set

$$N(C, \psi) = -\frac{1}{2\pi}\int_C d\arg d\psi - 1.$$

(The integral $\frac{1}{2\pi}\int_C d\arg d\psi$ is known as the *rotation index*, the *rotation number*, the *tangent winding number* or the *circulation* of the image curve $\psi(C)$ (see Chern [5], Hopf [13], Whitney [35]; cf. Francis [10] and Quine [27], too). We extend by linearity the definition of N to an arbitrary cycle on which ψ is single-valued holomorphic and $d\psi\neq 0$. Namely, if $C_1,\dots, C_s$ are such curves as above and $C=m_1C_1+\cdots+m_sC_s$ with integers $m_1,\dots, m_s$, then we define

$$N(C, \psi) = \textstyle\sum_{i=1}^{s} m_i N(C_i, \psi).$$

The following lemma is an easy consequence of Lemma 4.

LEMMA 7. *Let n, k be two integers with $n>k\geqq 1$ and suppose that $df_n\neq 0$ on $\partial R_k=\beta_k^{(1)}+\beta_k^{(2)}+\cdots+\beta_k^{(h_k)}$. Then*

$$N(\beta_k^{(i)}, f_n) \geqq 0 \qquad \textit{for} \quad i = 1, 2,\dots, h_k.$$

Equality holds for some j $(1\leqq j\leqq h_k)$ if and only if df_n has no surplus zeros on $\partial R_{n,k}^{(j)}$ and $df_n\neq 0$ on $R_{n,k}^{(j)}$. (Recall that $R_{n,k}^{(j)}$ is the connected component of $R_n\setminus \bar{R}_k$, which has $-\beta_k^{(j)}$ as one of its boundary contours. Cf. section 6.)

9. Let γ be an ideal boundary component of R and n a fixed integer $\geqq 1$. Then there exists a unique boundary contour γ_n of R_n such that $-\gamma_n$, together with γ and possibly with some other ideal boundary components of R, bounds a component of $R\setminus \bar{R}_n$. According to Marden-Rodin's terminology (cf. [20], p. 4), γ is a derivation of γ_n. We shall call γ_n the *n-th antiderivation* of γ.

PROPOSITION 6. *For each ideal boundary component γ of R,*

$$N(\gamma_n, f) \leqq N(\gamma_m, f)$$

so long as n, m are large enough and $n \geqq m$.

PROOF. Let γ_n' be the 1-cycle on ∂R_n such that $\gamma_n + \gamma_n' - \gamma_m = \partial R_{n,m}^{(i)}$ for some i $(1 \leqq i \leqq h_m)$ and assume that $df \neq 0$ on $\partial R_n \cup \partial R_m$. Then, by Lemma 3 and the definition of $N(\gamma_n, f)$, we have

$$N(\gamma_m, f) - N(\gamma_n, f) = Z(df,\ R_{n,m}^{(i)} \cup \partial R_{n,m}^{(i)}) + N(\gamma_n', f).$$

Since $Z(df,\ R_{n,m}^{(i)} \cup \partial R_{n,m}^{(i)})$ is non-negative and $N(\gamma_n', f)$ is, by Lemma 7, also non-negative, we conclude that $N(\gamma_m, f) \geqq N(\gamma_n, f)$. q.e.d.

COROLLARY. *With each ideal boundary component γ of R we can associate a non-negative integer*

$$N(\gamma, f) := \lim_{n \to \infty} N(\gamma_n, f).$$

$N(\gamma, f)$ is determined by γ and f, and does not depend on the exhaustion $\{R_n\}_{n=1}^{\infty}$.

The following proposition is an immediate consequence of Corollary 1 to Lemma 6.

PROPOSITION 7. *For each k large enough,*

$$W = N(\partial R_k, f).$$

This proposition, together with Lemma 7, again yields that W is a non-negative integer (cf. section 6).

10. Now we can prove

PROPOSITION 8. *Only finitely many (and actually at most W) ideal boundary components γ satisfy*

$$(*) \qquad N(\gamma, f) \geqq 1.$$

PROOF. Suppose the contrary and take $W+1$ (distinct) ideal boundary components $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(W+1)}$ of R, each of which satisfies condition $(*)$. We can take a number k sufficiently large so that (i) the k-th antiderivation $\gamma_k^{(i)}$ of $\gamma^{(i)}$ are different from each other, (ii) $N(\gamma_k^{(i)}, f) = N(\gamma^{(i)}, f)$, $i = 1, 2, \dots, W+1$, and (iii) all the zeros of df are contained in R_k. It follows immediately that

$$N(\partial R_k, f) \geqq \sum_{i=1}^{W+1} N(\gamma_k^{(i)}, f) \geqq W + 1.$$

This contradicts Proposition 7, however. q.e.d.

The following lemma is easy to prove:

LEMMA 8. *Let R_k be as above and β a boundary contour of R_k with $N(\beta, f) \neq 0$. Then there is an ideal boundary component γ of R such that*

(i) $N(\gamma, f) \neq 0$, *and*

(ii) *γ is a derivation of β.*

Using this lemma we can prove the next refinement of Proposition 8 without difficulty.

PROPOSITION 9.

$$W = \sum_\gamma N(\gamma, f),$$

where the right hand side is the sum of $N(\gamma, f)$ over all ideal boundary components γ of R.

Note that if γ is an isolated point-like boundary component, $N(\gamma, f)$ is exactly the branch order of f at that point. In particular, Theorem 1 reduces to the classical Riemann-Hurwitz formula if the given R is what we call a Riemann surface of finite type.

11. As we have shown above, for almost all ideal boundary components γ of R, condition

$$(**) \qquad N(\gamma, f) = 0$$

is satisfied. Let γ be an ideal boundary component satisfying (**) and m an integer such that R_m contains all the zeros of df and $N(\gamma_m, f) = 0$.

Let $\Omega(\gamma_m)$ be the component of $R \setminus \bar{R}_m$ whose relative boundary is precisely $-\gamma_m$ and set

$$\Omega_n(\gamma_m) = \Omega(\gamma_m) \cap R_n, \quad n > m.$$

(If i is the index for which $\gamma_m = \beta_m^{(i)}$, then $\Omega_n(\gamma_m)$ is identical with $R_{n,m}^{(i)}$ in section 6.) If n is large enough, $df_n \neq 0$ on ∂R_m and $N(\gamma_m, f_n) = 0$, since df_n converges to df locally uniformly on $R \setminus \bar{R}_1$. Then by Lemma 7 df_n has no surplus zeros on $\partial\Omega_n(\gamma_m)$ and $df_n \neq 0$ on $\Omega_n(\gamma_m)$. It is now obvious that f_n maps each contour of $\partial\Omega_n(\gamma_m) \cap \partial R_n$ onto (the two edges of) a vertical slit in a one-to-one manner. Two edges of the slit are still distinguished.

