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Study of three-dimensional algebras with straightening laws which are Gorenstein domains II

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Introduction

In the previous paper [6] we determined all the partially ordered sets (poset for short) on which there exist three dimensional homogeneous Gorenstein ASL (algebra with straightening laws) domains over a field. Now, in this second part, we shall analyze normality and rationality of these algebras.

Our main purpose in this paper is to prove the following

THEOREM. *Let k be an algebraically closed field of arbitrary characteristic.*

(i) *The non-normal three dimensional homogeneous Gorenstein ASL domain over k is, up to isomorphism as ASL, either* [6, Example g)] *or* Example b) *in* §3.

(ii) *Every three dimensional homogeneous Gorenstein ASL domain over k is rational, that is, the quotient field of this algebra is a purely transcendental extension of the base field k.*

The basic methods in our proof are the calculations of singularities and the theory of "branches" (see §4). The former is useful to find out the non-normal ASL domains, while the latter plays an essential role for the proof of rationality.

Moreover, the calculations of singularities enable us to classify all the homogeneous ASL domains on the poset C_6 (§1). This classification is accomplished by means of some expressions of these algebras as subalgebras of the Veronese subring $k[x, y, z]^{(3)}$ (see (2.4) in §2). We will continue our classification in our further work.

Apart from the above results, this paper contains several lemmas, especially Lemma 10 in §4, which give criteria for a quasi-ASL (§1) to be an ASL. Using these lemmas it is easy to see that all the examples appeared in [6] are ASL.

§1. Notation and preliminaries

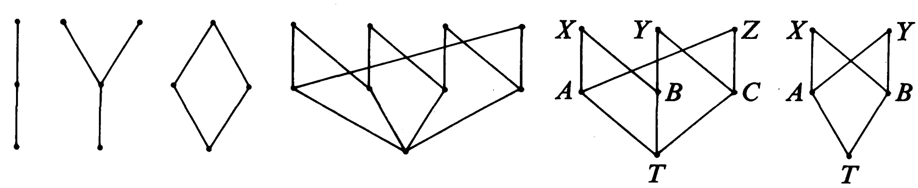
We shall refer to [6] for the basic definitions and terminologies on commutative algebras and combinatorics and, unless otherwise stated, keep the notation in [6]. We here summarize additional notation and results which are not contained in [6].

(1.1) Let k be a field and H a finite poset. We denote by $k[v(H)]=k[v_\alpha; \alpha \in H]$ the polynomial ring in $\sharp(H)$ variables over k. Suppose that $R=\oplus_{n\geq 0}R_n$ is a homogeneous ASL on H over k and that $\alpha \nsim \beta$, $\alpha\beta=\sum r_i\gamma_i\delta_i$ $(\gamma_i \leq \delta_i,\ 0\neq r_i \in k)$. Then we define

$$f_{\alpha\beta} = v_\alpha v_\beta - \sum r_i v_{\gamma_i} v_{\delta_i}.$$

We have $R\cong k[v(H)]/(f_{\alpha\beta};\ \alpha\nsim\beta)$. By abuse of notation, we say that R is an ASL defined by $\alpha\beta=\sum r_i\gamma_i\delta_i$ $(\alpha\nsim\beta)$ if $R\cong k[v(H)]/(f_{\alpha\beta};\ \alpha\nsim\beta)$ (cf. §2, Lemma 3). Now, if R is a domain we denote by Proj(R) the projective variety in the projective space $\mathbf{P}_k^{\sharp(H)-1}$ defined by the prime ideal $(f_{\alpha\beta};\ \alpha\nsim\beta)$ in $k[v(H)]$. Also, if $\delta\in H$ and U_δ is the affine open set $v_\delta\neq 0$ in $\mathbf{P}_k^{\sharp(H)-1}$, then we denote by $\mathscr{A}_\delta(R)$ the affine variety $U_\delta \cap \mathrm{Proj}(R)$ in the affine space $\mathbf{A}_k^{\sharp(H)-1}$. Let $(f_{\alpha\beta})_{\delta=1}$ be the polynomial obtained by substituting $v_\delta=1$ in $f_{\alpha\beta}$. Then we denote by $R_{\delta=1}$ the k-algebra $k[v(H-\{\delta\})]/((f_{\alpha\beta})_{\delta=1};\ \alpha\nsim\beta)$. For simplicity we sometimes write α instead of v_α if there is no confusion.

(1.2) We give names to the posets, which appear frequently, as follows:



A_1 B_2 $\bar{\mathrm{B}}_2$ C_8 C_6 C_4

It is easy to see that every homogeneous ASL domain on the poset A_1, B_2 and $\bar{\mathrm{B}}_2$ over k is normal and rational. Moreover, by [6, Example b)] the homogeneous ASL domain on the poset C_8 is unique up to isomorphism as ASL and isomorphic to the Segre product $k[s^2, st, t^2]\sharp k[a^2, ab, b^2]$, which is normal and rational. So, our investigation will be on the ASL domains on the posets C_6 and C_4 with branches of height 3.

(1.3) Let k be a field, R a k-algebra and H a poset contained in R which generates R as a k-algebra. Then we call R a *quasi-ASL* on H over k if, for every α and β with $\alpha\nsim\beta$, $\alpha\beta$ is expressed as a linear combination of standard monomials which satisfies the axiom (ASL–2).

(1.4) If R is a homogeneous quasi-ASL on H and R' is an ASL on H, then $\dim_k R_n \leq \dim_k R'_n$ for every $n\geq 0$ and R is an ASL if and only if $\dim_k R_n=\dim_k R'_n$ for every n.

(1.5) When a quasi-ASL turns out to be an ASL? The following lemmas are partial answers to this problem.

LEMMA 1. *Let H be a poset with a unique minimal element T and R a quasi-ASL on H over a field k. Suppose that T is a non-zero divisor of R and that the quotient algebra $R/(T)$ is an ASL on the poset $H-\{T\}$. Then R is, in fact, an ASL on H over k.*

For the proof we must check the axiom (ASL–1), but it is almost obvious.

LEMMA 2. *Let $R=\bigoplus_{n\geq 0} R_n$ be a noetherian graded ring and $x\in R_n$ for some $n>0$. Assume the following conditions:*

(i) *R_x is of pure dimension d.*

(ii) *$R/(x)$ is of pure dimension $d-1$.*

(iii) $\mathrm{Ass}_R(R)\cap \mathrm{Ass}_R(R/(x))=\emptyset$.

Then R is of pure dimension d and x is a non-zero divisor of R. (we say that R is of pure dimension d if $\dim R/\mathfrak{p}=d$ for every $\mathfrak{p}\in \mathrm{Ass}_R(R)$.)

