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On connection formulas for a fourth order hypergeometric system

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§1. Introduction

We shall be concerned with a connection problem for the fourth order Fuchsian differential system (hypergeometric system)

$$(1.1) \qquad (t-B)\frac{dx}{dt} = Ax \quad (t \in \boldsymbol{C}),$$

where B is a 4 by 4 diagonal matrix of the form

$$B = \mathrm{diag}\,[\lambda_0, \lambda_0, \lambda_1, \lambda_2]$$

and $A=[a_{jk}]$ is a 4 by 4 matrix similar to a diagonal matrix of the form

$$\mathrm{diag}\,[\mu_1, \mu_1, \mu_2, \mu_2]$$

with $a_{12}=a_{21}=0$.

Obviously, (1.1) has only four regular singularities at $t=\lambda_l$ ($l=0, 1, 2$) and $t=\infty$ in the whole complex plane.

Denoting the diagonal elements a_{jj} of A by ρ_j ($j=1, 2, 3, 4$), we here assume the following:

[A_0] There are no straight lines through all finite singularities λ_0, λ_1 and λ_2.

[A_1: Generic condition] None of the quantities

$$\rho_j,\ \rho_j - \rho_k,\ \mu_p,\ \mu_1 - \mu_2 \quad (j,\ k=1, 2, 3, 4, j\neq k;\ p=1, 2)$$

is an integer. This implies that there exist no logarithmic solutions.

[A_2: Irreducible condition] None of the quantities

$$\rho_j + \rho_3 - \mu_1 - \mu_2, \quad \rho_j - \mu_p \quad (j=1, 2; p=1, 2)$$

is an integer.

[A_3] None of the quantities

$$\rho_1 + \rho_2 - \mu_1 - \mu_2, \quad \rho_k - \mu_p \quad (k=3, 4; p=1, 2)$$

is an integer.

The hypergeometric system (1.1) corresponds to a section of the Appell function F_3. In the paper [6] T. Sasai calculates the monodromy group of the above hypergeometric system and then shows that the group is irreducible under the condition [A_2]. We observe that, under the conditions [A_1] and [A_2], the quantities

$$d_0 = a_{13}a_{24} - a_{14}a_{23}, \quad d_1 = a_{31}a_{42} - a_{32}a_{41}$$

and all elements of A except a_{12} and a_{21} never vanish.

Now the purpose of this paper is to evaluate explicit values of connection coefficients between solutions near $t=\infty$ and the finite singularities $t=\lambda_l$ $(l=0, 1, 2)$. The general theory of such connection problems for hypergeometric systems is developed by M. Kohno [3] (see also [4]). This work is a quite good example of his method.

In the later consideration, we may assume without loss of generality that there hold

$$\begin{cases} \arg(\lambda_1-\lambda_0) < \arg(\lambda_2-\lambda_0) < \arg(\lambda_1-\lambda_0) + \pi, \\ \arg(\lambda_0-\lambda_1) > \arg(\lambda_2-\lambda_1) > \arg(\lambda_0-\lambda_1) - \pi, \\ \arg(\lambda_0-\lambda_2) < \arg(\lambda_1-\lambda_2) < \arg(\lambda_0-\lambda_2) + \pi. \end{cases}$$

§ 2. Solutions near the finite singularities

2.1. Solutions near $t=\lambda_0$

Near $t=\lambda_0$, there exist two non-holomorphic solutions of (1.1) of the form

$$\hat{X}_{0j}(t) = (t-\lambda_0)^{\rho_j} \sum_{m=0}^{\infty} \hat{G}_{0j}(m)(t-\lambda_0)^m \quad (|t-\lambda_0|<R_0; j=1, 2), \tag{2.1}$$

where $R_0=\min(|\lambda_l-\lambda_0|;\ l=1, 2)$. The coefficient vectors $\hat{G}_{0j}(m)$ $(m\geqq 0; j=1, 2)$ are uniquely determined up to a constant factor by the recursion formulas

$$\begin{cases} (B-\lambda_0)(m+1+\rho_j)\hat{G}_{0j}(m+1) = (m+\rho_j-A)\hat{G}_{0j}(m) \quad (m\geqq 0) \\ (B-\lambda_0)\rho_j\hat{G}_{0j}(0) = 0 \quad (j=1, 2). \end{cases} \tag{2.2}$$

On the other hand, there exist holomorphic solutions of (1.1) of the form

$$X_0(t) = \sum_{m=0}^{\infty} G_0(m)(t-\lambda_0)^m \quad (|t-\lambda_0|<R_0),$$

where the coefficient vectors $G_0(m)$ $(m \geqq 0)$ are characterized by the recursion formulas replacing $m+\rho_j$ by m in (2.2). In particular, $G_0(0)$ is characterized by

$$\begin{bmatrix} \rho_1 & 0 & a_{13} & a_{14} \\ 0 & \rho_2 & a_{23} & a_{24} \end{bmatrix} G_0(0) = 0.$$

Then there exist two linearly independent holomorphic solutions of (1.1) near $t=\lambda_0$.

In order to seek the explicit forms of $\hat{G}_{0j}(m)$ $(m \geqq 0; j=1, 2)$ and $G_0(m)$ $(m \geqq 0)$, we regard m as a complex variable in (2.2). Then, eliminating the components $\hat{g}^3_{0j}$, $\hat{g}^4_{0j}$ and $\hat{g}^{3-j}_{0j}$ of $\hat{G}_{0j}(m)=[\hat{g}^k_{0j}(m)]^4_{k=1}$ in (2.2) to obtain the second order linear difference equation for $\hat{g}^j_{0j}$ and putting

$$\hat{g}^j_{0j}(m) = \frac{\Gamma(m+\rho_j-\mu_1)\Gamma(m+\rho_j-\mu_2)}{(\lambda_1-\lambda_0)^m(\lambda_2-\lambda_0)^m\Gamma(m+1+\rho_j)\Gamma(m+1)}\hat{\hat{g}}^j_{0j}(m),$$

$$m+1=z$$

and

$$\hat{\hat{g}}^j_{0j}(m) = \hat{\hat{g}}^j_{0j}(z-1) = \tilde{g}^j_{0j}(z),$$

we can easily see that $\tilde{g}^j_{0j}(z)$ satisfies the following second order linear difference equation:

$$\begin{aligned} (2.3)\quad & (z+2+2\rho_j-\rho_1-\rho_2-2)f(z+2) \\ & -[\{(\lambda_1-\lambda_0)+(\lambda_2-\lambda_0)\}(z+1)+(\lambda_1-\lambda_0)(\rho_j+\rho_3-\mu_1-\mu_2-1) \\ & +(\lambda_2-\lambda_0)(\rho_j+\rho_4-\mu_1-\mu_2-1)]f(z+1)+(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)zf(z)=0 \\ & \qquad\qquad (j=1, 2). \end{aligned}$$

Taking account of the Fuchs' relation (trace relation) $\sum_{k=1}^4 \rho_k=2(\mu_1+\mu_2)$, i.e.,

$$\rho_j+\rho_3-\mu_1-\mu_2-1+\rho_j+\rho_4-\mu_1-\mu_2-1 = 2\rho_j-\rho_1-\rho_2-2,$$

we can immediately see that (2.3) is just of the normal form of the so-called hypergeometric difference equation [1; p. 69(103)]. P. M. Batchelder defines six particular solutions and investigates their global behavior in great details. Among them, we here choose suitable solutions so that they satisfy the initial condition in (2.2) and moreover have no poles in the right half plane. As such solutions we can first take the entire solutions of (2.3)

$$\ell^{0j}(z) = (\lambda_1-\lambda_0)^{z-1}{}_2F_1\left(\begin{matrix} 1-z, & \rho_j+\rho_4-\mu_1-\mu_2 \\ & 2\rho_j-\rho_1-\rho_2 \end{matrix} \middle| \chi\right) \qquad (j=1, 2),$$

where $\chi=1-(\lambda_2-\lambda_0)/(\lambda_1-\lambda_0)$ and ${}_2F_1\left(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix}\middle| w\right)^{*)}$ denotes the hypergeometric function. As to the asymptotic behavior of $\ell^{0j}(z)$ for sufficiently large values of z, P. M. Batchelder gives the following result:

$$\ell^{0j}(z) \sim \begin{cases} -S_2^j(z), & \theta_0-\pi < \arg z < \theta_0, \\ S_1^j(z), & \theta_0 < \arg z < \theta_0+\pi, \end{cases}$$

where $\theta_0=\tan^{-1}\{\log|1-\chi|/\arg(1-\chi)\}$ $(\tan^{-1}0=0)$ and $S_\nu^j(z)$ $(\nu=1, 2)$ are formal solutions of (2.3) of the form

$$S_\nu^j(z) = (\lambda_\nu-\lambda_0)^z z^{-(\rho_j+\rho_{5-\nu}-\mu_1-\mu_2)}(s_\nu+s'_\nu z^{-1}+\cdots) \ (\nu=1, 2).$$