Now, by identifying a pair of points on each contour at which f_n assumes the same value, we get a simply connected surface $\tilde{\Omega}_n(\gamma_m)$ with $\partial\tilde{\Omega}_n(\gamma_m) = -\gamma_m$ such that the function $f_n \mid \Omega_n(\gamma_m)$ can be extended holomorphically onto $\tilde{\Omega}_n(\gamma_m)$. If we denote the extended function by $\tilde{f}_n$, then $d\tilde{f}_n \neq 0$ on $\tilde{\Omega}_n(\gamma_m)$.

REMARKS. (1) If $N(\gamma, f) = 0$, then $N(\gamma_n, f_n) = 0$ for every sufficiently large n

(cf. Proposition 6). This again proves the last half of Lemma 7.

(2) If γ satisfies $N(\gamma, f)=0$, then for every n large enough γ_n has a neighborhood $U(\gamma_n)$ on which f_n is univalent. ($U(\gamma_n)$ is an annular domain on $\bar{R}_n$ with the boundary $\gamma_n - \delta_n$, δ_n being a simple closed curve on R_n homologous to γ_n.) To show this, choose m and n $(>m)$ so large that $N(\gamma_m, f_n)=0$ holds. Since the holomorphic function $\tilde{f}_n$ on $\tilde{\Omega}_n(\gamma_m)$ maps the compact set γ_n $(\subset \tilde{\Omega}_n(\gamma_m))$ onto a vertical segment (of positive length), the local univalence of $\tilde{f}_n$ on γ_n yields the univalence of $\tilde{f}_n$ on some neighborhood of γ_n, and hence the univalence of f_n on some $U(\gamma_n)$.

(2′) The univalence of f_n on γ_n does *not* always imply the univalence of f_n near γ_n, as a simple example $\left\{\frac{1}{2}\left(z+\frac{1}{z}\right)\right\}^3$ on $\{|z|<1\}$ shows. Cf. [33].

12. In general, $f(\gamma_m)$ is a closed analytic curve with some multiple points. Unless $f(\gamma_m)$ traces the same curve more than once, the number of self-intersections of the curve $f(\gamma_m)$ is finite. In any case, $f(\gamma_m)$ divides $\hat{C}$ into finitely many simply connected domains. To be more precise, let $|\gamma_m|$ be the set of points which lie on the curve γ_m and $f(|\gamma_m|)$ its image. Then the set $\hat{C} \setminus f(|\gamma_m|)$ consists of a finite number of connected components which are all simply connected. Let K be any one of them.

For $\varepsilon>0$ we set $K_\varepsilon=\{w \in K \mid \operatorname{dist}(w, \partial K)>\varepsilon\}$, where $\operatorname{dist}(w, \partial K)$ stands for the distance between w and ∂K. We can choose ε so small that K_ε is a connected, simply connected non-void set. Let $\nu(K)$ be the winding number of $f(-\gamma_m)$ with respect to K.

13. Now we shall go back to the consideration of f_n. If n is large enough, K_ε has the index $\nu(K)$ with respect to $f_n(-\gamma_m)$ (cf. Proposition 4). Then $\tilde{f}_n \mid \tilde{f}_n^{-1}(K_\varepsilon)$ covers K_ε exactly $\nu(K)$ times; to each $w \in K_\varepsilon$ there correspond $\nu(K)$ (distinct) points on $\tilde{\Omega}_n(\gamma_m)$ which are mapped to w and no other points are mapped to w.

Let

$$\tilde{f}_n^{-1}(K_\varepsilon) = X_1 \cup X_2 \cup \cdots \cup X_s$$

be the decomposition of $\tilde{f}_n^{-1}(K_\varepsilon)$ into the connected components (with respect to $\tilde{\Omega}_n(\gamma_m)$). (Precisely speaking, X_i and s depend on ε and n, and hence we should have written as $X_i(\varepsilon, n)$ and $s(\varepsilon, n)$ instead of X_i and s, respectively.)

Denote by $\nu_i(w)$ the cardinal number of $X_i \cap \tilde{f}_n^{-1}(w)$, $w \in K_\varepsilon$. Each $\nu_i(w)$ is obviously a lower semicontinuous function of w. It is also clear that

$$\nu_1(w) + \nu_2(w) + \cdots + \nu_s(w) = \nu(K)$$

for all $w \in K_\varepsilon$. It follows that $\nu_i(w)$ is continuous in w. Hence

$$\nu_i(w) = \text{const.} \quad \text{on} \quad K_\varepsilon,\ i = 1, 2, \ldots, s,$$

for each $\nu_i(w)$ is integer-valued. (Similar argument can be found in a paper by Gulliver, Osserman and Royden [11].) Thus the covering $\tilde{f}_n | X_i$: $X_i \to K_\varepsilon$ is unlimited; no relative boundary point appears.

On the other hand, the covering $\tilde{f}_n | X_i$: $X_i \to K_\varepsilon$ is unbranched, because $d\tilde{f}_n$ does not vanish on the closure of $\tilde{\Omega}_n(\gamma_m)$. Since K_ε is simply connected, the monodromy theorem now implies that each covering map $\tilde{f}_n | X_i$ is a homeomorphism. We have therefore proved that

$$\nu_1 = \nu_2 = \cdots = \nu_s = 1 \quad \text{and} \quad s = \nu(K)$$

This means that each X_i covers K_ε exactly once. Namely:

PROPOSITION 10. *Let γ be an ideal boundary component of R satisfying condition (**). Then for sufficiently large m and n ($>m$) and for any sufficiently small $\varepsilon>0$, $\tilde{f}_n^{-1}(K_\varepsilon)$ consists of $\nu(K)$ components X_i ($i=1, 2,\dots, \nu(K)$) with respect to $\tilde{\Omega}_n(\gamma_m)$ and the covering f_n: $X_i \cap \Omega_n(\gamma_m) \to K_\varepsilon$ is (at most) one-sheeted, $i=1, 2,\dots, \nu(K)$. ($X_i \cap \Omega_n(\gamma_m)$ may not be connected.)*

