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# Notes on Wishartness and independence of multivariate quadratic forms in correlated normal vectors

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**Summary.** This note concerns with a multivariate analogue of Ogasawara and Takahashi's theorem for the distribution of a quadratic form in correlated normal variates. Also we discuss on the independence of two such multivariate quadratic forms.

## 1. Introduction

Quadratic forms in univariate normal variates have been discussed for their distributions and independence between them by many authors, for example, Cochran [1], Craig [2], [3], Hotelling [5], Matusita [10], Sakamoto [15], Ogawa [13], [14], Kawada [6], Lancaster [9], Laha [8], and so on. Khatri [7] and Hogg [4] have extended the discussion to the multivariate case. Ogasawara and Takahashi [12] have considered the distribution of a quadratic form in correlated univariate normal variates, where their correlation matrix is positive semi-definite (p.s.d.). Khatri [7] and Hogg [4] have also treated the correlated and multivariate case but their correlation matrix is positive definte (p.d.).

In this note, we generalize Ogasawara and Takahashi's result to the multivariate case for the Wishartness in both central and noncentral cases. We also discuss the independence between two multivariate quadratic forms in the correlated case.

Let $V \sim W_p(\Sigma, n)$, $n \geq p$. Then it is well known that $\boldsymbol{a}'V\boldsymbol{a}/\boldsymbol{a}'\Sigma\boldsymbol{a} \sim \chi^2$ for all $\boldsymbol{a}$ such that $\boldsymbol{a}'\Sigma\boldsymbol{a} \neq 0$. If the converse of this proposition holds, multivariate generalization of Ogasawara and Takahashi's theorem is immediately obtained and nothing remains to do. It is however known from the counter-example given by Mitra [11] that the converse is not true. Thus we need to give a seperate proof of this generalization.

We also note on the independence of two multivariate quadratic forms in the correlated case, where the correlation matrix is p.s.d.

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## 2. Three known results

To establish the main theorems for the multivariate quadratic forms in the case of a p.s.d. correlation matrix, we use the following three known results:

LEMMA 2.1. (Ogasawara and Takahashi's theorem) *Let* $\boldsymbol{y}=(y_1,\dots,y_n)'$ *be a vector of n observations on a random variable y. Suppose* $\boldsymbol{y}\sim N_n(\boldsymbol{\xi},\Psi)$, *where* $\Psi$: $n\times n$ *is an arbitrary p.s.d. covariance matrix of rank* $\nu\leq n$. *Then the central quaratic form* $q(\boldsymbol{y})=(\boldsymbol{y}-\boldsymbol{\xi})'A(\boldsymbol{y}-\boldsymbol{\xi})$, *where* $A$: $n\times n$ *is symmetric, has a* $\chi^2$*-distribution if and only if*

$$\Psi A\Psi A\Psi = \Psi A\Psi \tag{2.1}$$

*holds. The degree of freedom (d.f.) of this* $\chi^2$*-distribution is tr* $(A\Psi)$.

LEMMA 2.2. (Khatri's theorem) *Let* $Y$: $p\times n=[\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$, $\boldsymbol{y}_1$*'s being independently distributed as* $N_p(\boldsymbol{\mu}_i,\Sigma)$, $\Sigma>0$. *Consider the second degree polynomial* $Q(Y)=YAY'+(1/2)(LY'+YL')+C$, *where* $A$: $n\times n$ *is symmetric,* $L$ *is* $p\times n$, *and* $C$: $n\times n$ *is symmetric. Then* $Q(Y)$ *has (noncentral) Wishart distribution if and only if*

$$\text{(i)}\quad A^2=A,\qquad \text{(ii)}\quad L=LA,\qquad \text{(iii)}\quad C=\frac{1}{4}LAL'. \tag{2.2}$$

LEMMA 2.3. (Khatri's theorem) *Consider two second degree polynomials in* $Y=[\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$, $\boldsymbol{y}_i$*'s being independently distributed as* $N_p(\boldsymbol{\mu}_i,\Sigma)$, $\Sigma>0$.

$$Q_1(Y)=YAY'+\frac{1}{2}(KY'+YK')+C,\qquad Q_2(Y)=YBY'+\frac{1}{2}(LY'+YL')+G,$$

*where* $A$, $B$ *are* $n\times n$ *symmetric matrices,* $K$, $L$ *are* $p\times n$, *and* $C$, $G$ *are* $p\times p$ *symmetric matrices. Then* $Q_1(Y)$ *and* $Q_2(Y)$ *are independently distributed if and only if*

$$\text{(i)}\quad AB=0,\qquad \text{(ii)}\quad LA=KB=0,\qquad \text{(iii)}\quad KL'=0. \tag{2.3}$$

