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# A central connection problem for a normal system of linear differential equations

Mitsuhiko KOHNO and Toshiaki YOKOYAMA

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Every $2\times 2$ system of linear differential equations

$$X' = (\textstyle\sum_{n=0}^{\infty} A_n t^{-n})X, \quad A_0 \sim \operatorname{diag}(\lambda_1, \lambda_2) \qquad (\lambda_1 \neq \lambda_2)$$

is meromorphically equivalent to a normal system of the form

$$(1) \qquad tX' = \left\{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t + \begin{pmatrix} \mu_1 & c \\ c' & \mu_2 \end{pmatrix}\right\} X,$$

where we put $\mu_2 - \mu_1 = \alpha + \beta$ and $cc' = -\alpha\beta$. This Birkhoff system has two singularities, a regular singular point $t=0$ and an irregular singular point $t=\infty$, in the whole complex $t$-plane. It is easy to see that through the transformation of the form $X = e^{\lambda t} t^{\delta} Y$, (1) can be reduced to the confluent hypergeometic equation and then solutions near $t=0$ are expressed in terms of Kummer's functions. Hence, by means of properties and connection formulas of Kummer's functions, the central connection problem between solutions near $t=0$ and $t=\infty$ can be immediately solved [2, 3, 5]. From this, it was also shown in [2, 3] that the values $\{\lambda_1, \lambda_2\}$, $\{\mu_1, \mu_2\}$ and

$$\gamma_1 = c/\Gamma(1-\alpha)\Gamma(1-\beta), \quad \gamma_2 = c'/\Gamma(1+\alpha)\Gamma(1+\beta)$$

are invariants, where $\{\gamma_1, \gamma_2\}$ are closely related to the monodromy matrix or the Stokes multipliers.

Here we shall deal with the same problem for (1), however, consider the global behavior of particular solutions near $t=0$, which are defined directly as convergent power series, but not in terms of Kummer's functions.

In general, in order to solve such a problem completely, one has to know the asymptotic behavior of coefficients of convergent power series solutions [4]. For instance, in case the coefficients are rational functions of gamma functions (like almost all special functions), the problem can be easily solved. In the case considered, the coefficients satisfy the second order linear difference equations or, after a simple transformation, the so-called hypergeometric difference equations. There is the very interesting study of hypergeometric difference equations by P. M. Batchelder [1]. By making the best use of his detailed investigations, we can then solve the connection problem for (1). From this point of view, the Birkhoff system (1) is a quite good example of cases where the coefficients satisfy more than one order linear difference equations and the consideration in this paper will be extended to the solution of connection problems for more general differential equations.

Hereafter we assume for simplicity that

$$\left\{\begin{array}{l} |\lambda_1-\lambda_2|\geqq|\lambda_1|\geqq|\lambda_2|>0, \\ -\pi<\arg\lambda_2-\arg\lambda_1\leqq\pi \end{array}\right. \tag{2}$$

and $\alpha-\beta$ is not equal to an integer.

One can find a fundamental set of convergent power series solutions of the column vectorial form

$$X_i(t)=t^{\rho_i}\sum_{m=0}^{\infty}G_i(m)t^m \qquad (i=1,2), \tag{3}$$

where the characteristic exponents $\rho_i$ $(i=1, 2)$ are eigenvalues of the constant matrix $\begin{pmatrix}\mu_1 & c\\ c' & \mu_2\end{pmatrix}$, i.e.,

$$\rho_1=\frac{1}{2}(\mu_1+\mu_2)+\frac{1}{2}(\alpha-\beta), \quad \rho_2=\frac{1}{2}(\mu_1+\mu_2)-\frac{1}{2}(\alpha-\beta).$$