14. We shall now show

PROPOSITION 11. *Let γ be an ideal boundary component of R with $N(\gamma, f)=0$. Then for every sufficiently large m and for every component K of $\hat{C} \setminus f(|\gamma_m|)$, each connected component of $f^{-1}(K)$ in $\Omega(\gamma_m)$ covers K at most once.*

PROOF. Suppose that there is a connected component K of $\hat{C} \setminus f(|\gamma_m|)$, a connected component Y of $f^{-1}(K)$, and a point w_0 in K for which we can find two distinct points p and q on Y with $f(p)=f(q)=w_0$. We choose $\varepsilon>0$ so small that w_0 belongs to K_ε. Take a neighborhood U of w_0 such that $\bar{U} \subset K_\varepsilon$. We can then find a neighborhood U_p (resp. U_q) of p (resp. q) such that U_p (resp. U_q) is contained in $\Omega(\gamma_m)$ and f gives a homeomorphism from U_p (resp. U_q) onto U. We may assume that $\bar{U}_p \cap \bar{U}_q = \emptyset$. Then an elementary argument yields that U_p and U_q are contained in the same component $X_j = X_{j(\varepsilon,n)}$ provided n is large enough.

By Proposition 4 (ii) and a theorem of Hurwitz f_n must have a w_0-point p_n in U_p for every large n. Similarly f_n must have a w_0-point q_n in U_q. We have thus found two distinct points p_n and q_n on X_j such that $f_n(p_n)=f_n(q_n)=w_0$, which contradicts Proposition 10. q.e.d.

For convenience' sake, we collect the above results (in sections 10–14) together in the following

THEOREM 2. *Let f be a non-constant S-function on R. Then for almost every (=except for finitely many) ideal boundary component γ of R there is a regularly embedded connected neighborhood Ω of γ such that each complemen-*

tary component K of $f(|\partial\Omega|)$ is covered by any connected component of $f^{-1}(K)$ in Ω at most once.

The number of exceptional ideal boundary components does not exceed W. An ideal boundary component γ is exceptional if and only if $N(\gamma, f)\geqq 1$.

REMARKS. (1) If some $f(\gamma_m)$ is simple for every γ, then by Lemma 2 we have $W=0$, so that there is no exceptional ideal boundary component. This fact is also proved directly by using the argument principle.

(2) We shall later give an example of an open Riemann surface of genus $g>0$ such that every non-constant S-function f on it with $\mu(f)\leqq g$ is *never* univalent on any neighborhood of the ideal boundary. This shows, in particular, that exceptional ideal boundary components can actually appear. See Proposition 16.

IV. Realization of the ideal boundary as vertical slits

Now we shall realize ∂R in a larger surface $\tilde{R}$. Specifically, we shall construct a continuation $\tilde{R}$ of R and a holomorphic extension $\tilde{f}$ of f to $\tilde{R}$ such that

(1) $\tilde{R}$ is either a closed surface or the interior of a compact bordered surface,

(2) $\tilde{f}$ is an S-function on $\tilde{R}$,

(3) $\tilde{f}\colon \tilde{R}\to\hat{C}$ realizes most part of ∂R as a nice subset of $\tilde{R}$ and the rest of ∂R as the border of $\tilde{R}$; each ideal boundary component of R realized in $\tilde{R}$ is an analytic curve (or a single point) whose projection by $\tilde{f}$ is a vertical segment.

15. We shall need the following proposition later.

PROPOSITION 12. *Let $\omega\in\Gamma_e(R)$. Suppose, furthermore, that ω vanishes identically near the poles of f. Then*

$$\iint_R \omega\wedge d(\mathrm{Re} f) = 0.$$

PROOF. Let U be an open set on R such that U contains all the poles of f and $\omega\equiv 0$ on $\bar{U}$. We may assume that $u:=\mathrm{Re} f$ has the representation

$$du = \omega_{hm} + \omega'_{e0}$$

on $R\backslash\bar{U}$ (cf. [36]), where $\omega_{hm}\in\Gamma_{hm}(R)$ and $\omega'_{e0}\in\Gamma_{e0}(R)\cap\Gamma^1(R)$. On the other hand, Dirichlet principle yields that there are two differentials $\omega_{he}\in\Gamma_{he}(R)$ and $\omega''_{e0}\in\Gamma_{e0}(R)$ such that

$$\omega = \omega_{he} + \omega''_{e0}$$

on R.

Since $\omega\equiv 0$ on $\bar{U}$, the inner product $(du, \omega^*)_R$ exists and

$$(du, \omega^*)_R = (du, \omega^*)_{R\setminus U} = (\omega_{hm} + \omega'_{e0}, \omega^*_{he} + \omega''^*_{e0})_{R\setminus U}$$
$$= (\omega_{hm} + \omega'_{e0}, \omega^*_{he} + \omega''^*_{e0})_R.$$

The last term is equal to zero, since ω_{hm}, ω^*_{he}, ω'_{e0} and ω''^*_{e0} are pairwise orthogonal. q.e.d.

REMARKS. (1) The above proposition can be used to give an alternate proof of Proposition 2.

(2) As for the classical version of Proposition 12, see [3], p. 405 or [8], p. 48.

16. Suppose $N(\gamma, f)=0$ and take m and n $(>m)$ so large that $N(\gamma_m, f_n)=0$. Let K be, as before, any connected component of $\hat{C}\setminus f(|\gamma_m|)$, and $\nu(K)$ the index of K with respect to $f(-\gamma_m)$. Although $f^{-1}(K)$ does not always consist of finitely many connected components, each of them covers (part of) K at most once (see Theorem 2).