PROOF. Considering the primary decomposition of (0) in R, we can put $(0)=\mathfrak{a}\cap\mathfrak{b}$, where $R/\mathfrak{a}$ is of pure dimension d and $\dim R/\mathfrak{b}<d$. By condition (ii), x is not contained in any associated prime ideals of $R/\mathfrak{a}$. Hence x is a non-zero divisor of $R/\mathfrak{a}$. On the other hand, by (iii), $\dim R/(\mathfrak{b}, x)<d-1$. By the exact sequence

$$0\longrightarrow \mathfrak{a}\longrightarrow R\longrightarrow R/\mathfrak{a}\longrightarrow 0$$

and the "snake lemma", we get the exact sequence

$$0\longrightarrow \mathfrak{a}/x\mathfrak{a}\longrightarrow R/(x)\longrightarrow R/(\mathfrak{a}, x)\longrightarrow 0,$$

which implies $\mathrm{Ass}_R(\mathfrak{a}/x\mathfrak{a})\subset \mathrm{Ass}_R(R/(x))$. But as $\mathrm{Supp}(\mathfrak{a}/x\mathfrak{a})\subset \mathrm{Supp}(R/(\mathfrak{b}, x))$, $\dim_R \mathfrak{a}/x\mathfrak{a}<d-1$, hence $\mathrm{Ass}_R(\mathfrak{a}/x\mathfrak{a})=\emptyset$. So, $\mathfrak{a}/x\mathfrak{a}=(0)$ and by "Nakayama's lemma", $\mathfrak{a}=(0)$. Q.E.D.

§2. Gorenstein ASL domains on the poset C_6

In this section, we will classify the homogeneous ASL domains on the poset C_6. For this purpose, we first proceed with the discussions in [6, §5].

(2.1) We will begin with

LEMMA 3. *If R is a homogeneous ASL domain on the poset C_6 over a field k, then R is isomorphic to an ASL defined by the equations:*

type[I] $AB = TX,\quad BC = TY,\quad CA = TZ,$

$$AY = BZ = CX = T(tT+aA+bB+cC),$$

$$XY = tTB + aTX + bB^2 + cTY,\quad YZ = tTC + aTZ + bTY + cC^2,$$

$$ZX = tTA + aA^2 + bTX + cTZ,$$

type[II] $AB = TX, \quad BC = T^2, \quad CA = TZ,$

$$AY = T(t_0T + b_0B + c_0C), \quad BZ = TA, \quad CX = TA,$$

$$XY = t_0TB + b_0B^2 + c_0T^2, \quad YZ = t_0TC + b_0T^2 + c_0C^2, \quad ZX = A^2.$$

Conversely, every quasi-ASL defined by the equations of type[I] *or* type[II] *is an ASL, although it might not be a domain.*

PROOF. (i) We have shown in [6, §5] that we can put

case[I] $AB = TX, \quad BC = TY, \quad CA = TZ,$ or

case[II] $AB = TX, \quad BC = T^2, \quad CA = TZ.$

In [6, §5], we assumed the existence of the branch P from A. But if we assume that R is a domain, it is not difficult to show that the existence of the branch is not necessary.

(ii) Next, we will show that in the case[I], we may put $x=y=z=0$ in the equation

$$AY = BZ = CX = T(tT + aA + bB + xX + yY + zZ).$$

In fact, firstly, if we continue the linear changes

$$\varphi_C(C) = C - xT, \quad \varphi_Y(Y) = Y - xB, \quad \varphi_Z(Z) = Z - xA,$$

then we get $x=0$. Secondly, by the linear changes

$$\varphi_A(A) = A - yT, \quad \varphi_Z(Z) = Z - yC, \quad \varphi_X(X) = X - yB,$$

we have $y=0$. Finally, we also have $z=0$ by

$$\varphi_B(B) = B - zT, \quad \varphi_X(X) = X - zA, \quad \varphi_Y(Y) = Y - zC.$$

Also, in the case[II], we can show that we may put $y_0=0$ in

$$AY = T(t_0T + b_0B + c_0C + y_0y)$$

by the similar method.

(iii) If R is a quasi-ASL of type[I] or type[II] on the poset C_6, it is easy to see that $R/(T)$ is a homogeneous ASL on the poset $C_6 - \{T\}$. Also, for every $\alpha \in H$, R_α is of pure dimension 3. So, if R has an embedded prime ideal $\mathfrak{p}$, then $\mathfrak{p}$ should contain all $\alpha \in H$. Thus R satisfies the conditions of Lemma 2 in §1 and T is a non-zero divisor of R. So, R is an ASL on H by Lemma 1. Q.E.D.

LEMMA 4. *Let R be a homogeneous ASL of* type[I] *or* type[II] *on* C_6 *in* Lemma 3. *Then*

(i) *if R is of* type[I], *R is a domain if and only if either $t \neq 0$ or at least two of the coefficients a, b and c are not zero*;

(ii) *if R is of* type[II], *R is a domain if and only if* $(t_0, b_0, c_0)\neq(0, 0, 0)$.

PROOF. Since T is a non-zero divisor of R, R is a domain if and only if so is $R_{T=1}$. If R is of type[I] (resp. type[II]),

$$R_{T=1} = k[A, B, C]/(ABC-(t+aA+bB+cC))$$

$$(\text{resp. } R_{T=1} = k[A, B, C, Y]/(BC - 1, AY - (t_0+b_0B+c_0C)))\,.$$

Our result follows immediately from these expressions. Q. E. D.

(2.2) Now, let k be an algebraically closed field and R a homogeneous ASL domain on the poset C_6 over k. We will study the singularities on the projective variety $\mathrm{Proj}(R)\subset\mathbf{P}_k^6$. Recall that a singular point P on a surface X is of (A_n)-*type* (or an (A_n)-*singularity*) if the local ring $\mathcal{O}_P$ is analytically isomorphic to the local ring at the origin (0, 0, 0) on the surface $xy=z^{n+1}$ in $\mathbf{A}_k^3$.

Case I. Firstly, suppose that R is of type[I] and

$$AY= BZ = CX = T(tT+aA+bB+cC)\,.$$

Then $\mathrm{Proj}(R)$ is non-singular if and only if $t^2\neq 4abc$. If $t^2=4abc$, then $\mathrm{Proj}(R)$ has a unique singular point

$$(T, A, B, C, X, Y, Z) = (t, -2bc, -2ca, -2ab, ct, at, bt)$$

in case $\mathrm{char}(k)\neq 2$ and

$$(\sqrt{abc}, bc, ca, ab, c\sqrt{abc}, a\sqrt{abc}, b\sqrt{abc})$$

in case $\mathrm{char}(k)=2$. This singular point is an (A_1)-singularity.

Case II. Secondly, if R is of type[II] and

$$AY= T(t_0T+b_0B+c_0C)\,,$$

then $\mathrm{Proj}(R)$ has always a singularity of (A_1)-type at the point

$$P = (0, 0, 0, 0, 0, 1, 0)\,.$$

If $t_0^2\neq 4b_0c_0$, then the set of singular points $\mathrm{Sing}(\mathrm{Proj}(R))$ on $\mathrm{Proj}(R)$ is $\{P\}$. On the other hand, if $t_0^2=4b_0c_0$, then $\mathrm{Sing}(\mathrm{Proj}(R))=\{P, Q\}$, where

$$Q = (-t_0, 0, 2c_0, 2b_0, 0, 0, 0)$$

in case $\mathrm{char}(k)\neq 2$ and

$$Q = (\sqrt{b_0c_0}, 0, c_0, b_0, 0, 0, 0)$$

in case $\mathrm{char}(k)=2$. The singularity at Q is also of (A_1)-type.

