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Picard principle for linear elliptic differential operators

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Take the punctured Euclidean unit n-ball Ω: $0<|x|<1$ $(x=(x_1,\cdots,x_n),\ n\geq 2)$. Throughout this paper we regard Ω as the subspace of the punctured Euclidean n-space M: $0<|x|<\infty$, so that the topological notions such as boundaries and closures etc. are considered relative to the "whole" space M. Hence $|x|=1$ is the boundary $\partial\Omega$ of Ω, $x=0$ is the ideal boundary of Ω, and the relative closure $\bar{\Omega}$ of Ω is $\Omega\cup\partial\Omega$. Consider an elliptic partial differential equation

$$Lu(x) \equiv \Delta u(x) + b(x)\cdot\nabla u(x) + c(x)u(x) = 0 \tag{1}$$

on Ω, where $\Delta=\sum_{i=1}^{n}\partial^2/\partial x_i^2$, $\nabla=(\partial/\partial x_1,\cdots,\partial/\partial_{x_n})$, $\cdot$ the inner product, and the vector field $b(x)=(b_1(x),\cdots,b_n(x))$ is of class C^2 on $\bar{\Omega}=\{0<|x|\leq 1\}$ and the function $c(x)$ of class C^1 on $\bar{\Omega}$ which may not be of constant sign. Thus the operator L is smooth on $\bar{\Omega}$ and especially on $\partial\Omega$: $|x|=1$, but may, and actually will, have singularities at $x=0$. We are interested in the class $\mathscr{P}$ of the nonnegative solutions of (1) on Ω with vanishing boundary values on $\partial\Omega$. It is convenient to consider the normalized subclass $\mathscr{P}_1$ of $\mathscr{P}$ given by $\mathscr{P}_1=\{u\in\mathscr{P}:\ \int_{\partial\Omega}(\partial/\partial n_x)u(x)dS_x=1\}$, where $(\partial/\partial n_x)u(x)$ denotes the inner normal derivative of $u(x)$ at each point of $\partial\Omega$ whose existence is well known since $u(x)$ vanishes on $\partial\Omega$ (cf. e.g. Miranda [6]) and dS the surface element on $\partial\Omega$. Since $\mathscr{P}_1$ is convex, we can consider the set ex. $\mathscr{P}_1$ of extreme points of $\mathscr{P}_1$ and the cardinal number $\sharp(\text{ex.}\,\mathscr{P}_1)$ of ex. $\mathscr{P}_1$, which will be referred to as the *Picard dimension* of L at $x=0$, $\dim L$ in notation:

$$\dim L = \sharp(\text{ex.}\,\mathscr{P}_1). \tag{2}$$

We are particularly interested in the case $\dim L=1$. In this case we say, after Bouligand, that the *Picard principle* is valid for L at $x=0$. We will give a sufficient condition for its validity in terms of the orders of the growth of coefficients of L.

It can happen that $\dim L=0$. To prevent this trivial case we need to consider the existence of "Green's function" on Ω. For any point y fixed in Ω take a ball U: $|x-y|<a$ in Ω. If U is sufficiently small, then the Green's function (with respect to the Dirichlet problem) $g_U(x,y)$ on U for (1) with its pole y exists (cf. e.g. [6]). Consider a positive solution $u(x)$ of (1) on $\Omega-\{y\}$ satisfying the following two conditions: (i) $u(x)-g_U(x,y)$ is a solution of (1) on U; (ii) if $v(x)$

is a solution of (1) on Ω dominated by $u(x)$ on $\Omega-\{y\}$, then $v(x)$ is nonpositive on Ω. In view of (i) and (ii) such a function $u(x)$ is unique provided that it exists. In this case, we denote the function $u(x)$ by $G_\Omega(x, y)$ and call it as the *Green's function* on Ω for (1) with its pole y. It is easy to see that the existence of the Green's function $G_\Omega(x, y)$ does not depend on the choice of y. The boundary values of $G_\Omega(x, y)$ on $\partial\Omega$ is zero but $G_\Omega(x, y)$ may have diverse behaviors at $x=0$ depending on the operator L. It is also easy to see that $\dim L>0$ if and only if there exists the Green's function $G_\Omega(x, y)$ on Ω for (1) (cf. e.g. Nakai [7]). The purpose of this paper is to establish the following:

THEOREM. *Suppose the existence of the Green's function $G_\Omega(x, y)$ on Ω for* (1) *and assume that*

$$(3) \qquad \begin{aligned} |b(x)| &= O(|x|^{-1}), \qquad |\nabla\cdot b(x)| = O(|x|^{-2}), \\ |c(x)| &= O(|x|^{-2}) \qquad (x\to 0). \end{aligned}$$

Then the Picard principle is valid for L at $x=0$.