The coefficient vector $G_i(m)$ satisfies the system of linear difference equations

$$\left\{m+\rho_i-\begin{pmatrix}\mu_1 & c\\ c' & \mu_2\end{pmatrix}\right\}G_i(m)=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}G_i(m-1) \qquad (i=1, 2) \tag{4}$$

together with the initial conditions $G_i(0)\neq 0$ and $G_i(r)=0$ $(r<0)$. Putting $G_i(m)=\begin{pmatrix}f_{i1}(m)\\ f_{i2}(m)\end{pmatrix}$, we write down (4) in the componentwise

$$\left\{\begin{array}{l} (m+\rho_i-\mu_1)f_{i1}(m)-cf_{i2}(m)=\lambda_1 f_{i1}(m-1), \\ -c'f_{i1}(m)+(m+\rho_i-\mu_2)f_{i2}(m)=\lambda_2 f_{i2}(m-1) \end{array}\right.$$

and then obtain

$$\begin{aligned} &\{(m+\rho_i-\mu_1)(m+\rho_i-\mu_2)+\alpha\beta\}f_{i1}(m) \\ &\quad=\{\lambda_1(m+\rho_i-\mu_2)+\lambda_2(m-1+\rho_i-\mu_1)\}f_{i1}(m-1)-\lambda_1\lambda_2 f_{i1}(m-2) \\ &\qquad\qquad (i=1, 2). \end{aligned} \tag{5}$$

Now, since

$$\begin{cases} \rho_1 - \mu_1 = \alpha, & \rho_1 - \mu_2 = -\beta, \\ \rho_2 - \mu_1 = \beta, & \rho_2 - \mu_2 = -\alpha, \end{cases}$$

we may only consider the difference equation (5) for $i=1$, i.e., dropping the index $i$,

$$\text{(6)} \quad m(m+\alpha-\beta)f_1(m) = \{(\lambda_1+\lambda_2)m-\lambda_1\beta+\lambda_2(\alpha-1)\}f_1(m-1)-\lambda_1\lambda_2 f_1(m-2)$$

and for $i=2$, we merely interchange the role of the parameters $\alpha$ and $\beta$. If we put

$$\text{(7)} \qquad f_1(m) = \frac{\hat{f}(m+1)}{\Gamma(m+1)},$$

then we obtain

$$\text{(8)} \qquad (m+\alpha-\beta+1)\hat{f}(m+2) \\ = \{(\lambda_1+\lambda_2)(m+1) + \lambda_2(\alpha-1) - \lambda_1\beta\}\hat{f}(m+1) - \lambda_1\lambda_2 m\hat{f}(m),$$

which is just the normal form of hypergeometric difference equation (103) [1; 69p], where $\rho_i=\lambda_i$ $(i=1, 2)$, $\beta_1=\alpha-1$ and $\beta_2=-\beta$. P. M. Batchelder defines six solutions $\mathscr{k}_i(m)$, $\mathscr{g}_i(m)$ $(i=1, 2)$, $\ell(m)$ and $\mathscr{m}(m)$ and investigates their global befavior in great details. Among them, we here have to choose a suitable solution which fits our purpose. Since $f_1(m)$ must be defined in the right half $m$-plane and vanish for negative integral values of $m$ because of the supposed initial conditions, we choose, taking account of the form (7), $\hat{f}(m)$ which has no poles at nonpositive integers, as a solution of (8). Hence we take the entire solution

$$\ell(m) = \int_{\lambda_2}^{\lambda_1} s^{m-1}(s-\lambda_1)^{\alpha-1}(s-\lambda_2)^{-\beta}ds$$

which is defined under the assumption that $\operatorname{Re}\alpha>0$, $\operatorname{Re}(-\beta)>-1$. If the above assumption is not satisfied, then one may replace the path of integration by the so-called double loop circuit. From this we put

$$\text{(9)} \qquad \hat{f}(m) = \ell''(m) = \oint_{\lambda_2}^{\lambda_1} s^{m-1}(s-\lambda_1)^{\alpha-1}(s-\lambda_2)^{-\beta}ds \\ = (1-e^{2\pi i(\alpha-1)})(1-e^{2\pi i(-\beta)})\ell(m).$$