(2.3) Let H be a poset of rank 3 with a unique minimal element T and R a homogeneous ASL domain on H over an algebraically closed field k. Since $R/(T)$ is an ASL on $H-\{T\}$, $R/(T)$ is reduced. Hence R is normal if and only if the localization R_T is normal, that is, $R_{T=1}$ is normal. Note that $R_{T=1}$ is normal if and only if the set of singular points $\mathrm{Sing}(\mathscr{A}_T(R))$ on $\mathscr{A}_T(R) \subset \mathbf{A}_k^{\#(H)-1}$ is finite, since R is Cohen-Macaulay (see [6, §2]).

Consequently, summarizing the above calculations of singularities, we can conclude that every homogeneous ASL domain on the poset C_6 over an algebraically closed field is normal.

(2.4) Now, we will classify all the homogeneous ASL domains on the poset C_6 over an algebraically closed field k. Our classification is not based on the viewpoint of ASL (see [6, §4]) but of graded k-algebras.

To classify homogeneous ASL domains on C_6 we use the expression of R as a subring of the Veronese subring $S=k[x, y, z]^{(3)}$ or $S'=k[x, y, w]^{(4)}$ (in S', we put $\deg(x)=\deg(y)=1$ and $\deg(w)=2$).

Case I. Let R be of type[I] with

$$AY = BZ = CX = T(tT+aA+bB+cC).$$

If $(a, b, c)=(0, 0, 0)$, then we may assume $t=1$ and we can embed R into S by

$$T = xyz, \quad A = x^2y, \quad B = y^2z, \quad C = z^2x,$$
$$X = xy^2, \quad Y = yz^2, \quad Z = zx^2.$$

On the other hand, if $(a, b, c)\neq(0, 0, 0)$ then we may assume $c\neq 0$. Hence there exist $p, q, u, v \in k$ such that

$$qu + cpv = t, \quad pu = b, \quad qv = a,$$

and R can be embedded into S by

$$T = xyz, \quad A = xy\ell_1, \quad B = yz\ell_2, \quad C = zx\ell_3,$$
$$X = y\ell_1\ell_2, \quad Y = z\ell_2\ell_3, \quad Z = x\ell_3\ell_1,$$

where

$$\ell_1 = x + py, \quad \ell_2 = cz + qy, \quad \ell_3 = uz + vx.$$

Now, suppose that $\mathrm{Proj}(R)$ is non-singular, namely $t^2\neq 4abc$. Then the determinant

$$\begin{vmatrix} 0 & u & -p \\ c & 0 & 1 \\ -q & -v & 0 \end{vmatrix}$$

is not zero. So, we define a k-automorphism ψ of $k[x, y, z]$ by

$$\psi(x) = uy - pz, \quad \psi(y) = cx + z, \quad \psi(z) = - qx - vy.$$

The image $\psi(R)$ of R by this automorphism ψ is

$$\mathscr{R}_1 = k[xyz, x^2y, x^2z, y^2z, y^2x, z^2x, z^2y].$$

On the other hand, if $t^2 = 4abc$ then R is isomorphic to

$$\mathscr{R}_2 = k[xyz, xy(x+y), x^2z, y^2z, z^2x, z^2y, z^3],$$

as graded rings over k.

Case II. Let R be of type[II] with

$$AY = T(t_0T + b_0B + c_0C).$$

Then R can be embedded into S by

$$T = xyz, \quad A = xz\ell_1, \quad B = x^2y, \quad C = yz^2,$$
$$X = x^2\ell_1, \quad Y = y^2\ell_2, \quad Z = z^2\ell_1,$$

where ℓ_1, ℓ_2 are linear polynomials such that

$$\ell_1\ell_2 = b_0x^2 + t_0xz + c_0z^2.$$

As in Case I, we can show that R is isomorphic to

$$\mathscr{R}_3 = k[xyz, x^2y, x^2z, y^2z, z^2x, z^2y, x^3],$$

if $t_0^2 \neq 4b_0c_0$ and isomorphic to

$$\mathscr{R}_4 = k[xyz, x^2y, x^2z, y^2x, z^2x, z^2y, x^3],$$

if $t_0^2 = 4b_0c_0$.

If R is an ASL domain of type[II], R can also be embedded into $S' = k[x, y, w]^{(4)}$ ($\deg(w) = 2$) by

$$T = xyw, \quad A = xy\ell, \quad B = x^2w, \quad C = y^2w,$$
$$X = x^2\ell, \quad Y = w^2, \quad Z = y^2\ell,$$

where

$$\ell = t_0xy + b_0x^2 + c_0y^2.$$

However, it can be checked that the ASL domains of type[I] cannot be embedded into S'.

Hence, in particular, the ASL domains of type[I] and type[II] are not isomorphic to each other.

SUMMARY. We summarize the above discussions in the following table.

type	[I]		[II]	
singular locus of Proj(R)	$\emptyset$	one point	one point	two points
condition of coefficients	$t^2 \neq 4abc$	$t^2 = 4abc$	$t_0^2 \neq 4b_0c_0$	$t_0^2 = 4b_0c_0$
subring of $k[x, y, z]$ (3)	$\mathscr{R}_1$	$\mathscr{R}_2$	$\mathscr{R}_3$	$\mathscr{R}_4$
embedding into $k[x, y, w]$ (4)	impossible		possible	

EXAMPLE a) Assume char$(k) \neq 2$.

Let R_1 (resp. R_2) be a homogeneous ASL domain of type[I] on C_6 over k with

$$AY = BZ = CX = T^2$$

$$(\text{resp. } AY = BZ = CX = T(-A-B-C)).$$

Then, $R_1 \cong R_2$ as graded rings over k by the above arguments. Now [6, Example c)] shows that R_2 has three branches of height 3; however, there exists no branch from R_1. Hence R_1 is not isomorphic to R_2 as ASL.

§ 3. Gorenstein ASL domains on the poset C_4

In this section, we will find out all the non-normal homogeneous ASL domains on the poset C_4 over an algebraically closed field k.

(3.1) We will begin with two lemmas which correspond to Lemma 3 and Lemma 4 in §2.

LEMMA 5. *Let k be a field and $R = k[v(C_4)]/I$ a homogeneous quasi-ASL on C_4 defined by*

$$AB - T(tT + xX + yY), \quad XY - T(t'T + a'A + b'B) - aA^2 - bB^2.$$

Then R is an ASL on C_4 over k.

PROOF. Since R is a complete intersection the Poincaré series of R is

$$(*) \qquad (1 + 2\theta + \theta^2)/(1-\theta)^3$$

by Stanley [7, Cor. 3.3]. On the other hand, the Poincaré series of homogeneous ASL domains on C_4 is also $(*)$ by [6, §3]. Hence the set of standard monomials in R is linearly independent over k (see (1.4)). Q.E.D.