As a refinement of Proposition 2, we have

LEMMA 9. *$f|\Omega(\gamma_m)$: $\Omega(\gamma_m)\to K$ is $\nu(K)$-valent almost everywhere.*

PROOF. Let $\nu(w, \Omega(\gamma_m))$ be the number (counted with the multiplicity) of w-points on $\Omega(\gamma_m)$. We must prove that $\nu(w, \Omega(\gamma_m))\leqq \nu(K)$ and $\mathrm{meas}\{w\in K \mid \nu(w, \Omega(\gamma_m))<\nu(K)\}=0$. To do this, we first show that for any $w_0\in K$ there are at most $\nu(K)$ points on $\Omega(\gamma_m)$ which are mapped to w_0. If $p_1,\dots, p_\nu$ are all the w_0-points (repeated as ofen as their multiplicities) and n is large enough, there are ν points $p_1^{(n)},\dots, p_\nu^{(n)}$ such that $f_n(p_j^{(n)})=w_0$ and each $p_j^{(n)}$ is located near p_j, $j=1, 2,\dots, \nu$. This shows that $\nu\leqq \nu^*(w_0)$, the index of w_0 with respect to $f(-\gamma_m)$. Since $\nu^*(w_0)$ is clearly equal to $\nu(K)$, we have proved $\nu\leqq\nu(K)$.

Let $\partial R_m-\gamma_m=\sum_{i=2}^{s}\gamma_m^{(i)}$ be the decomposition into contours, $s=h_m$. For simplicity we set $\gamma_m=\gamma_m^{(1)}$. Let $\nu_i(w_0)$ denote the number of w_0-points on $\Omega(\gamma_m^{(i)})$, and $\nu_i^*(w_0)$ denote the index of w_0 with respect to $f(-\gamma_m^{(i)})$, $i=1, 2,\dots, s$. Then, as we have shown above,

$$\nu_i(w_0) \leqq \nu_i^*(w_0) = -\frac{1}{2\pi}\int_{\gamma_m^{(i)}} d\arg(f-w_0), \quad i = 1, 2,\dots, s.$$

Summing these equations up, we have

$$\nu_1(w_0) + \nu_2(w_0) + \cdots + \nu_s(w_0) \leqq -\frac{1}{2\pi}\int_{\partial R_m} d\arg(f-w_0) = \mu - \nu_0(w_0),$$

where $\nu_0(w_0)$ is the number of w_0-points on $\bar{R}_m$. Hence we have

$$\nu_0(w_0) + \nu_1(w_0) + \cdots + \nu_s(w_0) \leqq \mu.$$

Since, on the other hand, $\mu = \nu_0(w_0) + \nu_1(w_0) + \cdots + \nu_s(w_0)$ for almost every $w_0 \in K$ (see Proposition 2), we finally have

$$\nu_i(w_0) = \nu_i^*(w_0) \quad a.e. \quad w_0 \in K, \quad i = 1, 2, \ldots, s;$$

in particular, we have proved that $\nu(w, \Omega(\gamma_m)) = \nu(K)$ for almost all $w \in K$.

q.e.d.

17. In this and the next section we shall give a precise definition of "sheets" on K.

First of all, set

$$\Omega^K(\gamma_m) = \{p \in \Omega(\gamma_m) \,|\, f(p) \in K, \ \nu(f(p), \Omega(\gamma_m)) = \nu(K)\}.$$

Then, by Lemma 9, $\mathrm{meas}(K \setminus f(\Omega^K(\gamma_m))) = 0$. We shall prove that $\Omega^K(\gamma_m)$ is divided into $\nu(K)$ mutually disjoint open sets, each of which covers K at most once. To this end, we begin with

DEFINITION. Two points p, q on $\Omega^K(\gamma_m)$ are said to be *f_n-equivalent modulo K* if

(1) $p, q \in \Omega_n(\gamma_m)$, and

(2) there is an arc $\tilde{\alpha}$ in $\tilde{\Omega}_n(\gamma_m)$ joining p to q such that $\tilde{f}_n(\tilde{\alpha}) \subset K$.

(Note that (2) implies that n is so large for which $f_n(p)$ and $f_n(q)$ shall belong to K.)

The following lemma is easily proved by using a theorem of Hurwitz:

LEMMA 10. *Given $p \in \Omega^K(\gamma_m)$ there exists a neighborhood U of p and an integer n_0 such that every point q in U is f_n-equivalent to p modulo K for all $n \geqq n_0$.*

We can prove the following lemma without difficulty:

LEMMA 11. *f_n-equivalence modulo K is an equivalence relation.*

The next lemma follows immediately from Proposition 10.

LEMMA 12. *Let p, q be distinct points on $\Omega^K(\gamma_m)$ with $f(p) = f(q)$. Then there exists an n_0 such that, for all $n \geqq n_0$, p, q are not f_n-equivalent modulo K.*

18. We shall now pass to the mapping function f.

DEFINITION. Two points p, q on $\Omega^K(\gamma_m)$ are called *f-equivalent* modulo K if there is an integer n_0 such that p, q are f_n-equivalent modulo K for every $n \geqq n_0$.

Corresponding Lemmas 11 and 12, we have:

LEMMA 11′. *f-equivalence modulo K is an equivalence relation.*

LEMMA 12′. *If p, q are distinct points on $\Omega^K(\gamma_m)$ with $f(p)=f(q)$, then p is not f-equivalent to q modulo K.*

We also have

LEMMA 13. *Let $p_1, p_2,\dots, p_{\nu(K)}$ be the set of $\nu(K)$ distinct points on $\Omega^K(\gamma_m)$ with $f(p_1)=f(p_2)=\cdots=f(p_{\nu(K)})$. Then, for every $q\in\Omega^K(\gamma_m)$, there is a (unique) i $(1\leqq i\leqq\nu(K))$ such that q is f-equivalent to p_i modulo K.*

PROOF. Set $w_0=f(q)$ and $w_1=f(p_1)$ $(=f(p_j)$ for all $j=2, 3,\dots, \nu(K))$. By Lemma 10 we can find, for each $j=1, 2,\dots, \nu(K)$, a neighborhood U_j of p_j such that every point in U_j is f_n-equivalent to p_j modulo K if n is large enough, say, $n\geqq n_j$. Similarly we can find a neighborhood U_0 of q such that every point in U_0 is f_n-equivalent to q if n is large enough, say, $n\geqq n_0$. We may assume that $n_0\geqq n_j$, $j=1, 2,\dots, \nu(K)$, and that U'_j are mutually disjoint.

We let n_0 larger again (if necessary). Then for every $n\geqq n_0$ there is a $q^{(n)}\in U_0$ with $f_n(q^{(n)})=w_0$. Similarly, there are $p_j^{(n)}\in U_j$ such that $f_n(p_j^{(n)})=w_1$, $j=1, 2,\dots, \nu(K)$.