Actually we will prove the theorem under a weaker assumption that the condition (3) holds only on a certain sequence of annuli converging to $x=0$. To be more precise, let $\{A_m\}$ be a sequence of annuli in Ω with the condition [A] in the sense of Kawamura [5], that is, each A_m is given by $(1-a)r_m\leq|x|\leq(1+a)r_m$, where a is any fixed number in $(0, 1)$ and $\{r_m\}$ is an arbitrarily fixed sequence in $(0, (1+a)^{-1})$ with $r_{m+1}<((1-a)/(1+a))r_m$ for each m. Then we can *generalize* the above theorem by relaxing the condition (3) with the following:

$$(4) \qquad |b(x)| \leq c_0/|x|, \quad |\nabla\cdot b(x)| \leq c_1/|x|^2, \quad |c(x)| \leq c_2/|x|^2$$

on $\bigcup_{m=1}^\infty A_m$ with some $a>0$, where c_0, c_1 and c_2 are constants. We wish once more to stress the following: We only assume the condition (4) for the coefficients of L on an arbitrarily given $\bigcup_{m=1}^\infty A_m$ no matter how a is small and also we do not care how wildly these coefficients behave in the remaining disjoint annuli $\Omega-\bigcup_{m=1}^\infty A_m$. Our result generalizes the corresponding one by Kawamura [5] where additional assumptions $b(x)=0$ and $c(x)\leq 0$ are made.

Although, in the general framework, various conditions for the existence of the Green's function are known (cf. e.g. Ito [4] among others), it is difficult to determine the existence explicitely in terms of $b(x)$ and $c(x)$. We will remark in no. 7 by an example that (3) can not assure the existence of the Green's function, and therefore we have to assume it in addition to the condition (3).

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§1. The maximum-minimum principle

1. Let Ω_s: $0<|x|<s$ ($s\in(0, 1]$), and $\partial\Omega_s$: $|x|=s$, with $\Omega_1=\Omega$ and $\partial\Omega_1=\partial\Omega$. For each fixed s in $(0, 1]$ take an exhaustion $\{N\}$ of M, i.e. a directed net of relatively compact subregions N of M with $\cup N=M$, such that each N contains $(\partial\Omega)\cup(\partial\Omega_s)$ and its boundary ∂N is of class C^3. Hence $\{\Omega_s\cap N\}$ forms an exhaustion of Ω_s toward $x=0$. The Green's function on Ω_s for (1) with its pole y is denoted by $G_{\Omega_s}(x, y)$. We recall that the adjoint equation of (1) on Ω is given by

$$L^*u(x) \equiv \Delta u(x) - b(x)\cdot\nabla u(x) + (c(x)-\nabla\cdot b(x))u(x) = 0. \tag{5}$$

For each ψ in $C(\partial\Omega_s)$ denote by $D^*(\Omega_s\cap N;\, \partial\Omega_s)\psi$ the solution of (5) on $\Omega_s\cap N$ with boundary values ψ on $\partial\Omega_s$ and zero on ∂N. The solvability of the Dirichlet problem on the relatively compact subregion $\Omega_s\cap N$ in $\bar{\Omega}$ is assured by the existence of $G_\Omega(x, y)$ (cf. e.g. [6]). We also denote by $G_{\Omega_s\cap N}(x, y)$ the Green's function on $\Omega_s\cap N$ for (1) with its pole y in $\Omega_s\cap N$. Then by the Green's formula

$$(D^*(\Omega_s\cap N;\, \partial\Omega_s)\psi)(y) = \int_{\partial\Omega_s}\psi(x)\frac{\partial}{\partial n_x}G_{\Omega_s\cap N}(x, y)dS_x.$$