LEMMA 6. *In the same notation as in* Lemma 5, *R is a domain if and only if one of the following conditions is satisfied*:

(i) $(x, y) = (0, 0)$, $t \neq 0$, $(t', a', b', a, b) \neq (0, 0, 0, 0, 0)$,

(ii) $(x, y) \neq (0, 0)$, $xy = 0$, $t \neq 0$, $(t', a', b', a, b) \neq (0, 0, 0, 0, 0)$,

(iii) $(x, y) \neq (0, 0)$, $xy = 0$, $t = 0$, $t' \neq 0$,

(iv) $(x, y) \neq (0, 0)$, $xy = 0$, $t = 0$, $t' = 0$, $(a, a') \neq (0, 0)$, $(b, b') \neq (0, 0)$,

(v) $xy \neq 0$, $(t', a', b', a, b) \neq (0, 0, 0, 0, 0)$.

PROOF. If $x=0$, $y=0$ then

$$R_{T=1} \cong k[A, B, X, Y]/(AB - t, XY - (t' + a'A + b'B + aA^2 + bB^2)),$$

if $x=0$, $y \neq 0$ then

$$R_{T=1} \cong k[A, B, X]/(((AB - t)/y)X - (t' + a'A + b'B + aA^2 + bB^2)),$$

and if $x \neq 0$, $y \neq 0$ then

$$R_{T=1} \cong k[A, B, X]/((x/y)X^2 - ((AB - t)/y)X + (t' + a'A + b'B + aA^2 + bB^2)).$$

Since T is a non-zero divisor of R, R is a domain if and only if $R_{T=1}$ is a domain. Hence our result follows from the above expressions. Q. E. D.

(3.2) Now, we shall show that the non-normal homogeneous ASL domain on the poset C_4 over an algebraically closed field k is, up to isomorphism as ASL, either [6, Example g)] or

EXAMPLE b) The k-algebra

$$R = k[v(C_4)]/(AB - TY, XY - A^2 - B^2)$$

is a non-normal homogeneous Gorenstein ASL domain on the poset C_4, whose normalization is $R[TB/A]$. This algebra can be represented as a subalgebra of the Veronese subring $k[x, y, z]^{(2)}$ by putting $T=yz$, $A=xy$, $B=zx$, $X=y^2+z^2$ and $Y=x^2$, whose normalization is $k[x, y, z]^{(2)}$.

Let R be a homogeneous ASL domain on C_4 over an algebraically closed field k. Among these ASL, we will find out all which are not normal. Recall that as T is a non-zero divisor and $R/(T)$ is reduced, R is normal if and only if so is $R_{T=1}$.

As in [6, §6], we have only to consider the cases $(x, y) = (0, 0)$, $(x, y) = (0, 1)$ and $(x, y) = (1.1)$ in the expression of Lemma 5. Before treating each case separately, we will state some lemmas.

LEMMA 7. *Let R be a noetherian normal domain which satisfies Serre's* *condition* (S_3). *Let* r, s *be non-zero elements of* R *and put* $\mathfrak{a}=(r, s)$ *and* $R'=R[X]/(rX-s)$, *where* X *is an indeterminate over* R. *Then,*

(i) R' *is a domain if and only if* $\mathrm{ht}(\mathfrak{a})\geq 2$,

(ii) *if* R' *is a domain,* R' *is normal if and only if, for every prime ideal* $\mathfrak{m}$ *of* R *with* $\mathrm{ht}(\mathfrak{m})=2$ *and* $\mathfrak{m}\supset\mathfrak{a}$, $R_{\mathfrak{m}}$ *is regular and* $\mathfrak{a}R_{\mathfrak{m}}\not\subset\mathfrak{m}^2R_{\mathfrak{m}}$.

PROOF (i) If $\mathfrak{p}$ is a prime ideal of height 1 in R, $\mathfrak{p}\supset\mathfrak{a}$, and if $\mathfrak{p}R_{\mathfrak{p}}=\pi R_{\mathfrak{p}}(\pi\in\mathfrak{p})$, then $rX-s$ is divisible by π in $R_{\mathfrak{p}}[X]$. Hence R' is not a domain. Conversely, if $\mathrm{ht}(\mathfrak{a})=2$, then (r, s) is an R-regular sequence and it is easy to see that the R-algebra homomorphism φ: $R[X]\to R[s/r]$ defined by $\varphi(X)=s/r$ has $(rX-s)$ as its kernel.

(ii) We want to check Serre's conditions (S_2) and (R_1) for R'. By our assumption, R' obviously satisfies (S_2). Let $\mathfrak{q}$ be a prime ideal of height 1 in R'. If $\mathrm{ht}(\mathfrak{q}\cap R)=1$, then $R_{(\mathfrak{q}\cap R)}\subset R'_{\mathfrak{q}}$ and as $R_{(\mathfrak{q}\cap R)}$ is a discrete valuation ring, $R_{(\mathfrak{q}\cap R)}=R'_{\mathfrak{q}}$. If $\mathrm{ht}(\mathfrak{q}\cap R)=2$, $\mathfrak{m}=\mathfrak{q}\cap R\supset\mathfrak{a}$. If we put $R[X]_{\mathfrak{m}R[X]}=R_{\mathfrak{m}}(X)$, $R'_{\mathfrak{q}}=R_{\mathfrak{m}}(X)/(rX-s)$. If $R_{\mathfrak{m}}$ is regular and $\mathfrak{a}R_{\mathfrak{m}}\not\subset\mathfrak{m}^2R_{\mathfrak{m}}$, $R_{\mathfrak{m}}(X)$ is regular and $rX-s\notin(\mathfrak{m}R_{\mathfrak{m}}(X))^2$, hence $R'_{\mathfrak{q}}$ is regular. Conversely, if $R'_{\mathfrak{q}}$ is regular, then so is $R_{\mathfrak{m}}(X)$ and $rX-s\notin(\mathfrak{m}R_{\mathfrak{m}}(X))^2=\mathfrak{m}^2R_{\mathfrak{m}}(X)$. Q.E.D.

Note that Lemma 7 (ii) is a special case of S. Goto and K. Yamagishi [4], in which the normality of blowing up by parameter ideals is discussed.

LEMMA 8. *Let* R *be a noetherian normal domain, in which* 2 *is a unit and* $R'=R[Y]/(Y^2-r)$. *Then* R' *is normal if and only if* r *is not a square in* R *and for every prime ideal* $\mathfrak{p}$ *with* $\mathrm{ht}(\mathfrak{p})=1$, $r\notin\mathfrak{p}^{(2)}$. *In particular, if* R *is a unique factorization domain,* R' *is normal if and only if* r *is square-free.*

PROOF. In the case R is a unique factorization domain, we refer to Hartshorne [9, II, Ex. 6.4, P. 147]. The general case can be proved similarly. Q.E.D.

Case I. $(x, y)=(0, 0)$.
In this case,

$$R_{T=1}\cong(k[A, B]/(AB-t))[X, Y]/(XY-(t'+a'A+b'B+aA^2+bB^2)),$$

which is easily seen to be normal.