For each n $(\geqq n_0)$ we can find a number $j=j(n)$, $1\leqq j(n)\leqq\nu(K)$, such that $q^{(n)}$ is f_n-equivalent to $p_j^{(n)}$, which we shall write, for typographical reason, as $\tilde{p}_{j(n)}$. Indeed, the existence of such a point is easily follows from Proposition 10. Let $\tilde{\alpha}_n$ be an arc joining $q^{(n)}$ and $\tilde{p}_{j(n)}$ in $\tilde{\Omega}_n(\gamma_m)$ such that $\tilde{f}_n(\tilde{\alpha}_n)\subset K$. We may assume that $\tilde{f}_n(\tilde{\alpha}_n)$ is the same arc $\tilde{A}$, independent of $n\geqq n_0$. The arc $\tilde{A}$ joins w_0 with w_1 in K. Observe that $\tilde{p}_{j(n)}$ is obtained from $q^{(n)}$ by the analytic continuation along the arc $\tilde{A}$. In other words, $\tilde{p}_{j(n)}$ is the terminal point of the lift $\tilde{\alpha}_n$ of $\tilde{A}$ by the covering $\tilde{f}_n$: $\tilde{\Omega}_n(\gamma_m)\to\hat{C}$.

Now abbreviate $\tilde{p}_{j(n)}$ and $q^{(n)}$ for $n=n_0$ as $\tilde{p}_*$ and q^*, respectively. Then, there is another arc α_{n_0} joining $\tilde{p}_*$ with q^* in $\Omega_{n_0}(\gamma_m)$. Let A be the arc $f_{n_0}(\alpha_{n_0})$, which joins w_0 with w_1. (A is not always contained in K.) For each $n\geqq n_0$, there is an arc α_n, the lift of A from the initial point $q^{(n)}$ for the covering f_n: $\Omega_n(\gamma_m)\to\hat{C}$, which joins $q^{(n)}$ with some point $r^{(n)}\in\Omega_n(\gamma_m)$. Since f_n converges to f locally uniformly, $r^{(n)}$ belongs to the same U_j, provided that we let n_0 larger again. (Cf. the parmanence of functional relations.)

Since $\tilde{\Omega}_n(\gamma_m)$ is a simply connected domain, $\tilde{\alpha}_n$ and α_n are homotopic to each other and every inverse function element of $\tilde{f}_n^{-1}$ at w_0 is analytically continued along all the arcs which define the homotopy equivalence of $\tilde{A}$ to A. Hence, for each $\tilde{f}_n$: $\tilde{\Omega}_n(\gamma_m)\to\hat{C}$, the terminal points, $\tilde{p}_{j(n)}$ and $r^{(n)}$, of the lifts of $\tilde{A}$ and A must coincide. Therefore, $j(n)$ is the same j for all $n\geqq n_0$, so that $q^{(n)}$ is f_n-equivalent to $\tilde{p}_*$ for all $n\geqq n_0$. Since $q^{(n)}$ is f_n-equivalent to q, we have proved that q is f_n-equivalent to $\tilde{p}_*$ modulo K (cf. Lemma 11), $n\geqq n_0$. Hence q is f-equivalent to $\tilde{p}_*$ modulo K. q. e. d.

Combining these lemmas, we have the following refinement of Theorem 2.

PROPOSITION 13. *$\Omega^K(\gamma_m)$ consists of $\nu(K)$ disjoint open sets $Y_1, Y_2,\dots, Y_{\nu(K)}$, each of which is mapped by f onto an open dense subset of K univalently.*

Intuitively, the open set K is covered by $\nu(K)$ distinct jig-saw puzzles $Y_1, Y_2,\dots, Y_{\nu(K)}$. There are neither missing pieces nor extra ones.

19. We begin this section with the following

DEFINITION. An ideal boundary component γ of R is said to be *realized*, by the covering $f: R\to\hat{C}$, *as a univalent vertical slit*, if there exists a neighborhood U of γ such that f is univalent on U and γ corresponds to a vertical segment in the closure of the plane domain $f(U)$. As usual, we do not exclude the case where the segment reduces to a single point.

For example, if $N(\gamma, f)=0$ and n is large enough, the contour γ_n of R_n is realized, by the covering $f_n: R_n \setminus \bar{R}_1\to\hat{C}$, as a univalent vertical segment. Cf. Remark (2) in section 11.

Now, we are ready to prove

THEOREM 3. *Each non-exceptional ideal boundary component of R is, by means of the covering $f: R\to\hat{C}$, realized as a univalent vertical slit (which may reduce to a point).*

PROOF. Let γ be an ideal boundary component with $N(\gamma, f)=0$, and γ_m the m-th antiderivation of γ such that $N(\gamma_m, f)=0$. As before, let K be any connected component of $\hat{C}\setminus f(|\gamma_m|)$ and let $\nu(K)$ denote the winding number of $f(-\gamma_m)$ with respect to K. Then, by Proposition 13, there are $\nu(K)$ mutually disjoint open subsets $Y_1, Y_2,\dots, Y_{\nu(K)}$ of $\Omega(\gamma_m)$ such that $\Omega^K(\gamma_m)=Y_1\cup Y_2\cup\cdots\cup Y_{\nu(K)}$ and $f|Y_j: Y_j\to K$ is univalent for each $j=1, 2,\dots, \nu(K)$.

We claim that, for each j $(1\leqq j\leqq \nu(K))$, $K\setminus f(Y_j)$ consists of vertical segments (including the case of single points). To verify this, suppose there is a component γ_0 of $K\setminus f(Y_j)$ which is neither a vertical segment nor a point. Then, choose real numbers u_1, u_2 $(u_1<u_2)$ and v_0 suitably and construct a rectangular open set G on K whose boundary consists of part of γ_0 and three segments on the lines $u=u_1$, $u=u_2$ and $v=v_0$. ∂G may intersect some other components of $K\setminus f(Y_j)$; furthermore, $G\cap f(Y_j)$ may be disconnected. (Recall the classical case. See, *e.g.*, [34], pp. 223–224.)