Making use of $\ell^{0j}(z)$, we can now give explicitly the coefficient vectors $\hat{G}_{0j}(m)=[\hat{g}_{0j}^k(m)]_{k=1}^4$ $(j=1, 2)$ in (2.1) as follows:

$$\hat{g}_{0j}^j(m) = \frac{\Gamma(\rho_j+1)}{\Gamma(\rho_j-\mu_1)\Gamma(\rho_j-\mu_2)} \times \frac{\Gamma(m+\rho_j-\mu_1)\Gamma(m+\rho_j-\mu_2)}{(\lambda_1-\lambda_0)^m(\lambda_2-\lambda_0)^m\Gamma(m+1+\rho_j)\Gamma(m+1)}\ell^{0j}(m+1),$$

$$\hat{g}_{0j}^{3-j}(m) = \frac{1}{a_{j3}a_{j4}(2\lambda_j-\lambda_1-\lambda_2)}\Big[d_0(\lambda_1-\lambda_0)(\lambda_2-\lambda_0) \times \frac{(m+1+\rho_j)(m+1)}{(m+\rho_j-\mu_1)(m+\rho_j-\mu_2)}\hat{g}_{0j}^j(m+1) - \{(\lambda_{3-j}-\lambda_0)a_{13}a_{24}-(\lambda_j-\lambda_0)a_{14}a_{23}\}\hat{g}_{0j}^j(m)\Big],$$

$$\hat{g}_{0j}^{2+\nu}(m) = \frac{(-1)^{j+\nu}}{d_0}\{a_{3-j,5-\nu}m\hat{g}_{0j}^j(m)-a_{j,5-\nu}(m+2\rho_j-\rho_1-\rho_2)\hat{g}_{0j}^{3-j}(m)\} \qquad (\nu=1, 2; j=1, 2).$$

In particular

$$\hat{G}_{0j}(0) = e_j \quad (j=1, 2), \tag{2.4}$$

where e_j denotes the j-th unit vector. Further, we can define the coefficient vectors $G_{0j}(m)=[g_{0j}^k(m)]_{k=1}^4$ $(j=1, 2)$ of holomorphic solutions by

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*) Throughout this paper we regard ${}_2F_1\left(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix}\middle| w\right)$ and the generalized hypergeometric function ${}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3\\ \beta_1, \beta_2\end{matrix}\middle| w\right)$ as single valued holomorphic functions of w in $C/[1, \infty)$.

$$G_{0j}(m) = \frac{\Gamma(-\rho_j+1)\Gamma(\rho_j-\mu_1+1)\Gamma(\rho_j-\mu_2+1)}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(\rho_j+1)}\hat{G}_{0j}(m-\rho_j) \quad (j=1, 2).$$

Then, in particular, the initial values $G_{0j}(0)$ $(j=1, 2)$ are given by

(2.5)

$$\begin{cases} g_{0j}^{j}(0) = (\rho_j-\mu_1)(\rho_j-\mu_2)(\lambda_2-\lambda_0)^{\rho_j}{}_2F_1\left(\begin{matrix}\rho_j,\ \rho_j+\rho_4-\mu_1-\mu_2\\ 2\rho_j-\rho_1-\rho_2\end{matrix}\middle|\chi\right), \\ g_{0j}^{3-j}(0) = \dfrac{a_{4j}a_{3-j,4}\rho_j}{2\rho_j-\rho_1-\rho_2+1}\chi(\lambda_2-\lambda_0)^{\rho_j}{}_2F_1\left(\begin{matrix}\rho_j+1,\ \rho_j+\rho_4-\mu_1-\mu_2+1\\ 2\rho_j-\rho_1-\rho_2+2\end{matrix}\middle|\chi\right), \\ g_{0j}^{3}(0) = a_{3j}\rho_j(1-\chi)(\lambda_2-\lambda_0)^{\rho_j}{}_2F_1\left(\begin{matrix}\rho_j+1,\ \rho_j+\rho_4-\mu_1-\mu_2+1\\ 2\rho_j-\rho_1-\rho_2+1\end{matrix}\middle|\chi\right), \\ g_{0j}^{4}(0) = a_{4j}\rho_j(\lambda_2-\lambda_0)^{\rho_j}{}_2F_1\left(\begin{matrix}\rho_j+1,\ \rho_j+\rho_4-\mu_1-\mu_2\\ 2\rho_j-\rho_1-\rho_2+1\end{matrix}\middle|\chi\right) \qquad (j=1, 2). \end{cases}$$

The growth orders of $\hat{G}_{0j}(m)$ and $G_{0j}(m)$ $(j=1, 2)$ for sufficiently large values of m are as follows:

$$\hat{G}_{0j}(m),\ G_{0j}(m) = \begin{cases} O((\lambda_1-\lambda_0)^{-m}m^{\gamma}), & \theta_0-\pi<\arg m<\theta_0, \\ O((\lambda_2-\lambda_0)^{-m}m^{\gamma'}), & \theta_0<\arg m<\theta_0+\pi, \end{cases}$$

where γ and γ' are suitable constants.

Now we put

$$X_{0j}(t) = \textstyle\sum_{m=0}^{\infty} G_{0j}(m)(t-\lambda_0)^m \quad (|t-\lambda_0|<R_0;\ j=1, 2),$$

which are holomorphic solutions of (1.1) near $t=\lambda_0$. As will be seen in §4, $X_{0j}(t)$ $(j=1, 2)$ are linearly independent and then $X_{0j}(t)$ $(j=1, 2)$ and $\hat{X}_{0j}(t)$ $(j=1, 2)$ form a fundamental set of solutions of (1.1) near $t=\lambda_0$.

We next take another solutions of (2.3), which are called the principal solutions of the hypergeometric difference equation, of the form

$$\mathscr{k}_1^{0j}(z) = \frac{\Gamma(z)(\lambda_1-\lambda_0)^{z-1}}{\Gamma(z+\rho_j+\rho_4-\mu_1-\mu_2)}{}_2F_1\left(\begin{matrix}\rho_j+\rho_4-\mu_1-\mu_2,\ \rho_{3-j}+\rho_4-\mu_1-\mu_2+1\\ z+\rho_j+\rho_4-\mu_1-\mu_2\end{matrix}\middle|\frac{1}{\chi}\right),$$

$$\mathscr{k}_2^{0j}(z) = \frac{\Gamma(z)(\lambda_2-\lambda_0)^{z-1}}{\Gamma(z+\rho_j+\rho_3-\mu_1-\mu_2)}{}_2F_1\left(\begin{matrix}\rho_j+\rho_3-\mu_1-\mu_2,\ \rho_{3-j}+\rho_3-\mu_1-\mu_2+1\\ z+\rho_j+\rho_3-\mu_1-\mu_2\end{matrix}\middle|1-\frac{1}{\chi}\right)$$

$(j=1, 2)$

with the asymptotic behavior

$$\mathscr{h}_\nu^{0j}(z) \sim S_\nu^j(z), \quad |\arg z| < \frac{\pi}{2} + \varepsilon \quad ({}^\exists\varepsilon>0;\ \nu=1, 2).$$