## 3. Wishartness

A multivariate analogue of Ogasawara-Takahashi's theorem is stated as

THEOREM 3.1. *Let* $X=[\boldsymbol{x}_1,\dots,\boldsymbol{x}_n]$ *be a* $p\times n$ *observation matrix, where* $\boldsymbol{x}_r\sim N_p(\boldsymbol{\mu}_r,\Sigma)$, $\Sigma>0$. *Suppose that instead of independence among* $\boldsymbol{x}_r$*'s, they are correlated, i.e.,* $Cov(\boldsymbol{x}_r,\boldsymbol{x}_s)=\psi_{rs}\Sigma$, *where* $\psi_{rr}=1$. *Let* $\Psi=(\psi_{rs})$ *be p.s.d. with rank* $\nu\leq n$. *Then* $Q(X)=XAX'$, *where* $A$: $n\times n$ *is symmetric, is distributed according to a (noncentral) Wishart distribution if and only if*

$$\text{(i)}\ \Psi A\Psi A\Psi=\Psi A\Psi,\quad \text{(ii)}\ MA\Psi A\Psi=MA\Psi,\quad \text{(iii)}\ MA\Psi AM'=MAM', \tag{3.1}$$

*where* $M=[\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n]$. *The d.f. of this distribution is* $tr(A\Psi)$. *When* $\Psi>0$, *the conditions reduce to* $A\Psi A=A$.

PROOF. Since $\Sigma$ is p.d., let $Y=\Sigma^{-1/2}(X-M)=[\boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$. Then $\boldsymbol{y}_r\sim N_p(\boldsymbol{0}, I_p)$, $Cov(\boldsymbol{y}_r, \boldsymbol{y}_s)=\psi_{rs}I_p$, and

$$XAX' = \Sigma^{1/2}(Y+\Sigma^{-1/2}M)A(Y+\Sigma^{-1/2}M)'\Sigma^{1/2}.$$

Let $Y=[\boldsymbol{y}_1^*,\dots,\boldsymbol{y}_p^*]'$, where $\boldsymbol{y}_i^{*\prime}$ is the $i$th row vector of $Y$. Then $\boldsymbol{y}_i^*$'s are independently and identically distributed (i.i.d.) according to $N_n(\boldsymbol{0}, \Psi)$. We now represent $\boldsymbol{y}_i^*$ as

$$\boldsymbol{y}_i^* = C\boldsymbol{u}_i,\ i = 1,\dots, p,$$

where $C$ is an $n\times\nu$ matrix of rank $\nu$ such that $CC'=\Psi$, and $\boldsymbol{u}_i$'s are i.i.d. as $N_\nu(\boldsymbol{0}, I_\nu)$. Hence we put $U$: $\nu\times p=[\boldsymbol{u}_1,\dots,\boldsymbol{u}_p]$, then the columns of $U'$ are i.i.d. as $N_p(\boldsymbol{0}, I_p)$. Now we have

$$\begin{aligned}&(Y+\Sigma^{-1/2}M)A(Y+\Sigma^{-1/2}M)'\\&\quad = U'C'ACU + U'C'AM'\Sigma^{-1/2} + \Sigma^{-1/2}MACU + \Sigma^{-1/2}MAM'\Sigma^{-1/2},\end{aligned} \tag{3.2}$$

which is the second degree polynomial in $U'$. Hence it follows from Lemma 2.2 that the quadratic form in (3.2) has a noncentral Wishart distribution if and only if

$$\begin{aligned}&(C'AC)(C'AC) = (C'AC),\quad \Sigma^{-1/2}MAC = (\Sigma^{-1/2}MAC)(C'AC),\\&\Sigma^{-1/2}MAM'\Sigma^{-1/2} = (\Sigma^{-1/2}MAC)(C'AC)(C'AM'\Sigma^{-1/2}).\end{aligned}$$

Since $\Sigma$ is p.d., these conditions are clearly equivalent to those in (3.1), and hence $XAX'$ is distributed as a Wishart if and only if three conditions in (3.1) are satisfied. The d.f. of the resultant Wishart distribution is equal to the rank of $C'AC$, i.e., $\mathrm{tr}(C'AC)=\mathrm{tr}(ACC')=\mathrm{tr}(A\Psi)$. Q.E.D.

When $M=0$, we can extend the result to the case of a p.s.d. $\Sigma$, that is,

THEOREM 3.2. *Let* $X=[\boldsymbol{x}_1,\dots,\boldsymbol{x}_n]$ *be an observation matrix such that* $\boldsymbol{x}_r\sim N_p(\boldsymbol{0}, \Sigma)$, $\Sigma\geq 0$ *with rank* $\gamma\leq p$. *Suppose that* $Cov(\boldsymbol{x}_r, \boldsymbol{x}_s)=\psi_{rs}\Sigma$, *where* $\psi_{rr}=1$ *and* $\Psi=(\psi_{rs})\geq 0$ *with rank* $\nu\leq n$. *Then* $XAX'$, *where* $A$ *is an* $n\times n$ *symmetric matrix, is distributed as a (central) Wishart if and only if*

$$\Psi A\Psi A\Psi = \Psi A\Psi. \tag{3.3}$$

*The d.f. of this Wishart distribution is equal to* $tr(A\Psi)$. *When* $\Psi>0$, *the condition reduces to* $A\Psi A=A$.