The convergence of $\{(\partial/\partial n_x)G_{\Omega_s\cap N}(x, y)\}$ to $(\partial/\partial n_x)G_{\Omega_s}(x, y)$ as $N\to M$ is uniform on $\partial\Omega_s$ for each y fixed in Ω_s. Thus

$$\lim_{N\to M}(D^*(\Omega_s\cap N;\, \partial\Omega_s)\psi)(y) = \int_{\partial\Omega_s}\psi(x)\frac{\partial}{\partial n_x}G_{\Omega_s}(x, y)dS_x$$

exists. We set

$$D^*(\Omega_s;\, \partial\Omega_s)\psi = \lim_{N\to M}D^*(\Omega_s\cap N;\, \partial\Omega_s)\psi.$$

In particular, for $s=1$ and $\psi=1$ we set

$$e_\Omega(y) = (D^*(\Omega;\, \partial\Omega)1)(y). \tag{6}$$

Then the *associated equation* with (1) is, by definition, given by

$$\hat{L}u(x) \equiv \Delta u(x) + (2\nabla\log e_\Omega(x)-b(x))\cdot\nabla u(x) = 0 \tag{7}$$

on Ω. We denote by $\mathscr{B}(\Omega_s;\, \hat{L})$ the linear space of bounded solutions of (7) on Ω_s with continuous boundary values on $\partial\Omega_s$. Then the following relation holds (cf. Nakai [7]):

$$\mathscr{B}(\Omega_s;\, \hat{L}) = \{(D^*(\Omega_s;\, \partial\Omega_s)\psi)/e_\Omega;\, \psi\in C(\partial\Omega_s)\}. \tag{8}$$

In [7] the above relation was shown only for $s=1$ but we can easily see that it is valid for any s in $(0, 1]$ with trivial modifications.

2. Take any bounded solution v in $\mathscr{B}(\Omega; \hat{L})$. We set $M_s = \sup_{\Omega_s} v$ and $m_s = \inf_{\Omega_s} v$ for each s in $(0, 1]$. Then $M_s - v$ is the solution of (7) on Ω_s with nonnegative boundary values $M_s - v$ on $\partial\Omega_s$ and $v - m_s$ the solution of (7) on Ω_s with nonnegative boundary values $v - m_s$ on $\partial\Omega_s$. Setting $(M_s - v)e_\Omega|_{\partial\Omega_s} = \psi_s$ and $(v - m_s)e_\Omega|_{\partial\Omega_s} = \phi_s$, observe that $(D^*(\Omega_s \cap N; \partial\Omega_s)\psi_s)/e_\Omega$ is the solution of (7) on $\Omega_s \cap N$ with boundary values $M_s - v$ on $\partial\Omega_s$ and zero on ∂N, and $(D^*(\Omega_s \cap N; \partial\Omega_s)\phi_s)/e_\Omega$ the solution of (7) on $\Omega_s \cap N$ with boundary values $v - m_s$ on $\partial\Omega_s$ and zero on ∂N. The maximum principle for solutions of the equation (7) whose term of zero order differentiation is missing (cf. e.g. [6]) yields that

$$\max_{\partial\Omega_s}(M_s - v) \geq (D^*(\Omega_s \cap N; \partial\Omega_s)\psi_s)/e_\Omega,$$
$$\max_{\partial\Omega_s}(v - m_s) \geq (D^*(\Omega_s \cap N; \partial\Omega_s)\phi_s)/e_\Omega$$

on $\Omega_s \cap N$. Letting $N \to M$, we have

$$\max_{\partial\Omega_s}(M_s - v) \geq (D^*(\Omega_s; \partial\Omega_s)\psi_s)/e_\Omega,$$
$$\max_{\partial\Omega_s}(v - m_s) \geq (D^*(\Omega_s; \partial\Omega_s)\phi_s)/e_\Omega$$

on Ω_s. Thus it follows from (8) that

$$M_s - \min_{\partial\Omega_s} v \geq M_s - v, \qquad \max_{\partial\Omega_s} v - m_s \geq v - m_s,$$

i.e. $\max_{\partial\Omega_s} v \geq v \geq \min_{\partial\Omega_s} v$ on Ω_s. Consequently we have the following:

LEMMA 1. *For any s in $(0, 1]$ and each v in $\mathscr{B}(\Omega; \hat{L})$ the following maximum-minimum principle is valid*:

$$\sup_{\Omega_s} v = \max_{\partial\Omega_s} v, \qquad \inf_{\Omega_s} v = \min_{\partial\Omega_s} v.$$