Since these principal solutions have poles at $z=-r$ ($r=0, 1, 2,\dots$), they cannot become solutions of (2.2), that is, the initial condition in (2.2) is not satisfied. However they give the coefficient vectors $\tilde{G}_{0l}(m)=[\tilde{g}_{0l}^k(m)]_{k=1}^4$ ($l=1, 2$) of holomorphic solutions of (1.1) as follows:

$$\tilde{g}_{0l}^j(m) = \frac{a_{j,2+l}}{\Gamma(-\mu_1)\Gamma(-\mu_2)} \times \frac{\Gamma(m-\mu_1)\Gamma(m-\mu_2)}{(\lambda_1-\lambda_0)^{m-\rho_j}(\lambda_2-\lambda_0)^{m-\rho_j}\Gamma(m+1)\Gamma(m+1-\rho_j)} \frac{\mathscr{h}_{3-l}^{0j}(m+1-\rho_j)}{(\lambda_l-\lambda_0)^{\rho_j}},$$

$$\tilde{g}_{0l}^{3-j}(m) = \frac{1}{a_{j3}a_{j4}(2\lambda_j-\lambda_1-\lambda_2)}\Bigg[d_0(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)\frac{(m+1)(m+1-\rho_j)}{(m-\mu_1)(m-\mu_2)} \times \tilde{g}_{0l}^j(m+1) - \{(\lambda_{3-j}-\lambda_0)a_{13}a_{24}-(\lambda_j-\lambda_0)a_{14}a_{23}\}\tilde{g}_{0l}^j(m)\Bigg],$$

$$\tilde{g}_{0l}^{2+\nu}(m) = \frac{(-1)^\nu}{d_0}\{a_{1,5-\nu}(m-\rho_2)\tilde{g}_{0l}^2(m) - a_{2,5-\nu}(m-\rho_1)\tilde{g}_{0l}^1(m)\}$$

$$(\nu=1, 2;\ l=1, 2).$$

Here it is remarked that for each l ($l=1, 2$), the vectors $\tilde{G}_{0l}(m)$ defined as above by putting $j=1$ and $j=2$ are same ones. This fact is easily checked by means of formulas satisfied by the hypergeometric function ${}_2F_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}\middle| w\right)$. The initial values are as follows:

$$(2.6)\quad \begin{cases} \tilde{g}_{0l}^j(0) = \dfrac{a_{j,2+l}}{\Gamma(\rho_{2+l}-\mu_1-\mu_2+1)} \times {}_2F_1\left(\begin{matrix}\rho_j+\rho_{2+l}-\mu_1-\mu_2,\ \rho_{3-j}+\rho_{2+l}-\mu_1-\mu_2+1 \\ \rho_{2+l}-\mu_1-\mu_2+1\end{matrix}\middle| \chi_l\right), \\ \tilde{g}_{0l}^{2+l}(0) = \dfrac{1}{\Gamma(\rho_{2+l}-\mu_1-\mu_2)} \times {}_2F_1\left(\begin{matrix}\rho_1+\rho_{2+l}-\mu_1-\mu_2,\ \rho_2+\rho_{2+l}-\mu_1-\mu_2 \\ \rho_{2+l}-\mu_1-\mu_2\end{matrix}\middle| \chi_l\right), \\ \tilde{g}_{0l}^{5-l}(0) = \dfrac{a_{5-l,2+l}(1-\chi_l)}{\Gamma(\rho_{2+l}-\mu_1-\mu_2+1)} \times {}_2F_1\left(\begin{matrix}\rho_1+\rho_{2+l}-\mu_1-\mu_2+1,\ \rho_2+\rho_{2+l}-\mu_1-\mu_2+1 \\ \rho_{2+l}-\mu_1-\mu_2+1\end{matrix}\middle| \chi_l\right), \end{cases}$$

$$(j=1, 2;\ l=1, 2),$$

where $\chi_1=1-\chi^{-1}$ and $\chi_2=\chi^{-1}$. The growth orders of $\tilde{G}_{0l}(m)$ $(l=1, 2)$ for sufficiently large values of m are as follows:

$$\tilde{G}_{0l}(m) = O((\lambda_l-\lambda_0)^{-m}m^{\gamma}), \quad |\arg m| < \frac{\pi}{2}+\varepsilon \quad ({}^{\exists}\varepsilon>0;\ l=1, 2),$$

where γ is a suitable constant. Now we put

$$\tilde{X}_{0l}(t) = \sum_{m=0}^{\infty} \tilde{G}_{0l}(m)(t-\lambda_0)^m \quad (|t-\lambda_0|<R_0;\ l=1, 2), \tag{2.7}$$

and then obtain holomorphic solutions of (1.1) near $t=\lambda_0$. As will be seen in §4, $\tilde{X}_{0l}(t)$ is holomorphic at $t=\lambda_{3-l}$ $(l=1, 2)$, and moreover $\hat{X}_{0j}(t)$ $(j=1, 2)$ and $\tilde{X}_{0l}(t)$ $(l=1, 2)$ form a fundamental set of solutions of (1.1) near $t=\lambda_0$.

2.2. Solutions near $t=\lambda_l$ $(l=1, 2)$

Near $t=\lambda_l$ $(l=1, 2)$, there exist a non-holomorphic solution and holomorphic solutions of (1.1) of the form

$$\hat{X}_l(t) = (t-\lambda_l)^{\rho_{2+l}} \sum_{m=0}^{\infty} \hat{G}_l(m)(t-\lambda_l)^m \quad (|t-\lambda_l|<R_l), \tag{2.8}$$

and

$$X_l(t) = \sum_{m=0}^{\infty} G_l(m)(t-\lambda_l)^m \quad (|t-\lambda_l|<R_l),$$

respectively, where $R_l=\min(|\lambda_0-\lambda_l|, |\lambda_1+\lambda_2-2\lambda_l|)$. The coefficient vectors $\hat{G}_l(m)$ $(m\geqq 0)$ are uniquely determined up to a constant factor by the recursion formulas

$$\begin{cases} (B-\lambda_l)(m+1+\rho_{2+l})\hat{G}_l(m+1) = (m+\rho_{2+l}-A)\hat{G}_l(m) \quad (m\geqq 0), \\ (B-\lambda_l)\rho_{2+l}\hat{G}_l(0) = 0, \end{cases} \tag{2.9}$$

and the coefficient vectors $G_l(m)$ $(m\geqq 0)$ are characterized by the recursion formulas replacing $m+\rho_{2+l}$ by m in (2.9). In particular, $G_l(0)$ is characterized by

$$[a_{31}a_{32}\rho_3 a_{34}]G_1(0) = 0 \quad \text{or} \quad [a_{41}a_{42}a_{43}\rho_4]G_2(0) = 0.$$

Then there exist three linearly independent holomorphic solutions of (1.1) near $t=\lambda_l$.