Case II. $(x, y)=(0, 1)$.
In this case,

$$R_{T=1}\cong k[A, B, X]/(X(AB-t)-(t'+a'A+b'B+aA^2+bB^2)).$$

We will use Lemma 7 for $k[A, B]$ and X.

If $t\neq 0$, then there is no maximal ideal $\mathfrak{m}$ in $k[A, B]$ with $AB-t\in\mathfrak{m}^2$. Hence $R_{T=1}$ is normal by Lemma 7.

If $t=0$, then only maximal ideal $\mathfrak{m}$ with $AB\in\mathfrak{m}^2$ is $\mathfrak{m}=(A, B)$. As $t'+a'A+b'B+aA^2+bB^2\in(A, B)^2$ if and only if $t'=a'=b'=0$, R is not normal if and only if R is isomorphic to the ASL in Example b) as ASL.

Case III. $(x, y)=(1, 1)$.
In this case,

$$R_{T=1}\cong k[A, B, X]/(X^2-(AB-t)X+(t'+a'A+b'B+aA^2+bB^2)).$$

(i) If $\mathrm{char}(k)\neq 2$, $R_{T=1}$ is normal if and only if the polynomial

$$F(A, B) = (AB-t)^2 - 4(t'+a'A+b'B+aA^2+bB^2)$$

does not have a square factor by Lemma 8. If $F=G^2$ and $\deg(G)=2$, then the polynomial G must be $AB-t$, and we have $(t', a', b', a, b)=(0, 0, 0, 0, 0)$. This contradicts our assumption that R is a domain. Let $\ell=\alpha A+\beta B+\gamma$ $(\alpha, \beta, \gamma\in k)$ and assume that F is divisible by ℓ^2. If $\alpha\neq 0$, we may assume $\ell=A-\beta B-\gamma$. Then we have

$$F(\beta B+\gamma, B) = (\partial F/\partial A)(\beta B+\gamma, B) = 0$$

in $k[B]$. From $F(\beta B+\gamma, B)=0$, we have $\beta=0$ and from $(\partial F/\partial A)(\gamma, B)=0$, we have $\gamma=0$. So, $F(A, B)$ is divisible by A^2 and this implies $(t, t', a', b', b)=(0, 0, 0, 0, 0)$. In the same manner, if $\beta\neq 0$ in $\ell=\alpha A+\beta B+\gamma$, we have $\ell=B$ and $(t, t', a', b', a)=(0, 0, 0, 0, 0)$.

The non-normal ASL domains we have got are defined by

$$\begin{cases} AB = T(X+Y) \\ XY = A^2 \end{cases} \quad\text{and}\quad \begin{cases} AB = T(X+Y) \\ XY = B^2 \end{cases},$$

which are isomorphic to the one in [6, Example g)].

(ii) In case $\mathrm{char}(k)=2$, we will consider the following simultaneous equations:

(1) $F = X^2 + (t-AB)X + (t'+a'A+b'B+aA^2+bB^2) = 0,$

(2) $\partial F/\partial X = t - AB = 0,$

(3) $\partial F/\partial A = a' - BX = 0,$

(4) $\partial F/\partial B = b' - AX = 0.$

By (2), (3) and (4), we can eliminate B and X in (1) and we get

(5) $aA^4 + a'A^3 + t'A^2 + b'tA + (b')^2 + bt^2 = 0.$

Similarly, we have

(6) $bB^4 + b'B^3 + t'B^2 + a'tB + (a')^2 + at^2 = 0.$

Hence we have $\sharp(\mathrm{Sing}(\mathscr{A}_T(R))) < \infty$ if $(t, t', a', b', a) \neq (0, 0, 0, 0, 0)$ and $(t, t', a', b', b) \neq (0, 0, 0, 0, 0)$.

REMARK. The method in Case III (ii) is also valid in case $\mathrm{char}(k) \neq 2$.

SUMMARY. If R is an ASL domain on the poset C_4 over an algebraically closed field k, and if R is not normal, then R is isomorphic as ASL to the one defined by

$$AB = TY,\ XY = A^2 + B^2 \qquad \text{(Example b))}$$

or

$$AB = T(X+Y),\ XY = A^2 \qquad \text{([6, Example g)])}.$$

As it is easy to see that the latter example is not isomorphic to a subring of $k[x, y, z]^{(2)}$, these two examples are not isomorphic to each other as graded rings over k.

§4. Possibility of extensions as ASL domains

Let R be an ASL domain on a poset H of rank 3 over a field k. In this section, we will find the condition for R to have an extension R', which is an ASL domain on $H \cup \{P\}$, where P is a branch of height 3.

We denote by $\mathrm{Ind}(R)$ the set of standard monomials which appear in the right-hand sides of the straightening relations in (ASL–2). For a homogeneous element r of R, $[r]$ is the set of standard monomials appearing in the standard monomial expression of r as in [6, (1.3)].

(4.1) We will begin with

LEMMA 9. *Let H be a poset of rank 3 with a unique minimal element T, and $A \in H$ an element of height 2 which has a branch P of height 3. Suppose that R is a homogeneous ASL on H over a field k. Then, applying adequate fundamental transformations to R, we have T^2, TA, $A^2 \notin \mathrm{Ind}(R')$, where R' is the ASL subring on $H - \{P\}$.*

PROOF. Let $\alpha \in H$ be an element with $\alpha \nsim P$. Suppose that tTP and aAP ($t, a \in k$) appear in the right-hand side of the straightening relation of $P\alpha$. Note that $a = 0$ if $\alpha \nsim A$. Now apply the fundamental transformation

$$\varphi_\alpha(\alpha) = \alpha - tT - aA$$

and we have TP, $AP \notin [P\alpha]$. If we continue this operation for every $\alpha \in H$ with $\alpha \nsim P$, then we have the desired result. In fact, if $\alpha, \beta \in H - \{P\}$, $\alpha \nsim \beta$ and $\alpha \nsim P$, then T^2P, TAP, $A^2P \notin [(P\alpha)\beta]$, which mean T^2, TA, $A^2 \notin [\alpha\beta]$. Q.E.D.

REMARK (4.2) The following conditions are easily seen to be equivalent.

(1) T^2, TA, $A^2 \notin \mathrm{Ind}(R')$.

(2) The ideal $\mathfrak{p}$ of R' generated by $\{\alpha \in H - \{P\};\ \alpha \neq T, A\}$ is a prime ideal of height 1.

(3) The ideal $\mathfrak{P}$ of $k[v(H-\{P\})]$ generated by $\{v_\alpha;\ \alpha \neq T, A, P\}$ contains $f_{\alpha\beta}$ for every α, $\beta \in H - \{P\}$, $\alpha \nsim \beta$.

(4) If $\mathfrak{p}$ and $\mathfrak{P}$ are as in (2) and (3), then $k[v(H-\{P\})]/\mathfrak{P}$ and $R'/\mathfrak{p}$ are isomorphic to the polynomial ring of two variables.

The following corollary is deduced from the proof of Lemma 9.