§2. Transformations of solutions of $L^*u(x) = 0$

3. Consider the transformation $x = \mathrm{T}_m\zeta$ defined by $\mathrm{T}_m\zeta = r_m\zeta$ for each positive integer m. We set $\Delta = \mathrm{T}_m^{-1}(\mathrm{A}_m)$: $(1-a) \leq |\zeta| \leq (1+a)$, and $f_m(\zeta) = f(\mathrm{T}_m\zeta)$ for the coefficients and the solutions of (5). We assume that (4) holds on each A_m. Then it is easy to see that

$$L^*u(x) = 0 \qquad \text{on} \quad \mathrm{A}_m$$

is transformed to the equation

$$L_m^* u_m(\zeta) = 0 \qquad \text{on} \quad \Delta, \tag{9}$$

where

$$L_m^* u_m(\zeta) = \Delta u_m(\zeta) - r_m b_m(\zeta) \cdot \nabla u_m(\zeta) + (r_m^2 c_m(\zeta) - r_m \nabla \cdot b_m(\zeta)) u_m(\zeta),$$
$$|r_m b_m(\zeta)| \leq c_0/|\zeta|, \quad |r_m^2 c_m(\zeta) - r_m \nabla \cdot b_m(\zeta)| \leq (c_1 + c_2)/|\zeta|^2$$

on Δ (cf. Gilbarg-Serrin [1]).

We set $\xi=\zeta-\alpha$ for each α fixed in Γ: $|\zeta|=1$, and

$$u_m(\zeta) = (1-B|\xi|^2)U_{m,B}(\xi)$$

for a positive constant B (cf. [1]). Denote by $\bar{B}(\alpha, a)$ the closed ball in Δ with center α and radius a and by $\bar{B}(0, a)$ the closed ball with center at the origin and radius a. If $1-Ba^2>0$, then we can easily see that the equation

$$L_m^* u_m(\zeta) = 0 \qquad \text{on} \quad \bar{B}(\alpha, a)$$

is transformed to the equation

$$L_{m,B}^* U_{m,B}(\xi) = 0 \qquad \text{on} \quad \bar{B}(0, a), \tag{10}$$

where

$$L_{m,B}^* U_{m,B}(\xi) = \Delta U_{m,B}(\xi) - E_{m,B}(\xi)\cdot \nabla U_{m,B}(\xi) + F_{m,B}(\xi)U_{m,B}(\xi),$$

$$\begin{cases} E_{m,B}(\xi) = \dfrac{4B\xi}{1-B|\xi|^2} + r_m b_m(\alpha+\xi), \\[2ex] F_{m,B}(\xi) = \dfrac{2B}{1-B|\xi|^2}(-n+r_m\xi\cdot b_m(\alpha+\xi)) \\[2ex] \qquad\qquad + r_m^2 c_m(\alpha+\xi) - r_m \nabla\cdot b_m(\alpha+\xi). \end{cases} \tag{11}$$

4. *An estimate of $F_{m,B}(\xi)$.* We will choose a pair of positive constants a and B such that $F_{m,B}(\xi)\le 0$ on $\bar{B}(0, a)$ for any m. We first observe that the following inequalities hold on $\bar{B}(0, a)$:

$$\begin{cases} |r_m b_m(\alpha+\xi)| \le \dfrac{c_0}{1-|\xi|}, \\[2ex] |r_m\xi\cdot b_m(\alpha+\xi)| \le \dfrac{c_0|\xi|}{1-|\xi|}, \\[2ex] |r_m^2 c_m(\alpha+\xi) - r_m\nabla\cdot b_m(\alpha+\xi)| \le \dfrac{c_1+c_2}{(1-|\xi|)^2}. \end{cases} \tag{12}$$