In order to seek the explicit forms of $\hat{G}_l(m)$ $(m\geqq 0)$ and $G_l(m)$ $(m\geqq 0)$, we again regard m as a complex variable in (2.9). Then, eliminating $\hat{g}_l^1$, $\hat{g}_l^2$ and $\hat{g}_l^{5-l}$ of $\hat{G}_l(m)=[\hat{g}_l^k(m)]_{k=1}^4$ in (2.9) to obtain the third order linear difference equation for $\hat{g}_l^{2+l}$ and putting

$$\hat{g}_l^{2+l}(m) = \frac{\Gamma(m+\rho_{2+l}-\mu_1)\Gamma(m+\rho_{2+l}-\mu_2)}{(\lambda_0-\lambda_l)^m(\lambda_1+\lambda_2-2\lambda_l)^m\Gamma(m+1+\rho_{2+l})\Gamma(m+1)}\hat{\hat{g}}_l^{2+l}(m),$$

$m+1=z$

and

$$\hat{\hat{g}}_l^{2+l}(m) = \hat{\hat{g}}_l^{2+l}(z-1) = \tilde{\tilde{g}}_l^{2+l}(z),$$

we can easily see that $\tilde{\tilde{g}}_l^{2+l}(z)$ satisfies the following third order linear difference equation:

$$(2.10)\quad \begin{aligned}&(z+2+\beta_1)(z+2+\beta_2)f(z+2)\\ &\quad -[\hat{\lambda}_0\{2(z+1+\beta_1)(z+1+\beta_2)-(\alpha_1+\alpha_2-1)(z+1)-\beta_1\beta_2-\alpha_1\alpha_2\}\\ &\qquad\qquad +\hat{\lambda}_1(z+1+\alpha_1)(z+1+\alpha_2)]f(z+1)\\ &\quad +\hat{\lambda}_0[\hat{\lambda}_0(z+1+\beta_1+\beta_2-\alpha_1-\alpha_2)+\hat{\lambda}_1\{2(z+1)+\alpha_1+\alpha_2-1\}]zf(z)\\ &\quad -\hat{\lambda}_0^2\hat{\lambda}_1 z(z-1)f(z-1)=0,\end{aligned}$$

where

$$\beta_\nu = \rho_{2+l} - \mu_\nu - 2,\quad \alpha_\nu = \rho_{2+l} + \rho_\nu - \mu_1 - \mu_2 - 1 \quad (\nu=1, 2)$$

and

$$\hat{\lambda}_0 = \lambda_1 + \lambda_2 - 2\lambda_l,\quad \hat{\lambda}_1 = \lambda_0 - \lambda_l.$$

We shall now investigate this difference equation. By the Mellin transformation

$$f(z) = \int t^{z-1}\phi(t)dt,$$

the difference equation (2.10) is transformed into the following linear differential equation:

$$(2.11)\quad \begin{aligned}&(t-\hat{\lambda}_0)^2(t-\hat{\lambda}_1)\phi'' - (t-\hat{\lambda}_0)\{(\beta_1+\beta_2-1)(t-\hat{\lambda}_0)-(\alpha_1+\alpha_2-1)(\hat{\lambda}_1-\hat{\lambda}_0)\}\phi'\\ &\quad +\{\beta_1\beta_2(t-\hat{\lambda}_0)-\alpha_1\alpha_2(\hat{\lambda}_1-\hat{\lambda}_0)\}\phi = 0.\end{aligned}$$

This equation is the second order Fuchsian differential equation with three singularities $t=\hat{\lambda}_0$, $\hat{\lambda}_1$ and ∞, whose solutions are expressed in terms of the hypergeometric function ${}_2F_1$. We here note that none of the quantities

$$\alpha_1 - \alpha_2,\quad \alpha_\nu\ (\nu=1, 2),\quad \beta_\nu\ (\nu=1, 2),\quad \beta_1 + \beta_2 - \alpha_1 - \alpha_2$$

is an integer under the assumptions [A_1], [A_2] and [A_3]. Hence we have the solutions of (2.10) as follows:

$$\mathscr{k}_{0j}(z) = \int_0^{\hat{\lambda}_0} t^{z-1}\phi_{0j}(t)dt \quad (j=1, 2),$$

$$\mathscr{k}_1(z) = \int_0^{\hat{\lambda}_1} t^{z-1}\phi_{12}(t)dt,$$

and

$$\ell(z) = \int_{\hat\lambda_1}^{\hat\lambda_0} t^{z-1}\phi_{12}(t)dt, \tag{2.12}$$

where

$$\phi_{0j}(t) = (t-\hat\lambda_0)^{\alpha_j}{}_2F_1\left(\begin{matrix}\alpha_j-\beta_1,\ \alpha_j-\beta_2\\ 2\alpha_j-\alpha_1-\alpha_2+1\end{matrix}\,\middle|\,\frac{t-\hat\lambda_0}{\hat\lambda_1-\hat\lambda_0}\right)\quad (j=1,\ 2)$$

and

$$\phi_{12}(t) = (t-\hat\lambda_0)^{\alpha_2}(t-\hat\lambda_1)^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}{}_2F_1\left(\begin{matrix}\beta_1-\alpha_1+1,\ \beta_2-\alpha_1+1\\ \beta_1+\beta_2-\alpha_1-\alpha_2+2\end{matrix}\,\middle|\,\frac{t-\hat\lambda_1}{\hat\lambda_0-\hat\lambda_1}\right)$$

are the solutions of (2.11). In the above integrals, we take the paths of integration as the straight lines. As to the arguments of their integrands, we take $\arg t=\arg\hat\lambda_0$ and $\arg(t-\hat\lambda_0)=\arg\hat\lambda_0+\pi$ in $k_{0j}(z)$ $(j=1, 2)$, and $\arg t=\arg\hat\lambda_1$ and $\arg(t-\hat\lambda_1)=\arg\hat\lambda_1+\pi$ in $k_1(z)$. If $\arg\hat\lambda_0>\arg\hat\lambda_1$, we take $\arg(t-\hat\lambda_0)$ between $\arg\hat\lambda_0+\pi$ and $\arg\hat\lambda_0+2\pi$ in $k_1(z)$; if $\arg\hat\lambda_0<\arg\hat\lambda_1$, we take $\arg(t-\hat\lambda_0)$ between $\arg\hat\lambda_0$ and $\arg\hat\lambda_0+\pi$ in $k_1(z)$. In $\ell(z)$, let $\arg t$ go from $\arg\hat\lambda_1$ to $\arg\hat\lambda_0$ and if $\arg\hat\lambda_0>\arg\hat\lambda_1$, we take for $\arg(t-\hat\lambda_0)$ the value which lies between $\arg\hat\lambda_0+\pi$ and $\arg\hat\lambda_0+2\pi$, and for $\arg(t-\hat\lambda_1)$ the value between $\arg\hat\lambda_1$ and $\arg\hat\lambda_1+\pi$; if $\arg\hat\lambda_0<\arg\hat\lambda_1$, we take for $\arg(t-\hat\lambda_0)$ the value between $\arg\hat\lambda_0$ and $\arg\hat\lambda_0+\pi$, and for $\arg(t-\hat\lambda_1)$ the value between $\arg\hat\lambda_1+\pi$ and $\arg\hat\lambda_1+2\pi$. When the above integrals are divergent, we regard them as "the finite part of a divergent integral."

From the termwise integration, we can easily see that $k_{0j}(z)$ $(j=1, 2)$ and $\ell(z)$ are expressed in terms of the generalized hypergeometric function ${}_3F_2$ as follows:

$$\begin{cases} k_{0j}(z) = (e^{\pi i}\hat\lambda_0)^{\alpha_j}\Gamma(\alpha_j+1)\hat\lambda_0\tilde{k}_{0j}(z), \\[2ex] \tilde{k}_{0j}(z) = \dfrac{\hat\lambda_0^{z-1}\Gamma(z)}{\Gamma(z+1+\alpha_j)}\,{}_3F_2\left(\begin{matrix}\alpha_j+1,\ \alpha_j-\beta_1,\ \alpha_j-\beta_2\\ z+1+\alpha_j,\ 2\alpha_j-\alpha_1-\alpha_2+1\end{matrix}\,\middle|\,\dfrac{1}{\hat\chi}\right), \end{cases}$$

$$\begin{cases} \ell(z) = (\hat\lambda_0-\hat\lambda_1)^{\beta_1+\beta_2-\alpha_1-\alpha_2+2}(\hat\lambda_1-\hat\lambda_0)^{\alpha_2}\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2+2) \\ \qquad\qquad \times\dfrac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{\Gamma(\beta_1+2)\Gamma(\beta_2+2)}\tilde{\ell}(z), \\[2ex] \tilde{\ell}(z) = \hat\lambda_0^{z-1}{}_3F_2\left(\begin{matrix}1-z,\ \alpha_1+1,\ \alpha_2+1\\ \beta_1+2,\ \beta_2+2\end{matrix}\,\middle|\,\hat\chi\right), \end{cases}$$

where $\hat\chi=1-\hat\lambda_1/\hat\lambda_0=\chi_l$ and the arguments of $\hat\lambda_0-\hat\lambda_1$ and $\hat\lambda_1-\hat\lambda_0$ in $\ell(z)$ are in accordance with $\arg(t-\hat\lambda_1)$ and $\arg(t-\hat\lambda_0)$ in the integrand of (2.12), respectively.