As to the asymptotic behavior of $\ell(m)$ for sufficiently large values of $m$, P. M. Batchelder gives the following result: If $\arg\lambda_1 \geqq \arg\lambda_2$, then

$$\text{(10)} \qquad \ell(m) \sim \begin{cases} S_1(m) & -\pi+\psi_1 < \arg m < \psi_1, \\ -S_2(m) & \psi_1 < \arg m < \pi+\psi_1, \end{cases}$$

where $\psi_1=\tan^{-1}\{\log(|\lambda_1|/|\lambda_2|)/\arg(\lambda_1/\lambda_2)\}$, and if $\arg\lambda_1<\arg\lambda_2$, then

$$(11) \qquad \ell(m) \sim \begin{cases} -S_2(m) & -\pi-\phi_0 < \arg m < -\phi_0, \\ S_1(m) & -\phi_0 < \arg m < \pi-\phi_0, \end{cases}$$

where $\phi_0 = \tan^{-1}\{\log(|\lambda_1|/|\lambda_2|)/\arg(\lambda_2/\lambda_1)\}$. The $S_i(m)$ $(i=1, 2)$ are formal solutions of (8) of the form

$$\begin{cases} S_1(m) = \lambda_1^m m^{-\alpha}\left\{s_1 + \frac{s_1'}{m} + \frac{s_1''}{m^2} + \cdots\right\}, \\ S_2(m) = \lambda_2^m m^{\beta-1}\left\{s_2 + \frac{s_2'}{m} + \frac{s_2''}{m^2} + \cdots\right\}, \end{cases}$$

where $s_1 = (-\lambda_1)^{\alpha-1}(\lambda_1-\lambda_2)^{-\beta}\Gamma(\alpha)$ and $s_2 = (-\lambda_2)^{-\beta}(\lambda_2-\lambda_1)^{\alpha-1}\Gamma(-\beta+1)$. From the definition (7) and (9), and the above result, we have thus known completely the behavior of the coefficient $G_1(m)$.

Now we proceed to the analysis of the connection problem. There exist formal solutions of (1) of the form

$$(12) \qquad Y^k(t) = e^{\lambda_k t} t^{\mu_k} \sum_{s=0}^{\infty} H^k(s) t^{-s} \qquad \left( H^k(0) = \begin{pmatrix} \delta_{1k} \\ \delta_{2k} \end{pmatrix};\ k=1, 2 \right),$$

$\delta_{ij}$ being the Kronecker's delta. According to our general theory [4], we can define the functions

$$(13) \qquad F_i^k(m) = \sum_{s=0}^{\infty} H^k(s) g_i^k(m+s) \qquad (i, k=1, 2),$$

where

$$(14) \qquad g_i^k(m) = \frac{\lambda_k^{m+\rho_i-\mu_k}}{\Gamma(m+\rho_i-\mu_k+1)} \qquad (i=1, 2),$$

and can then prove that for each $i$, $F_i^k(m)$ $(k=1, 2)$ form a fundamental set of solutions of (4). Hence, by the theory of linear difference equations, we have

$$(15) \qquad G_i(m) = \sum_{k=1}^{2} T_i^k(m) F_i^k(m) \qquad (i = 1, 2),$$

where the $T_i^k(m)$ are periodic functions of period 1.

By means of the behavior of $G_i(m)$ and $F_i^k(m)$, we can then determine the $T_i^k(m)$ explicitly, which are actually constants and become the connection coefficients (the Stockes multipliers). Consider the $T_1^k(m)$ $(k=1, 2)$, since the $T_2^k(m)$ $(k=1, 2)$ can be given by interchanging $\alpha$ and $\beta$. From the Cramer rule it immediately follows that

$$T_1^k(m) = |G_1(m), F_1^{k'}(m)|/|F_1^k(m), F_1^{k'}(m)| \qquad (k \neq k';\ k, k'=1, 2).$$

They are holomorphic in the right half $m$-plane, since the denominator is the Casoratian, which does not vanish. We now investigate the behavior of the

$T_1^k(m)$ in some period strip $N \leqq \operatorname{Re} m < N+1$, $N$ being a sufficiently large positive integer. Taking account of the asymptotic behavior

$$F_1^k(m) = H^k(0) g_1^k(m)\{1+O(m^{-1})\}$$

in the right half $m$-plane, we have

$$\begin{aligned}(16) \qquad T_1^k(m) &= |G_1(m)/g_1^k(m),\ H^{k\prime}(0)\{1+O(m^{-1})\}| \\ &\quad \div |H^k(0),\ H^{k\prime}(0)|\{1+O(m^{-1})\}.\end{aligned}$$