Define a C^2-function $\Psi(u, v)$ on K by

$$\Psi(u, v) = \begin{cases} (u-u_1)^2(u-u_2)^2(v-v_0)^2 & \text{on } G \\ 0 & \text{on } K\setminus G. \end{cases}$$

Because of the univalence of f on Y_j, f pulls $\Psi(u, v)$ back to a C^2-function $\tilde{\Psi}(p)=\Psi(u(p), v(p))$ on $\tilde{G}_j=f^{-1}(G)\cap Y_j$. If we set $\tilde{\Psi}(p)=0$ for all $p\in R\setminus\tilde{G}_j$, $\tilde{\Psi}(p)$ is a C^2-function on R and $\omega=d\tilde{\Psi}$ clearly satisfies the assumption of Proposition 12. Nevertheless,

$$\iint_R \omega \wedge d(\mathrm{Re} f) = -2\iint_G (u-u_1)^2(u-u_2)^2(v-v_0)dudv \neq 0,$$

which violates the conclusion of the proposition.

Using Theorem 2 (deform γ_m a little, if necessary), we can now conclude that every ideal boundary component γ' whose m-th antiderivation is the same γ_m is realized by f as a univalent vertical slit. q.e.d.

20. By "welding" non-exceptional ideal boundary components in an obvious manner, we obtain a new surface $\tilde{R}$ of genus g. To be more precise, we only need to identify the two edges of the realized slits and go back to the initial surface R. Since there exist only finitely many exceptional ideal boundary components, the resulting surface $\tilde{R}$ is finitely connected. In fact, the number of the boundary components of $\tilde{R}$ is at most W (cf. Theorem 2). The set $\tilde{R}\setminus R$ has a vanishing area (see Proposition 2).

It should be noted that the function f can be naturally extended onto $\tilde{R}$ as a meromorphic function $\tilde{f}$ which is holomorphic on $\tilde{R}\setminus R$. This follows immediately from the construction of $\tilde{R}$.

The surface $\tilde{R}$ is either a closed surface or (is conformally equivalent to) a closed surface with finitely many mutually disjoint closed disks and/or points removed. If it is a closed surface, we are through; $\tilde{R}$ is a maximal continuation of R. If $\tilde{R}$ has punctures, the punctured points are easily recovered, so that we obtain a new surface without punctures, which we continue to denote by the same latter $\tilde{R}$. It is evident that $\tilde{f}$ again extends holomorphically to the punctures. The extended function which is also denoted by $\tilde{f}$ has the vanishing first derivative at each puncture. In other words, punctures are branch points of the (new) covering $\tilde{f}$: $\tilde{R}\to\hat{C}$. If γ is the ideal boundary component of R which corresponds to a puncture of $\tilde{R}$, it is evident that the branch order at that point is $N(\gamma, f)$.

21. After all the punctures have been recovered, $\tilde{R}$ can be assumed to be the interior of a compact bordered surface. We shall first prove

PROPOSITION 14. *The function $\tilde{f}$ is an S-function on $\tilde{R}$.*

PROOF. Since $\mathrm{Re}\,(df)$ is distinguished, we can find a relatively compact open subset U of R and differentials $\omega_{hm}\in\Gamma_{hm}(R)$, $\omega_{e0}\in\Gamma_{e0}(R)\cap\Gamma^1(R)$ such that U contains all the singularities of f, and $\mathrm{Re}\,(df)=\omega_{hm}+\omega_{e0}$ on $R\setminus\overline{U}$. (See the end of section 1; cf. also [36].)

We set

$$\tilde{\omega} = \begin{cases} \operatorname{Re}(d\tilde{f}) & \text{on} \quad \tilde{R} \setminus \bar{U} \\ \omega_{hm} + \omega_{e0} & \text{on} \quad \bar{U}. \end{cases}$$

Then $\tilde{\omega}$ is a closed C^1-differential on $\tilde{R}$. Since $\tilde{\omega}$ is clearly square integrable on $\tilde{R}$ (cf. Proposition 2), it belongs to $\Gamma_c^1(\tilde{R})$. ($\tilde{\omega}$ is in fact harmonic on $\tilde{R} \setminus \bar{U}$.)

By the Dirichlet principle on $\tilde{R}$ (see section 1) we can find $\tilde{\omega}_h \in \Gamma_h(\tilde{R})$ and $\tilde{\omega}_{e0} \in \Gamma_{e0}(\tilde{R}) \cap \Gamma^1(\tilde{R})$ such that

$$\tilde{\omega} = \tilde{\omega}_h + \tilde{\omega}_{e0}$$

on $\tilde{R}$. We claim that $\tilde{\omega}_h$ belongs to $\Gamma_{hm}(\tilde{R})$. To verify this, let $\tilde{\omega}_{hse} \in \Gamma_{hse}(\tilde{R})$. Since the restriction of any element of $\Gamma_{hse}(\tilde{R})$ onto R obviously belongs to $\Gamma_{hse}(R)$ and $\tilde{R} \setminus R$ has a vanishing measure, we have

$$\begin{aligned}(\tilde{\omega}_h, \tilde{\omega}^*_{hse})_{\tilde{R}} &= (\tilde{\omega} - \tilde{\omega}_{e0}, \tilde{\omega}^*_{hse})_{\tilde{R}} = (\tilde{\omega}, \tilde{\omega}^*_{hse})_{\tilde{R}} \\ &= (\tilde{\omega}, \tilde{\omega}^*_{hse})_R = (\omega_{hm} + \omega_{e0}, \tilde{\omega}^*_{hse}|R)_R = 0.\end{aligned}$$

Hence ω_h is orthogonal to $\Gamma^*_{hse}(\tilde{R})$, so that $\tilde{\omega}_h \in \Gamma_{hm}(\tilde{R})$.

Since $\operatorname{Re}(d\tilde{f})$ is equal to $\tilde{\omega}$ on $\tilde{R} \setminus \bar{U}$, we have obtained the representation

$$\operatorname{Re}(d\tilde{f}) = \tilde{\omega}_{hm} + \tilde{\omega}_{e0}$$

on $\tilde{R} \setminus \bar{U}$ with $\tilde{\omega}_{hm} \in \Gamma_{hm}(\tilde{R})$ and $\tilde{\omega}_{e0} \in \Gamma_{e0}(\tilde{R}) \cap \Gamma^1(\tilde{R})$. This shows that $\operatorname{Re}(d\tilde{f})$ is distinguished on $\tilde{R}$, so that $\tilde{f}$ is an S-function on $\tilde{R}$. q.e.d.