Here we see that $\mathscr{l}(z)$ is an entire solution of (2.10) and $\mathscr{k}_{0j}(z)$ $(j=1, 2)$ have poles at $z=-r$ $(r=0, 1, 2,\ldots)$.

As to the global behavior of $\mathscr{k}_{0j}(z)$ $(j=1, 2)$, $\mathscr{k}_1(z)$ and $\mathscr{l}(z)$ for sufficiently large values of z, we have the following

PROPOSITION 1. *For sufficiently large values of* z,

$$\mathscr{k}_{0j}(z) \sim S_{0j}(z), \quad |\arg z| < \frac{\pi}{2} + \varepsilon \quad ({}^{\exists}\varepsilon>0;\, j=1, 2) \tag{2.13}$$

$$\mathscr{k}_1(z) \sim S_1(z), \quad |\arg z| < \frac{\pi}{2} + \varepsilon \quad ({}^{\exists}\varepsilon<0), \tag{2.14}$$

where $S_{0j}(z)$ $(j=1, 2)$ *and* $S_1(z)$ *are the formal solutions of* (2.10) *of the form*

$$S_{0j}(z) = \hat{\lambda}_0^z z^{-(\alpha_j+1)}(s_{0j}+s'_{0j}z^{-1}+\cdots) \quad (j=1, 2),$$

$$S_1(z) = \hat{\lambda}_1^z z^{-(\beta_1+\beta_2-\alpha_1-\alpha_2+2)}(s_1+s'_1 z^{-1}+\cdots).$$

As to $\mathscr{l}(z)$, *if* $\arg\hat{\lambda}_0>\arg\hat{\lambda}_1$, *then*

$$\mathscr{l}(z) = \begin{cases} O(\hat{\lambda}_0^z z^{\gamma}), & -\dfrac{\pi}{2} - \varepsilon < \arg z < \hat{\theta} \\ O(\hat{\lambda}_1^z z^{\gamma'}), & \hat{\theta} < \arg z < \dfrac{\pi}{2} + \varepsilon \quad ({}^{\exists}\varepsilon>0), \end{cases} \tag{2.15}$$

and if $\arg\hat{\lambda}_0<\arg\hat{\lambda}_1$, *then*

$$\mathscr{l}(z) = \begin{cases} O(\hat{\lambda}_1^z z^{\gamma'}), & -\dfrac{\pi}{2} - \varepsilon < \arg z < \hat{\theta} \\ O(\hat{\lambda}_0^z z^{\gamma}), & \hat{\theta} < \arg z < \dfrac{\pi}{2} + \varepsilon \quad ({}^{\exists}\varepsilon >0), \end{cases} \tag{2.16}$$

where $\hat{\theta}=\tan^{-1}\{\log|1-\hat{\chi}|/\arg(1-\hat{\chi})\}$ $(\tan^{-1}0=0)$ *and* γ *and* γ' *are suitable constants.*

From this proposition, we see that $\mathscr{k}_{0j}(z)$ $(j=1, 2)$ and $\mathscr{k}_1(z)$ are the principal solutions of (2.10) in the right half plane. This proposition follows from the next lemma, of which the proof is omitted here and referred to [5; p. 4] for example:

LEMMA. *Let* $\psi(\xi)$ *be holomorphic in* $D=\{\xi;\, |\mathrm{Im}\,\xi|<\eta,\ \mathrm{Re}\,\xi<\eta'\}$ $(\eta, \eta'>0)$, *and have the growth order*

$$\psi(\xi) = O(\xi^{\gamma}) \quad (\gamma\text{: a constant})$$

as $\xi\to\infty$, $\xi\in D$. *Then the function* $\Psi(z)$ *of a complex variable* z *defined by*

$$\Psi(z) = \frac{1}{2\pi i}\int_C e^{z\xi}\,\xi^{\alpha}\psi(\xi)d\xi \quad (\operatorname{Re} z>0),$$

where the path of interation C is the contour in D from ∞ along the negative real axis, around $\xi=0$ in the positive sense and back to ∞ along the negative real axis, and the argument of ξ goes from $-\pi$ to π, has an asymptotic expansion

$$\Psi(z) \sim z^{-(\alpha+1)}\sum_{r=0}^{\infty}\frac{d_r}{\Gamma(-\alpha-r)}z^{-r}$$

in the right half plane $|\arg z|<\pi/2$, where $\psi(\xi)=\sum_{r=0}^{\infty}d_r\xi^r$ at $\xi=0$.

PROOF of PROPOSITION 1. We here consider $\mathscr{h}_1(z)$ only. For $\mathscr{h}_{0j}(z)$ $(j=1, 2)$, we can have the similar discussions to obtain the above results. Putting $t=\hat{\lambda}_1\tau$ in $\mathscr{h}_1(z)$, we have

$$\mathscr{h}_1(z) = \hat{\lambda}_1^z\int_0^1 \tau^{z-1}(\tau-1)^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}\hat{\phi}_{12}(\tau)d\tau,$$

where $\hat{\phi}_{12}(\tau)=(\tau-1)^{-(\beta_1+\beta_2-\alpha_1-\alpha_2+1)}\phi_{12}(\hat{\lambda}_1\tau)$. Putting $\tau=e^{\xi}$ again, we have

$$\begin{aligned}(2.17)\qquad \mathscr{h}_1(z) &= \hat{\lambda}_1^z\int_{-\infty}^0 e^{z\xi}\xi^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}\psi(\xi)d\xi\\ &= (e^{-2\pi i(\beta_1+\beta_2-\alpha_1-\alpha_2)}-1)^{-1}\hat{\lambda}_1^z\int_C e^{z\xi}\xi^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}\psi(\xi)d\xi,\end{aligned}$$

where

$$\psi(\xi) = \left(\frac{e^{\xi}-1}{\xi}\right)^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}\hat{\phi}_{12}(e^{\xi}) = \xi^{-(\beta_1+\beta_2-\alpha_1-\alpha_2+1)}\phi_{12}(\hat{\lambda}_1e^{\xi}),$$

and the path of integration and the value of $\arg\xi$ are the same in Lemma. We observe that the above $\psi(\xi)$ satisfies the conditions for $\psi(\xi)$ in Lemma. Then, in the right half plane $|\arg z|<\pi/2$, $\mathscr{h}_1(z)$ has an asymptotic expansion

$$\mathscr{h}_1(z) \sim \hat{\lambda}_1^z z^{-(\beta_1+\beta_2-\alpha_1-\alpha_2+2)}\sum_{r=0}^{\infty}\hat{d}_r z^{-r},$$

of which the right hand side must be a formal solution of (2.10).