So we have only to investigate the behavior of $G_1(m)/g_1^k(m)$ near both ends $\operatorname{Im} m \to \pm\infty$ of the period strip. For simplicity, dropping the constant factor $(1-e^{2\pi i\alpha})(1-e^{-2\pi i\beta})$, we first consider the behavior of

$$\begin{cases} f_1(m) = \dfrac{\ell(m+1)}{\Gamma(m+1)} \\ f_2(m) = \dfrac{1}{c}\{(m+\alpha)f_1(m) - \lambda_1 f_1(m-1)\} \\ \qquad\quad = \dfrac{1}{c}\dfrac{1}{\Gamma(m)}\left\{\left(1+\dfrac{\alpha}{m}\right)\ell(m+1) - \lambda_1 \ell(m)\right\}. \end{cases}$$

From (10) and (11), if $\arg\lambda_1 \geqq \arg\lambda_2$ ($\arg\lambda_1 < \arg\lambda_2$), we have

$$\text{(i)} \quad \begin{cases} f_1(m) \sim \dfrac{S_1(m+1)}{\Gamma(m+1)} = \dfrac{\lambda_1^{m+1} m^{-\alpha}}{\Gamma(m+1)}\left\{s_1 + \dfrac{d_1}{m} + \cdots\right\} & (d_1 = s_1' - s_1\alpha) \\ f_2(m) \sim \dfrac{1}{c}\dfrac{\lambda_1^{m+1} m^{-\alpha}}{\Gamma(m+1)}\left\{\dfrac{e_1}{m} + \cdots\right\} & (e_1\colon \textit{constant}) \end{cases}$$

near the lower (upper) end of the period strip and

$$\text{(ii)} \quad \begin{cases} f_1(m) \sim -\dfrac{S_2(m+1)}{\Gamma(m+1)} = -\dfrac{\lambda_2^{m+1} m^{\beta-1}}{\Gamma(m+1)}\left\{s_2 + \dfrac{d_2}{m} + \cdots\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad (d_2 = s_2' + (\beta-1)s_2) \\ f_2(m) \sim -\dfrac{1}{c}\dfrac{\lambda_2^m m^{\beta-1}}{\Gamma(m)}\left\{(\lambda_2-\lambda_1)s_2 + \dfrac{e_2}{m} + \cdots\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad (e_2\colon \textit{constant}) \end{cases}$$

near the upper (lower) end of the period strip. Hence in case (i) we have

$$G_1(m)/g_1^1(m) = \lambda_1^{1-\alpha}\begin{pmatrix} s_1 \\ 0 \end{pmatrix} + O(m^{-1}),$$

$$G_1(m)/g_1^2(m) = m^{-(\alpha+\beta)}\left(\frac{\lambda_1}{\lambda_2}\right)^m \left\{\begin{pmatrix} s_1\lambda_2^\beta \\ 0 \end{pmatrix} + O(m^{-1})\right\}$$

and in case (ii)

$$G_1(m)/g_1^1(m) = m^{(\alpha+\beta)}\left(\frac{\lambda_2}{\lambda_1}\right)^m\left\{\begin{pmatrix} 0 \\ \dfrac{\lambda_1-\lambda_2}{c\lambda_1^\alpha}s_2 \end{pmatrix}+O(m^{-1})\right\},$$

$$G_1(m)/g_1^2(m) = \frac{\lambda_2^\beta(\lambda_1-\lambda_2)}{c}\begin{pmatrix} 0 \\ s_2 \end{pmatrix} + O(m^{-1}).$$

Substituting these results into (16), we can see that the $T_1^k(m)$ $(k=1, 2)$ behave like

$$\text{(17)} \quad \begin{cases} T_1^1(m) = \lambda_1^{1-\alpha}s_1\{1+O(m^{-1})\}, \\ T_1^1(m) = \left(\dfrac{\lambda_2}{\lambda_1}\right)^m m^{\alpha+\beta}O(m^{-1}) \end{cases}$$

$$\text{(18)} \quad \begin{cases} T_1^2(m) = \dfrac{\lambda_2^\beta(\lambda_1-\lambda_2)}{c}s_2\{1+O(m^{-1})\}, \\ T_1^2(m) = \left(\dfrac{\lambda_1}{\lambda_2}\right)^m m^{-(\alpha+\beta)}O(m^{-1}) \end{cases}$$

near both ends of the period strip. Now the transformation $z=e^{2\pi im}$ maps the unit period strip on the entire $z$-plane, where both ends of the strip correspond to $z=0$ and $z=\infty$. So the functions $T_1^k(z)=T_1^k(m)$ $(k=1, 2)$ are holomorphic at every point of $z$-plane except possibly at $z=0$ and $z=\infty$. However, from (17) and (18) we can see that the singularities are removable because