We are now in a position to use the following well known

PROPOSITION 15. *Let R_0 be the interior of a compact bordered Riemann surface and f_0 an S-function on R_0. Then,* $\operatorname{Re} f_0$ *assumes a constant value on each boundary contour of R_0.*

For the proof of this proposition, see [2] and [18], for instance. Cf. Proposition 3, too.

It follows now from the above proposition that $\tilde{f}$ is a vertical slit mapping in the sense of Nehari (cf. [23] and [18]) and $\tilde{R}$ is realized, via $\tilde{f}$, as an (at most) μ-sheeted vertical slit covering surface of $\hat{C}$. Cf. [33], too.

22. In this section we shall use, by abuse of language, the term "compact" surfaces for the interior of compact bordered surfaces as well as closed surfaces. Then we have almost proved the following theorem:

THEOREM 4. *Let R be an open Riemann surface of finite genus g, and f a (non-constant) S-function on R.*

Then there exist a "compact" Riemann surface $\tilde{R}$ of genus g, a single-valued *meromorphic function $\tilde{f}$ on $\tilde{R}$, and a conformal injection ι: $R \to \tilde{R}$ which have the following properties:*

(1) *The set $\tilde{R} \setminus \iota(R)$ is of measure zero.*

(2) *$\tilde{f}$ is holomorphic on $\tilde{R} \setminus \iota(R)$.*

(3) *$\tilde{f} \circ \iota = f$ on R.*

(4) *Each component of $\tilde{R} \setminus \iota(R)$ is mapped by $\tilde{f}$ onto a univalent vartical line segment, which may reduce to a point; in particular, $d\tilde{f}$ vanishes nowhere on $\tilde{R} \setminus \iota(R)$.*

(5) Re$\tilde{f}$ *is constant on each boundary contour of $\tilde{R}$ (if any exists).*

(6) *$\tilde{f}$ is $(N(\gamma, f)+1)$-valent on a neighborhood of the boundary contour of $\tilde{R}$ which corresponds to the ideal boundary component γ of R.*

REMARK. The set $\tilde{R} \setminus \iota(R)$ is a Lebesgue null set of arcs on the trajectories of the meromorphic quadratic differential $-(d\tilde{f})^2$ on $\tilde{R}$. See Figure 1.

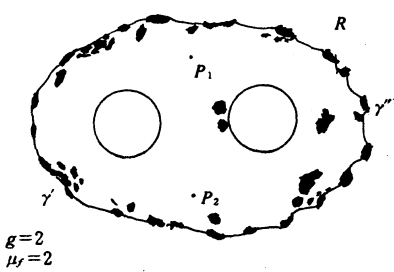

Figure 1 a)

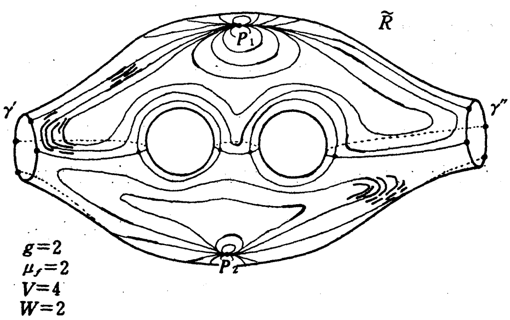

Figure 1 b)

$$\begin{tikzcd}
R \arrow[rr, "\iota"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowright"] & \tilde{R} \arrow[dl, "\tilde{f}"] \\
 & \hat{C} &
\end{tikzcd}$$

Fig. 2

The rest of the proof is easy and therefore omitted. Conditions (5) and (6) are, of course, superfluous when $\tilde{R}$ is a "compact" surface without boundary (=closed surface). In any case, the original surface R is obtained from a "compact" surface $\tilde{R}$ by deleting a finite or infinite number of "univalent vertical slits" of vanishing total area. The given S-function f on R is naturally extended to an S-function on a larger "compact" surface $\tilde{R}$ of the same genus.

The above theorem gives a generalization of the classical result due to Koebe, known as the *generalized uniformization theorem* or the *fundamental theorem in the theory of conformal mapping*. Cf. *e.g.*, [15], [16], and [8].

V. Remarks and supplements

23. We shall here give a simple example which shows that exceptional ideal boundary components in Theorems 3 and 4 can actually appear.

Let R_0 be the hyperelliptic Riemann surface of genus g which is defined by

$$w^2 = (z-a_1)(z-a_2)\cdots(z-a_{2g+2}),$$

where a_i are real numbers with $0<a_1<a_2<\cdots<a_{2g+2}$. Denote by J_0 the hyperelliptic involution of R_0. Note that we can realize R_0 as the two sheeted covering surface of $\hat{C}$.

Take a real number a_0 with $a_0<a_1$ and consider the set

$$\sigma = \{(z, w)\in R_0 \mid \operatorname{Im} z=0,\ a_0\leqq \operatorname{Re} z\leqq a_1\},$$

which is obviously J_0-invariant. Finally we set

$$R = R_0 \setminus \sigma.$$

Then R is an open Riemann surface of genus g with a single (ideal) boundary component. Note that no interior point of R lies over a_1.

It is easy to see that R admits an automorphism J of order two which is induced by J_0. It fixes the $2g+1$ points $\tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_{2g+2}$ on R which lie over $a_2, a_3, \ldots, a_{2g+2}$, respectively.

24. Let f be a non-constant S-function on R. If we set

$$F = f - f\circ J,$$

then F is also an S-function on R, since $\operatorname{Re} F$ assumes a constant value on the boundary. (The constant is, in fact, zero; but we do not need this property.)

Suppose now that f has poles at r_0 points of $\{\tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_{2g+2}\}$, $0\leqq r_0\leqq 2g+1$, and let μ_0 be the total order of the poles of f at these points, $r_0\leqq\mu_0\leqq\mu=\mu(f)$. Then the degree of the polar divisor $(F)_\infty$ of F does not exceed

$$2(\mu-\mu_0)+\mu_0 = 2\mu-\mu_0.$$

On the other hand, at the remaining $(2g+1)-r_0$ points of $\{\tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_{2g+2}\}$, F clearly vanishes, so that we have unless $F\not\equiv 0$,

$$2\mu - \mu_0 \geqq 2g + 1 - r_0,$$

since the number of zeros of F is not greater than the number of poles of F (see Proposition 1). It follows immediately that $F \equiv 0$ whenever $g \geqq \mu$ ($\geqq 1$).