If we choose a and B so as to satisfy $a<1/(1+c_0)$ and $0<1-Ba^2$, then

$$F_{m,B}(\xi) \le \frac{2Bn}{1-B|\xi|^2}\left(-1+\frac{c_0|\xi|}{1-|\xi|}\right) + \frac{c_1+c_2}{(1-|\xi|)^2} \tag{13}$$

on $\bar{B}(0, a)$. Observe that

$$0 < 2n(1-a)(1-(c_0+1)a) \le (c_1+c_2)|\xi|^2$$
$$\qquad + 2n(1-|\xi|)(1-(c_0+1)|\xi|), \quad -1 + c_0|\xi|/(1-|\xi|) < 0$$

on $\bar{B}(0, a)$. We can choose a and B in such a way that the right hand side of (13) is not positive on $\bar{B}(0, a)$ or equivalently

$$\frac{c_1+c_2}{(c_1+c_2)|\xi|^2+2n(1-|\xi|)(1-(c_0+1)|\xi|)} \leq B < \frac{1}{a^2}.$$

In fact it suffices to choose a and B so as to satisfy

$$\frac{c_1+c_2}{2n(1-a)(1-(c_0+1)a)} \leq B < \frac{1}{a^2}. \tag{14}$$

Then $a>0$ must satisfy

$$0 < 2n - 2n(c_0+2)a + (2n(c_0+1)-(c_1+c_2))a^2,$$

or

$$0 < a < \frac{2}{(c_0+2)+[c_0^2+2n^{-1}(c_1+c_2)]^{1/2}}. \tag{15}$$

We compile the above argument in:

LEMMA 2. *If a satisfies* (15) *and B satisfies* (14), *then*

$$F_{m,B}(\xi) \leq 0$$

on $\bar{B}(0, a)$ for each positive integer m.

In this case the following inequalities are valid on $\bar{B}(0, a)$:

$$|E_{m,B}(\xi)| \leq \frac{4Ba}{1-Ba^2} + \frac{c_0}{1-a},$$

$$-F_{m,B}(\xi) \leq \frac{2Bn}{1-Ba^2}\left(1+\frac{c_0 a}{1-a}\right) + \frac{c_1+c_2}{(1-a)^2}.$$

We remark that the last two inequalities in the above lemma can be derived from (11) and (12).

§3. A boundary Harnack principle

5. Choose any a satisfying (15) and let $\{r_m\}$ be an arbitrary sequence in $(0, (1+a)^{-1})$ with $r_{m+1}<((1-a)/(1+a))r_m$ for each m. Denote by $\{A_m\}$ the sequence of annuli on Ω with the condition [A] which is determined by $\{r_m\}$ and a. Then our boundary Harnack principle for L^* and $\hat{L}$ can be stated as follows (cf Kawamura[5]):

LEMMA 3. *Suppose the existence of the Green's function on Ω for* (1) *and assume that* (4) *holds on $\cup_{m=1}^{\infty} A_m$. Let u be any nonnegative solution of* (5) *on Ω_s with an arbitrary s in* $(0, 1]$. *Then there exists a constant $K>1$ which is independent of m, s and u such that*

$$u(x_1) \leq Ku(x_2)$$

for any x_1 and x_2 on $|x|=r_m$ and for each r_m satisfying $r_m(1+a)<s$. Consequently, each nonnegative v in $\mathscr{B}(\Omega_s, \hat{L})$ satisfies

$$v(x_1) \leq K^2 v(x_2)$$

for any x_1 and x_2 as above.

Using the notation in no. 3, the positive solution $u(x)$ of (5) on A_m is transformed to the solution $u_m(\zeta)$ of (9) on Δ and $u_m(\zeta)$ on $\bar{B}(\alpha, a)$ with each α in Γ is transformed to the solution $U_{m,B}(\xi)$ of (10) on $\bar{B}(0, a)$ for each m such that $r_m(1+a)<s$, where the constant B is chosen so as to satisfy (14). Then it follows from Lemma 2 and the Harnack inequality (Serrin [8]) that there exists a positive constant k which depends only on a, c_0, c_1, c_2, B and n and is independent of m and s such that $U_{m,B}(\xi_1) \leq kU_{m,B}(\xi_2)$ for any ξ_1 and ξ_2 in $\bar{B}(0, a/4)$. Returning to $u_m(\zeta)$,

$$u_m(\zeta_1) \leq \frac{4^2 k}{4^2 - Ba^2} u_m(\zeta_2) \tag{16}$$

for any ζ_1 and ζ_2 in $\bar{B}(\alpha, a/4)$. Since Γ is compact, we can find a positive integer N and the points $\{\zeta_j\}_{j=1}^N$ in Γ such that $\Gamma \subset \cup_{j=1}^N \bar{B}(\zeta_j, a/4)$. Applying the inequality (16) at most N times