COROLLARY. *Let R be as in* Lemma 9 *and $\mathfrak{p}$ be the prime ideal defined in* Remark (4.2) (2). *Then, after suitable fundamental transformations on R, we may assume that $P\mathfrak{p} \subset R'$.*

EXAMPLE c) As an application of Lemma 9 we shall prove that the homogeneous ASL domain R in Example b) has no branch of height 3.

Suppose, on the contrary, that there exists a branch P of height 3 from C_4 under the relation

$$AB = TY, \quad XY = A^2 + B^2.$$

We may assume P is a branch from A. Then, by the proof of Lemma 9, after suitable linear changes:

$$\varphi_B(B) = B - sT, \quad \varphi_X(X) = X - aA - bB - tT, \quad \varphi_Y(Y) = Y - a'A - b'B - t'T,$$

we have T^2, TA, $A^2 \notin \mathrm{Ind}(R)$. Hence

$$s = a', \quad t' = 0, \quad aa' = 1, \quad ta' + t'a = 0, \quad tt' = s^2,$$

but $s = a'$, $t' = 0$ and $tt' = s^2$ imply $a' = 0$, which contradicts $aa' = 1$. Q.E.D.

(4.3) The following lemma is useful to construct a homogeneous ASL domain on a poset H of rank 3 with branches of height 3.

LEMMA 10. *Let H be a poset of rank* 3 *with a unique minimal element T and P a branch from A, where* $\mathrm{ht}(A) = 2$ *and* $\mathrm{ht}(P) = 3$. *Suppose that R is a homogeneous quasi-ASL on H over a field k and that the subring $R' = k[H - \{P\}]$ is an ASL on $H - \{P\}$ over k. Then, if T, A, P are algebraically independent over k, R is, in fact, a homogeneous ASL on H over k.*

PROOF. Let $\{\alpha_1, \dots, \alpha_n\}$ be the subset of H consisting of the elements which are incomparable with P, namely,

$$\{\alpha_1,\dots,\alpha_n\} = H - \{T, A, P\}.$$

Since R is a homogeneous quasi-ASL on H over k, we have

$$\begin{pmatrix} P+f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & P+f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \cdots & P+f_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix},$$

where each f_{ij} (resp. g_i) is a linear combination of T, A (resp. T^2, TA, A^2). Since T, A, P are algebraically independent over k, the determinant of the matrix in the left-hand side is not zero. Hence, by "Cramer's formula", each α_i is expressed as

$$(*) \qquad \alpha_i = \frac{q_2^{(i)}P^{n-1} + q_3^{(i)}P^{n-2} + \cdots + q_{n+1}^{(i)}}{P^n + p_1 P^{n-1} + \cdots + p_n},$$

where p_j, $q_j^{(i)}$ are linear combinations of T^j, $T^{j-1}A,\dots, A^j$.

Now we shall show that R satisfies the axiom (ASL–1). Assume that we have a relation

$$(**) \qquad \phi_i P^{m-i} + \cdots + \phi_{m-1}P + \psi_m = 0 \qquad (m > i),$$

where $\phi_j \in k[T, A]$ is a linear combination of T^j, $T^{j-1}A,\dots, A^j$ and $\psi_m \in R'_m$ is a linear combination of standard monomials in R'. If we substitute $(*)$ in the equality $(**)$ and clear the denominators, then we see that $\phi_i = 0$ since T, A, P are algebraically independent over k and ϕ_i is the coefficient of the highest degree in P. Continuing this process, we have $\phi_i = \cdots = \phi_{m-1} = 0$ and then $\psi_m = 0$. Q. E. D.

(4.4) Now, we will show that the condition in Remark (4.2) is a sufficient condition for a normal ASL domain R' on $H - \{P\}$ to be a subring of an ASL domain R on H.

If S is a Cohen-Macaulay domain and $\mathfrak{a}$ is an ideal in S, it is known that there exists $x \notin S$ in the quotient field of S such that $\mathfrak{a}x \subset S$ if and only if $\mathrm{ht}(\mathfrak{a}) \leq 1$. In our case such an element x can be chosen to be homogeneous of degree 1.

LEMMA 11. *Let $R = \oplus_{n\geq 0} R_n$ be a graded Gorenstein domain over a field $k = R_0$ and put $Q(R) = \oplus_{n\in\mathbf{Z}} Q_n$, where*

$$Q(R) = \{y/x;\ y \in R,\ 0 \neq x \in R_n \text{ for some } n\}.$$

If $\mathfrak{a}$ is a graded ideal in R of pure height 1, *then the minimal number n such that there exists an element $x \in Q_n - R_n$ with $x\mathfrak{a} \subset R$ is $a(R) - a(R/\mathfrak{a})$, where $a(R)$ is defined in* [3, (3.1.4)].

PROOF. An element $x \in Q = Q(R)$ with $x\mathfrak{a} \subset R$ induces an element $\varphi \in$

$\underline{\mathrm{Hom}}_R(R/\mathfrak{a}, Q/R)$ and $x \notin R$ if and only if $\varphi \neq 0$. As Q is an injective object in the category of graded R-modules, the exact sequence

$$0 \longrightarrow R \longrightarrow Q \longrightarrow Q/R \longrightarrow 0$$

implies

$$\underline{\mathrm{Hom}}_R(R/\mathfrak{a}, Q/R) \cong \underline{\mathrm{Ext}}^1_R(R/\mathfrak{a}, R).$$

By our assumption, the canonical module K_R of R is isomorphic to $R(a(R))$ as graded R-modules. Hence we get

$$\underline{\mathrm{Ext}}^1_R(R/\mathfrak{a}, R) \cong \underline{\mathrm{Ext}}^1_R(R/\mathfrak{a}, K_R(-a(R)) \cong K_{(R/\mathfrak{a})}(-a(R))$$

by [3, (2.2.9)]. By the definition of $a(R/\mathfrak{a})$, the minimal degree of non-zero homogeneous element of $K_{(R/\mathfrak{a})}(-a(R))$ is $-a(R/\mathfrak{a})+a(R)$. Q.E.D.

COROLLARY. *If R' is a homogeneous Gorenstein ASL domain with $a(R)=-1$, and the ideal $\mathfrak{p}=(\alpha;\ \alpha \in H-\{P\},\ \alpha \neq T, A)$ satisfies the condition of* Remark (4.2), *then there exists an element $\eta \in Q(R')-R'$ of degree* 1 *such that $\eta\mathfrak{p} \subset R'$.*

PROOF. In our case, $a(R)=-1$ and $a(R/\mathfrak{p})=-2$. Q.E.D.

REMARK. (1) Every homogeneous ASL domain R on the poset C_4, C_6 (may have branches) or C_8 is Gorenstein and $a(R)=-1$.

(2) The element η as in the corollary is unique up to constant multiplication and modulo R, since by the proof of Lemma 9,

$$\underline{\mathrm{Hom}}_R(R/\mathfrak{p}, Q/R) \cong K_{(R/\mathfrak{p})}(1) \cong (R/\mathfrak{p})(-1).$$

Now we can state the main result in this section.