In (2.17), we put $\xi=e^{i\theta}\zeta$ $(\theta\in\boldsymbol{R})$. Then we have

$$\begin{aligned}&\int_C e^{z\xi}\xi^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}\psi(\xi)d\xi\\ &\quad= e^{i\theta(\beta_1+\beta_2-\alpha_1-\alpha_2+2)}\int_{e^{-i\theta}C} e^{z'\zeta}\zeta^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}\hat{\psi}(\zeta)d\zeta,\end{aligned}$$

where $z'=e^{i\theta}z$ and $\hat{\psi}(\zeta)=\psi(e^{i\theta}\zeta)$. Now, if $|\theta|$ is sufficiently small, $\hat{\psi}(\zeta)$ satisfies the condition for $\psi(\zeta)$ in Lemma and the integral

$$\int_C e^{z'\zeta}\zeta^{\beta_1+\beta_2-\alpha_1-2+1}\hat{\psi}(\zeta)d\zeta$$

is well-defined for $|\arg z'|<\pi/2$. Moreover, from Cauchy's theorem,

$$\int_C e^{z'\zeta}\zeta^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}\hat{\psi}(\zeta)d\zeta = \int_{e^{-i\theta}C} e^{z'\zeta}\zeta^{\beta_1+\beta_2-\alpha_1-\alpha_2+1}\hat{\psi}(\zeta)d\zeta$$

holds for $z \in \{|\arg z|<\pi/2\} \cap \{|\arg z'|<\pi/2\}$. Hence $\mathscr{k}_1(z)$ has an asymptotic expansion

$$\begin{aligned}\mathscr{k}_1(z) &\sim \hat{\lambda}_1^z e^{i\theta(\beta_1+\beta_2-\alpha_1-\alpha_2+2)} z'^{-(\beta_1+\beta_2-\alpha_1-\alpha_2+2)} \textstyle\sum_{r=0}^{\infty} (e^{i\theta r}\hat{d}_r) z'^{-r} \\ &= \hat{\lambda}_1^z z^{-(\beta_1+\beta_2-\alpha_1-\alpha_2+2)} \textstyle\sum_{r=0}^{\infty} \hat{d}_r z^{-r}\end{aligned}$$

in the half plane $|\arg z'|<\pi/2$, i.e., $-\pi/2-\theta<\arg z<\pi/2-\theta$. This proves (2.14).

As to $\mathscr{l}(z)$, there exist constants c_1 and c_2, $(c_1, c_2)\neq(0, 0)$, such that

$$\begin{aligned}\mathscr{l}(z) &= \int_{\hat{\lambda}_1}^{\lambda_0} t^{z-1}\phi_{12}(t)dt = \int_0^{\lambda_0} t^{z-1}\phi_{12}(t)dt - \int_0^{\hat{\lambda}_1} t^{z-1}\phi_{12}(t)dt \\ &= \int_0^{\lambda_0} t^{z-1}(c_1\phi_{01}(t)+c_2\phi_{02}(t))dt - \int_0^{\hat{\lambda}_1} t^{z-1}\phi_{12}(t)dt \\ &= c_1\mathscr{k}_{01}(z) + c_2\mathscr{k}_{02}(z) - \mathscr{k}_1(z).\end{aligned}$$

Then, comparing the determining factors of the asymptotic expansions of $\mathscr{k}_{0j}(z)$ $(j=1, 2)$ and $\mathscr{k}_1(z)$, we have (2.15) and (2.16). Thus Proposition 1 is established.

Now, making use of the above $\mathscr{l}(z)$, we can give the coefficient vectors $\hat{G}_l(m)=[\hat{g}_l^k(m)]_{k=1}^4$ in (2.8) as follows:

$$\begin{aligned}\hat{g}_l^{2+l}(m) &= \frac{\Gamma(\rho_{2+l}+1)}{\Gamma(\rho_{2+l}-\mu_1)\Gamma(\rho_{2+l}-\mu_2)} \\ &\quad\times \frac{\Gamma(m+\rho_{2+l}-\mu_1)\Gamma(m+\rho_{2+l}-\mu_2)}{(\lambda_0-\lambda_l)^m(\lambda_1+\lambda_2-2\lambda_l)^m\Gamma(m+1)\Gamma(m+1+\rho_{2+l})}\bar{\mathscr{l}}(m+1), \\ \hat{g}_l^{5-l}(m) &= \frac{1}{(\lambda_{3-l}\lambda_0)a_{2+l,5-l}}\left\{\frac{(m-1+\rho_{2+l}-\mu_1)(m-1+\rho_{2+l}-\mu_2)}{m+\rho_{2+l}}\right. \\ &\qquad\qquad \left.\times\,\hat{g}_l^{2+l}(m-1)-(\lambda_0-\lambda_l)m\hat{g}_l^{2+l}(m)\right\}, \\ \hat{g}_l^{\nu}(m) &= (-1)^{l+\nu}d_l^{-1}[a_{5-l,3-\nu}(m+\rho_{2+l}+\rho_\nu-\mu_1-\mu_2)\hat{g}_l^{2+l}(m) \\ &\quad - a_{2+l,3-\nu}\{(m+\rho_{2+l}+\rho_\nu-\mu_1-\mu_2)\hat{g}_l^{5-l}(m) \\ &\qquad - (\lambda_1+\lambda_2-2\lambda_l)(m+1+\rho_{2+l})\hat{g}_l^{5-l}(m+1)\}] \\ &\qquad\qquad\qquad\qquad (\nu=1, 2),\end{aligned}$$

in particular,

$$\hat{G}_l(0) = e_{2+l}. \tag{2.18}$$

Further, we can define the coefficient vectors $G_{l1}(m)=[g^k_{l1}(m)]^4_{k=1}$ of a holomorphic solution by

$$G_{l1}(m) = \frac{\Gamma(-\rho_{2+l}+1)\Gamma(\rho_{2+l}-\mu_1+1)\Gamma(\rho_{2+l}-\mu_2+1)}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(\rho_{2+l}+1)}(\lambda_0-\lambda_l)^{-\rho_{2+l}}\hat{G}_l(m-\rho_{2+l}).$$

Then, in particular, the initial value $G_{l1}(0)$ is given by

$$\left\{\begin{aligned}
g^{2+l}_{l1}(0) &= (\rho_{2+l}-\mu_1)(\rho_{2+l}-\mu_2) \\
&\quad\times {}_3F_2\left(\left.\begin{matrix}\rho_{2+l},\ \rho_{2+l}+\rho_1-\mu_1-\mu_2,\ \rho_{2+l}+\rho_2-\mu_1-\mu_2 \\ \rho_{2+l}-\mu_1,\ \rho_{2+l}-\mu_2\end{matrix}\ \right|\ \chi_l\right), \\
g^{5-l}_{l1}(0) &= a_{5-l,2+l}\rho_{2+l}(1-\chi_l) \\
&\quad\times {}_3F_2\left(\left.\begin{matrix}\rho_{2+l}+1,\ \rho_{2+l}+\rho_1-\mu_1-\mu_2+1,\ \rho_{2+l}+\rho_2-\mu_1-\mu_2+1 \\ \rho_{2+l}-\mu_1+1,\ \rho_{2+l}-\mu_2+1\end{matrix}\ \right|\ \chi_l\right), \\
g^{\nu}_{l1}(0) &= a_{\nu,2+l}\rho_{2+l} \\
&\quad\times {}_3F_2\left(\left.\begin{matrix}\rho_{2+l}+1,\ \rho_{2+l}+\rho_\nu-\mu_1-\mu_2,\ \rho_{2+l}+\rho_{3-\nu}-\mu_1-\mu_2+1 \\ \rho_{2+l}-\mu_1+1,\ \rho_{2+l}-\mu_2+1\end{matrix}\ \right|\ \chi_l\right) \\
&\qquad\qquad (\nu=1,\ 2).
\end{aligned}\right. \tag{2.19}$$

On the other hand, $\not{h}_{0j}(z)$ $(j=1, 2)$ give the coefficient vectors $G_{l,1+j}(m)=[g^k_{l,1+j}(m)]^4_{k=1}$ $(j=1, 2)$ of holomorphic solutions as follows:

$$\begin{aligned}
g^{2+l}_{l,1+j}(m) &= \frac{a_{2+l,j}(\lambda_0-\lambda_l)^{-\rho_{2+l}}}{\Gamma(-\mu_1)\Gamma(-\mu_2)} \\
&\quad\times\frac{\Gamma(m-\mu_1)\Gamma(m-\mu_2)\bar{\not{h}}_{0j}(m+1-\rho_{2+l})}{(\lambda_0-\lambda_l)^{m-\rho_{2+l}}(\lambda_1+\lambda_2-2\lambda_l)^{m-\rho_{2+l}}\Gamma(m+1)\Gamma(m+1-\rho_{2+l})} \\
g^{5-l}_{l,1+j}(m) &= \frac{1}{(\lambda_{3-l}-\lambda_0)a_{2+l,5-l}} \\
&\quad\times\left\{\frac{(m-1-\mu_1)(m-1-\mu_2)}{m}g^{2+l}_{l,1+j}(m-1)-(\lambda_0-\lambda_l)(m-\rho_{2+l})g^{2+l}_{l,1+j}(m)\right\}, \\
g^{\nu}_{l,1+j}(m) &= (-1)^{l+\nu}d_1^{-1}[a_{5-l,3-\nu}(m+\rho_\nu-\mu_1-\mu_2)g^{2+l}_{l,1+j}(m) \\
&\qquad -a_{2+l,3-\nu}\{(m+\rho_\nu-\mu_1-\mu_2)g^{5-l}_{l,1+j}(m) \\
&\qquad\qquad -(\lambda_1+\lambda_2-2\lambda_l)(m+1)g^{5-l}_{l,1+j}(m+1)\}] \\
&\qquad\qquad\qquad (\nu=1,\ 2;\ j=1,\ 2).
\end{aligned}$$