PROOF. First of all, we write $\boldsymbol{x}_r=B\boldsymbol{y}_r$, where $\boldsymbol{y}_r\sim N_\gamma(\boldsymbol{0}, I_\gamma)$ and $B$ is a $p\times\gamma$ matrix of rank $\gamma$ such that $BB'=\Sigma$. Hence $XAX'=BYAY'B'$, where $Y=[\boldsymbol{y}_1,\dots,$

$\boldsymbol{y}_n]$, and $Cov(\boldsymbol{y}_r, \boldsymbol{y}_s)=\psi_{rs}I_\gamma$. Now let $Y=[\boldsymbol{y}_1^*,\dots, \boldsymbol{y}_\gamma^*]'$, where $\boldsymbol{y}_i^*$ is the $i$th row vector of $Y$ with $n$ components. Then $\boldsymbol{y}_i^*$'s are i.i.d. according to $N_n(\mathbf{0}, \Psi)$ and are represented as $\boldsymbol{y}_i^*=C\boldsymbol{u}_i$, $i=1,\dots, \gamma$, where $\boldsymbol{u}_i$'s are i.i.d. as $N_\nu(\mathbf{0}, I_\nu)$, and $C$ is an $n\times\nu$ matrix of rank $\nu$ such that $CC'=\Psi$. Hence if we put, as in the proof of Theorem 3.1, $U=[\boldsymbol{u}_1,\dots, \boldsymbol{u}_\gamma]$, then the columns of $U'$ are i.i.d. as $N_\gamma(\mathbf{0}, I_\gamma)$ and we have

$$YAY' = U'C'ACU.$$

Hence again by Lemma 2.2, $(C'AC)(C'AC)=(C'AC)$ or equivalently $\Psi A\Psi A\Psi=\Psi A\Psi$ is a necessary and sufficient condition for $YAY'$ to have a Wishart distribution with d.f. of $\mathrm{tr}(C'AC)=\mathrm{tr}(A\Psi)$, i.e. $YAY'\sim W_\gamma(I_\gamma, \theta)$, where $\theta=\mathrm{tr}(A\Psi)$. Thus if the condition (3.3) is satisfied, then $YAY'\sim W_\gamma(I_\gamma, \theta)$ and hence $XAX'=B(YAY')B'\sim W_p(\Sigma, \theta)$. Conversely, suppose $XAX'=(v_{ij})$ is distributed according to a central Wishart distribution $W_p(\Lambda, m)$, where $v_{ij}=\boldsymbol{x}_i^{*\prime}A\boldsymbol{x}_j^*$, $\boldsymbol{x}_i^{*\prime}$ being the $i$th row of $X$. Then it follows that the $i$th diagonal element $v_{ii}$ $(i=1,\dots, p)$ is distributed as $\lambda_{ii}\chi^2$, where $\Lambda=(\lambda_{ij})$. Since $\boldsymbol{x}_i^*\sim N_n(\mathbf{0}, \Psi)$, we have from Lemma 2.1 the condition $\Psi A\Psi A\Psi=\Psi A\Psi$. It turns out that $YAY'$ in (3.4) has $W_\gamma(I_\gamma, \theta)$, which implies $XAX'\sim W_p(\Sigma, \theta)$. Hence $\Lambda=\Sigma$ and $m=\theta$. Q. E. D.

## 4. Independence of two quadratic forms

THEOREM 4.1. *Let* $X=[\boldsymbol{x}_1,\dots, \boldsymbol{x}_n]$ *be* $p\times n$ *observation matrix such that* $\boldsymbol{x}_r\sim N_p(\boldsymbol{\mu}_r, \Sigma)$, $\Sigma>0$. *Suppose* $Cov(\boldsymbol{x}_r, \boldsymbol{x}_s)=\Psi_{rs}\Sigma$, *where* $\psi_{rr}=1$ *and* $\Psi=(\psi_{rs})\geq 0$ *with rank* $\nu\leq n$. *Then two quadratic forms* $XAX'$ *and* $XBX'$ *are independently distributed if and only if*

$$\text{(i) } \Psi A\Psi B\Psi = 0, \quad \text{(ii) } MA\Psi B\Psi = MB\Psi A\Psi = 0, \quad \text{(iii) } MA\Psi BM' = 0, \tag{4.1}$$

*where* $A$, $B$ *are* $n\times n$ *symmetric matrices and* $M=[\boldsymbol{\mu}_1,\dots, \boldsymbol{\mu}_n]$. *When* $\Psi>0$, *the conditions reduce to* $A\Psi B=0$. *When* $M=0$ *and* $\Psi\geq 0$, *the conditions reduce to* $\Psi A\Psi B\Psi=0$.

PROOF. If we express $XAX'$ and $XBX'$ in terms of $U$ and $C$ as in (3.2), the conditions in (4.1) can be obtained immediately from Lemma 2.3.

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