$$u_m(\zeta_1) \leq \left(\frac{4^2 k}{4^2 - Ba^2}\right)^N u_m(\zeta_2)$$

for any ζ_1 and ζ_2 in Γ. Setting

$$K = \left(\frac{4^2 k}{4^2 - Ba^2}\right)^N,$$

it is clear that K is independent of m and s. Changing ζ to the original variables x, we obtain $u(x_1) \leq Ku(x_2)$ for any x_1 and x_2 on $|x|=r_m$ and for each r_m such that $r_m(1+a)<s$. Then the boundary Harnack principle for $\hat{L}$ follows instantly from that for L^* (cf. [5]). This completes the proof.

6. *The proof of the theorem.* The essence of our proof is almost identical with Kawamura [5] once preparations in nos. 1–5 are established. But we will give it here for the sake of completeness. The proof will be given for the theorem in the sharper form mentioned in the introductory part with the condition (4) only assumed on $\cup_{m=1}^\infty \mathrm{A}_m$, where $\{\mathrm{A}_m\}$ is an arbitrary sequence of annuli in Ω with the condition [A]. We may assume that a satisfies (15).

To assert the validity of the Picard principle for L at $x=0$ it suffices to show that $\lim_{x\to 0} v(x)$ exists for every v in $\mathscr{B}(\Omega; \hat{L})$ in view of the duality theorem (Nakai [7], cf. also Heins [3] and Hayashi [2]). Let $\{r_m\}$ be the sequence in $(0, (1+a)^{-1})$ which determines $\{A_m\}$. For each v in $\mathscr{B}(\Omega; \hat{L})$ we set $M(r_m)=\sup_{\Omega_m} v(x)$, where Ω_m: $0<|x|<r_m$. Then $M(r_m)-v(x)$ is a positive element of $\mathscr{B}(\Omega_n; \hat{L})$ for each n: $n>m$. It follows from Lemmas 1 and 3 that

$$\sup_{\Omega_n}(M(r_m)-v(x)) \leq K^2(\inf_{\Omega_n}(M(r_m)-v(x)))$$

for each n: $n>m$. Letting $n\to\infty$ for a fixed m,

$$M(r_m) - \liminf_{x\to 0} v(x) \leq K^2(M(r_m)-\limsup_{x\to 0} v(x)).$$

As $m\to\infty$, we deduce that $\limsup_{x\to 0} v(x)-\liminf_{x\to 0} v(x)=0$, i.e. $\lim_{x\to 0} v(x)$ exists for each v in $\mathscr{B}(\Omega; \hat{L})$. This completes the proof.

7. *An example.* Let $x=(r, \Theta)=(r, \theta_1,\cdots, \theta_{n-1})$: $|x|=r$, be the spherical coordinates on $\bar{\Omega}$ and denote by $d\Theta$ the surface element of $\partial\Omega$. Consider

$$(17) \qquad Lu(x) \equiv \Delta u(x) - (n-2)\left(\frac{x_1}{|x|^2},\cdots, \frac{x_n}{|x|^2}\right)\cdot\nabla u(x) + \frac{1}{|x|^2}u(x) = 0$$

on Ω. Suppose that there exists a positive solution $h(x)=h(r, \Theta)$ of (17) on Ω with boundary values zero on $\partial\Omega$. The function $h^*(r)$ defined by $\int_{\partial\Omega} h(r, \Theta)d\Theta$ is a positive solution of (17) on Ω. The linear space consisting of rotation free solutions, i.e. solutions with the variable r only, of (17) on Ω is generated by $\cos(\log r)$ and $\sin(\log r)$. Thus $h^*(r)=\alpha\cos(\log r)+\beta\sin(\log r)$ for some constants α and β. This contradicts $h^*(r)>0$ on Ω. Hence there exists no positive solutions of (17) on Ω with vanishing boundary values on $\partial\Omega$. Then it follows from a criterion (cf. [7], p. 285) that there exists no Green's function on Ω for (17). On the other hand, it is easily checked that $|b(x)|=(n-2)/|x|$, $|\nabla\cdot b(x)|=(n-2)^2/|x|^2$ and $c(x)=1/|x|^2$ on Ω. Therefore the assumption of the existence of the Green's function in our theorem can not be dispensed with.

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