In this definition, the initial values are as follows:

$$
(2.20)\quad \begin{cases}
g^{2+l}_{l,1+j}(0) = \dfrac{a_{2+l,j}}{\Gamma(\rho_j-\mu_1-\mu_2+1)} \\
\qquad \times {}_3F_2\left(\begin{matrix}\rho_{2+l}+\rho_j-\mu_1-\mu_2,\ \rho_j-\mu_1+1,\ \rho_j-\mu_2+1\\ \rho_j-\mu_1-\mu_2+1,\ 2\rho_j-\rho_1-\rho_2+1\end{matrix}\ \middle|\ \dfrac{1}{\chi_l}\right), \\
g^{5-l}_{l,1+j}(0) = \dfrac{a_{5-l,j}}{\Gamma(\rho_j-\mu_1-\mu_2+1)}\left(1-\dfrac{1}{\chi_l}\right) \\
\qquad \times {}_3F_2\left(\begin{matrix}\rho_{2+l}+\rho_j-\mu_1-\mu_2+1,\ \rho_j-\mu_1+1,\ \rho_j-\mu_2+1\\ \rho_j-\mu_1-\mu_2+1,\ 2\rho_j-\rho_1-\rho_2+1\end{matrix}\ \middle|\ \dfrac{1}{\chi_l}\right), \\
g^{j}_{l,1+j}(0) = \dfrac{1}{\Gamma(\rho_j-\mu_1-\mu_2)} \\
\qquad \times {}_3F_2\left(\begin{matrix}\rho_{2+l}+\rho_j-\mu_1-\mu_2,\ \rho_j-\mu_1,\ \rho_j-\mu_2\\ \rho_j-\mu_1-\mu_2,\ 2\rho_j-\rho_1-\rho_2\end{matrix}\ \middle|\ \dfrac{1}{\chi_l}\right), \\
g^{3-j}_{l,1+j}(0) = \dfrac{a_{2+l,j}a_{3-j,2+l}}{2\rho_j-\rho_1-\rho_2+1}\dfrac{1}{\Gamma(\rho_j-\mu_1-\mu_2+1)}\dfrac{1}{\chi_l} \\
\qquad \times {}_3F_2\left(\begin{matrix}\rho_{2+l}+\rho_j-\mu_1-\mu_2+1,\ \rho_j-\mu_1+1,\ \rho_j-\mu_2+1\\ \rho_j-\mu_1-\mu_2+1,\ 2\rho_j-\rho_1-\rho_2+2\end{matrix}\ \middle|\ \dfrac{1}{\chi_l}\right)
\end{cases}
$$

$$(j=1, 2).$$

As to the growth orders of $\hat{G}_l(m)$ $(l=1, 2)$ and $G_{li}(m)$ $(l=1, 2;\ i=1, 2, 3)$, noting that if $l=1$, then $\arg\hat{\lambda}_0<\arg\hat{\lambda}_1$; if $l=2$, then $\arg\hat{\lambda}_0>\arg\hat{\lambda}_1$, we have the following results:

$$
\hat{G}_1(m),\ G_{11}(m) = \begin{cases} O((\lambda_2-\lambda_1)^{-m}m^{\gamma'}), & -\dfrac{\pi}{2}-\varepsilon<\arg m<\theta_1 \\ O((\lambda_0-\lambda_1)^{-m}m^{\gamma}), & \theta_1<\arg m<\dfrac{\pi}{2}+\varepsilon \end{cases} \qquad ({}^{\exists}\varepsilon>0),
$$

$$
\hat{G}_2(m),\ G_{21}(m) = \begin{cases} O((\lambda_0-\lambda_2)^{-m}m^{\gamma}), & -\dfrac{\pi}{2}-\varepsilon<\arg m<\theta_2 \\ O((\lambda_1-\lambda_2)^{-m}m^{\gamma'}), & \theta_2<\arg m<\dfrac{\pi}{2}+\varepsilon \end{cases} \qquad ({}^{\exists}\varepsilon>0),
$$

$$
G_{l,1+j}(m) = O((\lambda_0-\lambda_l)^{-m}m^{\gamma''}),\quad |\arg m|<\frac{\pi}{2}+\varepsilon \qquad ({}^{\exists}\varepsilon>0;\ j=1, 2;\ l=1, 2),
$$

where $\theta_l=\tan^{-1}\{\log|1-\chi_l|/\arg(1-\chi_l)\}$ $(\tan^{-1}0=0;\ l=1, 2)$ and γ, γ' and γ''

are suitable constants. Now we put

(2.21) $$X_{li}(t) = \sum_{m=0}^{\infty} G_{li}(m)(t-\lambda_l)^m \quad (|t-\lambda_i|<R_l;\ i=1,\ 2,\ 3),$$

which are holomorphic solutions of (1.1) near $t=\lambda_l$. As we will see in §4, $X_{li}(t)$ $(i=1, 2, 3)$ and $\hat{X}_l(t)$ form a fundamental set of solutions of (1.1) and, in particular, $X_{l,1+j}(t)$ $(j=1, 2)$ are holomorphic at $t=\lambda_{3-l}$ $(l=1, 2)$.

§ 3. Solutions near $t=\infty$

Near $t=\infty$, there exist four linearly independent solutions of (1.1), i.e., a fundamental set of solutions of the form

$$Y^{pq}(t;\lambda) = (t-\lambda)^{\mu_p} \sum_{r=0}^{\infty} H^{pq}(r;\lambda)(t-\lambda)^{-r}$$
$$(|t-\lambda|>\max\{|\lambda_l-\lambda|;\ l=0,\ 1,\ 2\};\ p,\ q=1,\ 2),$$

where λ is an arbitrary complex number. The coefficient vectors $H^{pq}(r;\lambda)$ $(r\geqq 1;\ p,\ q=1, 2)$ are determined in terms of the recursion formulas

(3.1) $$(r-\mu_p+A)H^{pq}(r;\lambda) = (r-1-\mu_p)(B-\lambda)H^{pq}(r-1;\lambda) \quad (p,\ q=1,\ 2)$$

subject to the initial conditions

(3.2) $$(A-\mu_p)H^{pq}(0;\lambda) = 0 \quad (p,\ q=1,\ 2).$$

For each p $(p=1, 2)$, since rank $(A-\mu_p)=2$, we can choose as $H^{pq}(0;\lambda)$ $(q=1, 2)$ two linearly independent eigen-vectors of A. The explicit values of $H^{pq}(0;\lambda)$ $(p, q=1, 2)$ will be given in §4.2.

Now we shall prove the following

PROPOSITION 2. *For* $\lambda'\neq\lambda$, *if* $H^{pq}(0;\lambda')=H^{pq}(0;\lambda)$ $(p, q=1, 2)$, *then we have*

$$Y^{pq}(t;\lambda') = Y^{pq}(t;\lambda) \quad (p,\ q=1,\ 2)$$

for $t\in\{t;\ |t-\lambda'|>\max\{|\lambda_l-\lambda'|;\ l=0,\ 1,\ 2\},\ |t-\lambda|>\max\{|\lambda_l-\lambda|;\ l=0,\ 1,\ 2\},$ $|\arg(t-\lambda)-\arg(t-\lambda')|<\pi\}$.