PROPOSITION 1. *Let H be a poset of rank* 3 *with a unique minimal element T, P a branch of height* 3 *from A with* $\mathrm{ht}(A)=2$ *and R' a homogeneous ASL domain on $H-\{P\}$ over a field k which is normal Gorenstein with $a(R)=-1$. Then, there exists a homogeneous ASL domain on H preserving the ASL structure of R' on $H-\{P\}$ if and only if, after suitable fundamental transformations, the ideal $\mathfrak{p}=(\alpha \in H-\{P\};\ \alpha \neq T, A)$ satisfies the condition in* Remark (4.2).

PROOF. The "only if" part is already stated in Cor. to Lemma 9 in (4.1). To prove "if" part, we first prove $R=R'[\eta]$ (η is as in Cor. to Lemma 11) is a quasi-ASL domain on H over k.

In the following, we denote by $\beta, \beta', \dots$ the elements of height 2 which are different from A and $\xi, \xi', \dots$ the elements of height 3 in $H-\{P\}$. We proceed in several steps. We assume that $\eta\beta, \eta\xi \in R'$. As usual, let $[\eta\beta]$, $[\eta\xi]$ denote the set of standard monomials appearing in $\eta\beta, \eta\xi \in R'$ respectively.

(i) If $\beta^2 \in [\eta\beta]$, then taking $\eta' = \eta - c\beta$ $(c \in k)$ instead of η, we may assume $\beta^2 \notin [\eta\beta]$ for every $\beta \in H$, $\mathrm{ht}(\beta) = 2$, $\beta \neq A$.

(ii) Now, we fix $\xi \in H$, $\mathrm{ht}(\xi) = 3$ and assume that $\beta < \xi$ and $\beta\xi \in [\eta\beta]$. Then taking $\eta' = \eta - c\xi$ for some $c \in k$, we may assume $\beta\xi \notin [\eta\beta]$.

(iii) If β, $\beta' < \xi$ and if $\beta\xi \notin [\eta\beta]$, then $\beta'\xi \notin [\eta\beta']$. For, if $\beta'\xi \in [\eta\beta']$, then $\beta'\xi^2 \in [(\eta\beta')\xi] = [(\eta\xi)\beta']$ and $\xi^2 \in [\eta\xi]$. As $(\eta\beta)\xi = (\eta\xi)\beta$, this implies $\beta\xi \in [\eta\beta]$, which is a contradiction.

(iv) Thus we may assume $\beta\xi \notin [\eta\beta]$ for every $\beta < \xi$.

(v) $\xi^2 \notin [\eta\beta]$ for $\mathrm{ht}(\beta) = 2$ and $\mathrm{ht}(\xi) = 3$.
For, if $\xi^2 \in [\eta\beta]$, then $\xi^3 \in [(\eta\beta)\xi] = [(\eta\xi)\beta]$, which is impossible.

(vi) β'^2, $\beta'\xi \notin [\eta\beta]$ for $\mathrm{ht}(\beta) = 2$, $\beta \neq \beta'$, $\beta' < \xi$.
For, $\beta'^2 \in [\eta\beta]$ (resp. $\beta'\xi \in [\eta\beta]$) implies $\beta'^2\xi \in [(\eta\beta)\xi] = [(\eta\xi)\beta]$ (resp. $\beta'\xi^2 \in [(\eta\xi)\beta]$), which is impossible.

(vii) (i)$\sim$(vi) show that every standard monomial in $[\eta\beta]$ is divisible by T.

(viii) $\beta\xi'$, ξ'^2, $\beta^2 \notin [\eta\xi]$ for $\mathrm{ht}(\xi) = \mathrm{ht}(\xi') = 3$, $\mathrm{ht}(\beta) = 2$, $\beta < \xi'$.
For, if $\beta\xi'$ (resp. ξ'^2, β^2) $\in [\eta\xi]$, then $\beta^2\xi'$ (resp. $\beta\xi'^2$, β^3) $\in [(\eta\beta)\xi]$, which is impossible by (vii).

(ix) If $\mathrm{ht}(\xi) = 3$, $\xi \nsim A$, then A^2, $A\xi' \notin [\eta\xi]$ $(A < \xi')$.
In fact, if A^2 (resp. $A\xi'$) $\in [\eta\xi]$, then $A^2\xi'$ (resp. $A\xi'^2$) $\in [(\eta\xi)\xi'] = [(\eta\xi')\xi]$, which is impossible since $\xi \nsim A$.

This shows that $R = R'[\eta]$ is a quasi-ASL on H by putting $\eta = P$. Now, $\eta = P \notin R'$ implies P is not integral over R' since R' is normal. Then, by the first part of the proof of Lemma 10, the quotient field of R' is $k(T, A, P)$, and hence T, A, P turn out to be algebraically independent over k since $\dim(R) = \dim(R') = 3$. Note that we can use "Cramer's formula" if P is not integral over $k[T, A]$.

So, thanks to Lemma 10, we have the desired result. Q.E.D.

EXAMPLE d) The condition of normality in Prop. 1 is indispensable. In fact, the non-normal homogeneous Gorenstein ASL domain R with straightening relations

$$AB = T(X+Y), \quad XY = B^2$$

on the poset C_4 satisfies the condition in Remark (4.2). However, as we have seen in the proof of Prop. G in [6], there cannot exist a branch from A under these relations.

In this example, $\eta = TY/B \in Q(R) - R$ satisfies

$$\eta B = TY, \quad \eta X = TB, \quad \eta Y = -TB + AY,$$

while, we have

$$\eta^2 - A\eta + T^2 = 0,$$

namely, η is integral over $k[T, A]$.

§ 5. Rationality and normality of three dimensional homogeneous Gorenstein ASL domains

In this final section, we will prove the rationality of three dimensional homogeneous Gorenstein ASL domains over an algebraically closed field and we will find out all non-normal three dimensional homogeneous Gorenstein ASL domains.

(5.1) We begin with a general criterion of rationality for every homogeneous ASL domain on certain type of posets.

PROPOSITION 2. *Let R be a homogeneous ASL domain on a poset H of rank n over a field k. Assume that H has a chain $\alpha_1<\alpha_2<\cdots<\alpha_n$ of length n satisying the condition*

(♯) *if $\beta\in H$ and $\beta\le\alpha_j$, then $\beta=\alpha_i$ for some $i\le j$.*

Then R is rational over k. In fact, the quotient field of R is $k(\alpha_1,\dots,\alpha_n)$.

PROOF. If $H=\{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m\}$, then $\beta_i\nsim\alpha_n$ for every i and as in (4.3), Lemma 10, we have the system of linear equations

$$\begin{pmatrix} \alpha_n+f_{11} & f_{12} & \cdots & f_{1m} \\ f_{21} & \alpha_n+f_{22} & \cdots & f_{2m} \\ \vdots & \vdots & & \vdots \\ f_{m1} & f_{m2} & \cdots & \alpha_n+f_{mm} \end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_m\end{pmatrix}=\begin{pmatrix}g_1\\ g_2\\ \vdots\\ g_m\end{pmatrix},$$

where f_{ij} (resp. g_i) is a linear form (resp. homogeneous element of degree 2) of $k[\alpha_1,\dots,\alpha_{n-1}]$, which shows that $\beta_1,\dots,\beta_m\in k(\alpha_1,\dots,\alpha_n)$. Q.E.D.