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{\pm m} m^{\pm(\alpha+\beta)-1} = z^{\pm\frac{1}{2\pi}\arg\left(\frac{\lambda_2}{\lambda_1}\right)\pm\frac{1}{2\pi i}\log\left|\frac{\lambda_2}{\lambda_1}\right|}\left(\frac{\log z}{2\pi i}\right)^{\pm(\alpha+\beta)-1}$$

with

$$-\frac{1}{2} < \frac{1}{2\pi}\arg\left(\frac{\lambda_2}{\lambda_1}\right) \leqq \frac{1}{2}.$$

Hence from the Liouville theorem it follows that the $T_1^k(z)$ are constant and equal to the limiting values as $z\to 0$ or $z\to\infty$, that is,

$$\begin{cases} T_1^1(m) = \lambda_1^{1-\alpha}s_1 = (-1)^{\alpha-1}(\lambda_1-\lambda_2)^{-\beta}\Gamma(\alpha), \\ T_1^2(m) = \dfrac{\lambda_2^\beta(\lambda_1-\lambda_2)}{c}s_2 = \dfrac{1}{c}(-1)^{-\beta+1}(\lambda_2-\lambda_1)^\alpha\Gamma(1-\beta). \end{cases}$$

The required $T_1^k$ $(k=1, 2)$ are given by multiplying the above values by the factor $(1-e^{2\pi i\alpha})(1-e^{-2\pi i\beta})$, and moreover $T_2^k$ $(k=1, 2)$ are given by $T_1^k$ $(k=1, 2)$ in which $\alpha$ and $\beta$ are interchanged.

Combining the above with our previous results [4], we have the following

THEOREM. *A fundamental set of convergent power series solutions* $(X_1(t), X_2(t))$ *admits the asymptotic expansion*

$$(X_1(t), X_2(t)) \sim (Y^1(t), Y^2(t)) \begin{pmatrix} e^{2\pi i \alpha l_1} T_1^1 & e^{2\pi i \beta l_1} T_2^1 \\ e^{-2\pi i \beta l_2} T_1^2 & e^{-2\pi i \alpha l_2} T_2^2 \end{pmatrix} \tag{19}$$

*as* $t \to \infty$ *in the sector*

$$\mathscr{D}(l_1, l_2) = \left\{ |\arg \lambda_1 t - 2\pi l_1| < \frac{3}{2}\pi \right\} \cap \left\{ |\arg \lambda_2 t - 2\pi l_2| < \frac{3}{2}\pi \right\},$$

$l_1$, $l_2$ *being integers, where*

$$\begin{cases} T_1^1 = (1-e^{2\pi i \alpha})(1-e^{-2\pi i \beta})(-1)^{\alpha-1}(\lambda_1-\lambda_2)^{-\beta}\Gamma(\alpha), \\ T_1^2 = (1-e^{2\pi i \alpha})(1-e^{-2\pi i \beta})(-1)^{-\beta+1}(\lambda_2-\lambda_1)^{\alpha}\Gamma(1-\beta)/c, \\ T_2^1 = (1-e^{2\pi i \beta})(1-e^{-2\pi i \alpha})(-1)^{\beta-1}(\lambda_1-\lambda_2)^{-\alpha}\Gamma(\beta), \\ T_2^2 = (1-e^{2\pi i \beta})(1-e^{-2\pi i \alpha})(-1)^{-\alpha+1}(\lambda_2-\lambda_1)^{\beta}\Gamma(1-\alpha)/c. \end{cases} \tag{20}$$

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*Department of Mathematics,*
*Faculty of Science,*
*Hiroshima University*