We have proved: if $\mu \leqq g$, then f takes the same value at two points p and $J(p)$. Therefore f is the lift of a function f_J on the quotient surface $R/\langle J \rangle$. Since f_J is an S-function on $R/\langle J \rangle$, we can conclude that there are (at least) four points on σ at which f assumes the same value.

We have hence proved:

PROPOSITION 16. *There exists an open Riemann surface of genus g, whose ideal boundary cannot be univalently realized by any S-function of order less than $g+1$.*

25. Our method of proof for Theorem 4 can be used to give an affirmative answer to Mizumoto's conjecture (see [20], p. 47). Namely we have

THEOREM 5. *Only finitely many ideal boundary components of an open Riemann surface of finite genus are realized as non-univalent (and all others as univalent) slits under any mapping which Mizumoto studies in his paper* [20].

To show this it suffices to repeat our discussion *mutatis mutandis.* Some modification will be necessary, of course. For instance, the proof of Proposition 5 should be somewhat changed. Note that Propositions 1, 2, 3 and 4 hold for vertical-horizontal slit mappings. See [20].

Theorem 4 also gives a new viewpoint to Open Question 3 in Sario-Oikawa [31]. We can characterize principal functions on an open Riemann surface R of finite genus by means of *compact* (*bordered*) *continuation* of R instead of *compactification* as in [31]. For example, every $(Q)L_1$-principal function on R can be extended harmonically onto a "compact" surface $\tilde{R}$ and the extended function assumes a constant value on each component of $(\tilde{R} \cup \partial\tilde{R}) \setminus R$.

26. As was pointed out earlier (see Proposition 16), the resulting surface $\tilde{R}$ in Theorem 4 is generally not closed, but it may well have a non-void boundary. In such a case, it would be possible to give a compact continuation R^* of $\tilde{R}$ onto which $\tilde{f}$ extends holomorphically. R^* is then a compact continuation of the initial R onto which the given S-function f extends naturally. In particular, f *will be considered as an algebraic function*; this seems to be a notable, so far unknown, property of S-functions. Details will appear in [33].

It should be noted, however, that R^* is not uniquely determined by the given pair, R and f. There are infinitely many distinct ways of obtaining R^* from $\tilde{R}$, hence from the pair (R, f). For example, let R, R_0 be the surfaces in the preceding example (see section 23) and f_0 the projection mapping $f_0\colon R_0 \to \hat{C}$. Clearly

$\sqrt{(-1)}f_0 \mid R$ is a (non-constant) S-function on R. The pair $(R, \sqrt{(-1)}f_0 \mid R)$ can be obtained from the closed surface $(R_0, \sqrt{(-1)}f_0)$. However, $(R, \sqrt{(-1)}f_0 \mid R)$ can be also obtained from another surface R_0' which is defined by the equation

$$w^2 = (z-a_1')(z-a_2)\cdots(z-a_{2g+2}).$$

with $\operatorname{Im} a_1'=0$ and $a_0<a_1'<a_1$. If $a_0<a_1'<a_1''<a_1$, then the corresponding closed surfaces R_0' and R_0'' are conformally inequivalent, in general.

27. In this section we shall again discuss the Riemann-Hurwitz relation for S-functions.

As before, let R be an open Riemann surface of genus g and f a non-constant S-function on R. Let γ be any ideal boundary component of R. By Theorem 4 we can find a neighborhood $\tilde{U}(\gamma)$ of γ in $\tilde{R}$ on which

(i) $\tilde{f}$ is $(N(\gamma, f)+1)$-valent, and

(ii) $df \neq 0$.

It will be convenient to say that f has *boundary branch points* on γ whose *total order* is $N(\gamma, f)$. Using Proposition 9, we can rewrite Theorem 1 as follows:

THEOREM 6 (*Riemann-Hurwitz relation*). *Let f be a non-constant S-function on an open Riemann surface of genus g and μ the number of poles of f (counted with multiplicities). Let V (resp. W) be the total order of interior (resp. boundary) branch points. Then*

$$g = 1 - \mu + \frac{V}{2} + \frac{W}{2}.$$

As was noted before (see section 26; cf. [33], too), there is a compact continuation R^* of R and a holomorphic extension f^* of f onto R^*. Each boundary component γ of R with $N(\gamma, f) \neq 0$ is then realized as a slit (or a point) on which f^* has branch points in the classical sense. Furthermore, the total order of the branch points on γ $(\subset R^*)$ is exactly $N(\gamma, f)$. This shows that the above definition of the total order of boundary branch points on γ is reasonable.

28. For the completeness we remark here that there is certainly a non-constant S-function on R. Indeed, this is a simple consequence of the Riemann-Roch theorem for open Riemann surfaces. For any $g+1$ prescribed points p_1, $p_2, \ldots, p_{g+1}$ on R there is a non-constant S-function which is a multiple of the divisor $1/p_1p_2\cdots p_{g+1}$. For the details, see [18] and [30], for instance.

Another way to obtain a non-constant S-function is to consider a suitable linear combination of elementary differentials. This method is simpler than the first, but gives no good information on the dimension of the space of S-functions.

29. Since the S-function f on R is a single-valued meromorphic function whose real part is $(Q)L_1$-principal, its imaginary part is an L_0-principal function. (See [22], [29] and [36].) The function $\sqrt{(-1)}\,f$ gives a horizontal slit covering map of R.

If R is of planar character and μ is equal to one, then $\sqrt{(-1)}\,f$ is precisely a "Strömungsfunktion" considered by Hilbert, Courant and Koebe (cf. [6], [12] and [16]). The function $\sqrt{(-1)}f$ has, as its name shows, a significant physical meaning: $\sqrt{(-1)}f$ describes a *two-dimensional, irrotational, incompressible, perfect fluid flow.* See, *e.g.*, [8]. It is not the S-function f itself but $\sqrt{(-1)}f$ that is attached to such a flow. The function f describes the conjugate flow.

30. "Branch points" play an important role in the theory of minimal surfaces, too. See, for instance, Gulliver-Osserman-Royden [11] and Osserman [26]. An inequality due to Fenchel, Sasaki and Nitsche (see [26]) admits, as in the present study, the existence of boundary "branch points". This inequality also includes a quantity which is essentially the same as our $N(\gamma, f)$. See formula (7) in [26]. Though there is a substantial difference between immersions and covering surfaces — which Francis [10] called *polymersions* —, the total number of "branch points" is analogously estimated.

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