COROLLARY. *If H is a poset of rank 3 and if H has a branch of height 3, then every homogeneous ASL domain on H over k is rational.*

THEOREM 1. *Every three dimensional homogeneous Gorenstein ASL domain over an algebraically closed field is rational.*

PROOF. Let R be a homogeneous Gorenstein ASL domain on the poset H over an algebraically closed field k. If H has a branch of height 3, then R is rational over k by Cor. to Prop. 2. Also, our classification of ASL domains on C_8 and C_6 ([6, Example b)] and (2.4)) shows that every homogeneous ASL domain on C_8 or C_6 is rational. So, we may assume $H=C_4$ and R is defined by

$$AB=T(tT+xX+yY),\quad XY=T(t'T+a'A+b'B)+aA^2+bB^2.$$

As we have seen in the proof of Lemma 6, R is rational if one of x, y, a or b

is 0. If $xyab \neq 0$, then we may assume $x=y=1$. Now, we will check the condition of Prop. 1 for R. Let $\mathfrak{P}$ be the ideal in $k[v(C_4)]$ generated by

$$v_B - \beta v_T, \quad v_X - \xi_1 v_A - \xi_2 v_T, \quad v_Y - \eta_1 v_A - \eta_2 v_T.$$

Then, modulo $\mathfrak{P}$,

$$\begin{aligned} & v_A v_B - v_T(t v_T + v_X + v_Y) \equiv (\beta - \xi_1 - \eta_1) v_A v_T - (t + \xi_2 + \eta_2) v_T^2, \\ & v_X v_Y - v_T(t' v_T + a' v_A + b' v_B) - a v_A^2 - b v_B^2 \\ & \equiv (\xi_1 \eta_1 - a) v_A^2 + (\xi_1 \eta_2 + \xi_2 \eta_1 - a') v_T v_A + (\xi_2 \eta_2 - t' - b'\beta - b\beta^2) v_T^2. \end{aligned}$$

Thus, R satisfies the condition of Prop. 1 if and only if we can choose β, ξ_1, ξ_2, η_1, $\eta_2 \in k$ so that

$$(*) \qquad \begin{aligned} & \beta = \xi_1 + \eta_1, \quad t + \xi_2 + \eta_2, \quad \xi_1 \eta_1 = a, \\ & \xi_1 \eta_2 + \xi_2 \eta_1 = a', \quad \xi_2 \eta_2 - b'\beta - b\beta^2 = t'. \end{aligned}$$

As k is algebraically closed and $a \neq 0$, we can find the solutions of $(*)$ in k. By Prop. 1, R is birational to an ASL domain on $H \cup \{P\}$, which is rational by Cor. to Prop. 2. Q.E.D.

(5.2) Next we will consider the normality of R. As we have shown that all homogeneous ASL domains on C_8, C_6 and C_4 are normal except for Example b) and [6, Example g)], we have only to consider the case where the poset H has a branch P.

So, let H be a poset of rank 3 with a unique minimal element T and P a branch of height 3 from A. Suppose that R is a homogeneous ASL domain on H over a field k and that $R' = k[H - \{P\}]$ is the ASL subring of R. By Cor. to Lemma 9, we may assume $\alpha P \in R'$ for every $\alpha \in H$, $\alpha \neq T, A, P$.

REMARK. By our assumption

$$R_{\alpha=1} = R'_{\alpha=1}$$

for every $\alpha \in H - \{T, A, P\}$, since $P \in R_{\alpha=1}$.

LEMMA 12. *Let P_0 be the closed point in* Proj(R), *defined by the prime ideal* $(\alpha \in H; \alpha \neq P)$. *Then P_0 is a smooth point.*

Proof. From the proof of Lemma 10, the local ring of $R_{P=1}$ at the maximal ideal $(\alpha; \alpha \in H - \{P\})$ has (T, A) as its maximal ideal. Q.E.D.

(5.3) Now, we will show that the normality of R' implies the normality of R. We will state the result in more general situation.

PROPOSITION 3. *Let H be a poset of rank n which contains a maximal chain*

$\alpha_1<\alpha_2<\cdots<\alpha_n$ *satisfying the condition* (♯) *in* Prop. 2. *Let R be a homogeneous ASL domain on H over a field k which satisfies Serre's condition* (S_2) *and* $R'=k[H-\{\alpha_n\}]$ (*which is an ASL domain on* $H-\{\alpha_n\}$). *Then, if R' is normal, so is R.*

PROOF. First, we show that R' is an ASL on $H-\{\alpha_n\}$. In fact, if $\alpha_i\alpha_n \in [\alpha\beta]$ for some $\alpha, \beta \in H-\{\alpha_n\}$, $\alpha \nsim \beta$, then we may assume $\beta \nsim \alpha_n$ and $\alpha_i\alpha_n^2 \in [\alpha(\beta\alpha_n)]$, which is impossible.

After suitable fundamental transformations, we may assume $\alpha_n\beta \in R'$ and $R_{\beta=1}=R'_{\beta=1}$ for every $\beta \in H-\{\alpha_1,\dots, \alpha_n\}$.

Now, assume that R is not normal. Then the singular locus Sing(Proj(R)) of Proj(R) is a closed subvariety of codimension 1 of Proj(R). Let V be an irreducible component of codimension 1 of Sing(Proj(R)). Then, as R' is normal and as Proj(R) and Proj(R') are isomorphic outside

$$W=V_+(\mathfrak{b}),\ \mathfrak{b}=(\beta;\ \beta \in H-\{\alpha_1,\dots, \alpha_n\}),$$

we have $V \subset W$. But as in the proof of Prop. 2, W is contained in some quadric hypersurface of Proj($k[\alpha_1,\dots, \alpha_n]$) defined by g, where g is some quadric form in $k[\alpha_1,\dots, \alpha_{n-1}]$. If g is irreducible, then $V=W$ and if g is reducible, then V is a hyperplane defined by a linear form in $k[\alpha_1,\dots, \alpha_{n-1}]$. In either case, V contains the closed point $P_0 \in$ Proj(R) defined by the ideal ($\beta \in H$; $\beta \neq \alpha_n$). As P_0 is a smooth point, this is a contradiction. Q.E.D.

Summarizing our results for normality, we have

THEOREM 2. *Any non-normal three dimensional homogeneous Gorenstein ASL domain over an algebraically closed field is, up to isomorphism as ASL, either* [6, Example g)] *or* Example b) *in* (3.2).

REMARK. Let k be a field and $\bar{k}$ an algebraic closure of k. If R is an ASL domain on H over k and $\bar{R}=R\otimes_k\bar{k}$ is a normal ASL domain on H over k, then R is also normal by faithfully flat descent of normality.

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