PROOF. There exist c_{jk} $(j, k=1, 2)$ such that

$$Y^{pq}(t;\lambda') = \sum_{j,k=1}^{2} c_{jk} Y^{jk}(t;\lambda)$$

in the above domain. However, by the assumption $[A_1]$, i.e., $\mu_1-\mu_2\neq$integer, $c_{3-p,k}$ $(k=1, 2)$ must be zero. Hence we have

$$(t-\lambda')^{-\mu_p} Y^{pq}(t;\lambda') = \sum_{k=1}^{2} c_{pk}(t-\lambda')^{-\mu_p} Y^{pk}(t;\lambda).$$

We here let t tend to infinity in the above domain, obtaining

$$H^{pq}(0;\lambda') = \sum_{k=1}^{2} c_{pk} H^{pk}(0;\lambda).$$

Therefore, we have $c_{p,3-q}=0$ and $c_{pq}=1$. This completes the proof of Proposition 2.

The above result will be used in the last stage of §4.

§4. Connection coefficients

4.1. Evaluation of the connection coefficients (the summary of M. Kohno's paper [3])

Let λ be one of λ_l $(l=0, 1, 2)$ and

$$X(t) = (t-\lambda)^{\rho} \sum_{m=0}^{\infty} G(m)(t-\lambda)^m$$

be a solution of (1.1) near $t=\lambda$ obtained in §2, where ρ is 0 or a diagonal element of A and $G(\zeta)$ has the growth order

$$G(\zeta) = \begin{cases} O((\lambda'-\lambda)^{-\zeta}\zeta^{\gamma}), & -\dfrac{\pi}{2} - \varepsilon < \arg\zeta < \theta, \\ O((\lambda''-\lambda)^{-\zeta}\zeta^{\gamma'}), & \theta < \arg\zeta < \dfrac{\pi}{2} + \varepsilon \quad (^{\exists}\varepsilon>0) \end{cases}$$

Then $X(t)$ is represented by the Barnes-integral as follows:

$$X(t) = -\frac{1}{2\pi i}\int_C G(\zeta)\frac{\pi e^{-\pi i\zeta}}{\sin\pi\zeta}(t-\lambda)^{\zeta+\rho}d\zeta, \tag{4.1}$$

where the path of integration C is a Barnes-contour running along the straight line $\zeta=-ia$ from $+\infty-ia$ to $0-ia$, a curve from $-ia$ to ia and the straight line $\zeta=ia$ from $0+ia$ to $+\infty-ia$ such that the points $\zeta=m (m=0, 1, 2,\dots)$ lie to the right of C and the points $\zeta=-\rho+\mu_p-r$ $(r=0, 1, 2,\dots; p=1, 2)$ lie to the left of C. The constant a is taken as $a>\max\{|\mathrm{Im}(\mu_p-\rho)|; p=1, 2\}$. In fact, from the growth order of $G(\zeta)$, we easily see that if $|t-\lambda|<R=\min\{|\lambda'-\lambda|, |\lambda''-\lambda|\}$, then the integral (4.1) is absolutely convergent and equal to the sum of residues at $\zeta=m$ $(m=0, 1, 2,\dots)$. Hence (4.1) holds for $|t-\lambda|<R$.

Let ξ be an arbitrary negative number not being equal to $\mathrm{Re}(-\rho+\mu_p-r)$ $(r=0, 1, 2,\dots; p=1, 2)$. We take the positive integers N_p $(p=1, 2)$ such that

$$-(N_p+1) < \xi + \mathrm{Re}(\rho-\mu_p) < -N_p \quad (p=1, 2).$$

We now replace the path C in (4.1) by the rectilinear contour L_{ξ} which runs first from $+\infty-ia$ to $\xi-ia$, next from $\xi-ia$ to $\xi+ia$, and finally from $\xi+ia$ to $+\infty+ia$. Then we have

$$X(t) = -\frac{1}{2\pi i}\int_{L_\xi} G(\zeta)\frac{e^{-\pi i\zeta}}{\sin\pi\zeta}(t-\lambda)^{\zeta+\rho}d\zeta - \sum \mathrm{Res}\left[G(\zeta)\frac{\pi e^{-\pi i\zeta}}{\sin\pi\zeta}(t-\lambda)^{\zeta+\rho}\right], \tag{4.2}$$

where the summation covers all poles in the domain encircled by L_ξ and the curve from $-ia$ to ia of C. Since $G(-r)=0$ $(r=1, 2,\dots)$, $\zeta=-r$ $(r=1, 2,\dots)$ are no longer poles. Then the integrand in (4.1) has simple poles only at $\zeta=-\rho+\mu_p-r$ $(r=0, 1, 2,\dots, N_p;\ p=1, 2)$. Hence we have

$$\begin{aligned}
&\sum \mathrm{Res}\left[G(\zeta)\frac{\pi e^{-\pi i\zeta}}{\sin\pi\zeta}(t-\lambda)^{\zeta+\rho}\right]\\
&= \textstyle\sum_{p=1}^{2}\sum_{r=0}^{N_p} \lim_{\zeta\to-\rho+\mu_p-r}\left[(\zeta+\rho-\mu_p+r)G(\zeta)\frac{\pi e^{-\pi i\zeta}}{\sin\pi\zeta}(t-\lambda)^{\zeta+\rho}\right]\\
&= \textstyle\sum_{p=1}^{2}\sum_{r=0}^{N_p} \frac{\pi e^{-\pi i(\mu_p-\rho)}}{\sin\pi(\mu_p-\rho)}H^p(r)(t-\lambda)^{-r+\mu_p},
\end{aligned}$$

where

$$H^p(r) = \lim_{\zeta\to-\rho+\mu_p-r}[(\zeta+\rho-\mu_p+r)G(\zeta)] \quad (r\geqq 0;\ p=1, 2).$$

Now we shall show that the $H^p(r)$ $(r\geqq 0)$ satisfy the recursion formulas (3.1) and (3.2). Since $G(\zeta)$ is holomorphic at $\zeta=-\rho+\mu_p+1$, we have

$$\begin{aligned}
(A-\mu_p)H^p(0) &= \lim_{\zeta\to-\rho+\mu_p}[-(\zeta+\rho-A)(\zeta+\rho-\mu_p)G(\zeta)]\\
&= \lim_{\zeta\to-\rho+\mu_p}[-(\zeta+\rho-\mu_p)(B-\lambda)(\zeta+1+\rho)G(\zeta+1)] = 0.
\end{aligned}$$

For $r\geqq 1$, we have

$$\begin{aligned}
(r-\mu_p+A)H^p(r) &= \lim_{\zeta\to-\rho+\mu_p-r}[-(\zeta+\rho-A)(\zeta+\rho-\mu_p+r)G(\zeta)]\\
&= \lim_{\zeta\to-\rho+\mu_p-r}[-(B-\lambda)(\zeta+1+\rho)(\zeta+\rho-\mu_p+r)G(\zeta+1)]\\
&= (r-1-\mu_p)(B-\lambda)\lim_{\zeta+1\to-\rho+\mu_p-(r-1)}[(\zeta+1+\rho-\mu_p+r-1)G(\zeta+1)]\\
&= (r-1-\mu_p)(B-\lambda)H^p(r-1).
\end{aligned}$$

Hence there uniquely exist the constants T^{pq} $(p, q=1, 2)$ such that

$$-\frac{\pi e^{-\pi i(\mu_p-\rho)}}{\sin\pi(\mu_p-\rho)}H^p(r) = \textstyle\sum_{q=1}^{2} T^{pq}H^{pq}(r;\lambda) \quad (r\geqq 0;\ p=1, 2). \tag{4.3}$$

We therefore obtain

$$(4.4)\qquad X(t) = \sum_{p=1}^{2} \sum_{q=1}^{2} T^{pq}[(t-\lambda)^{\mu_p} \sum_{r=0}^{N_p} H^{pq}(r;\lambda)(t-\lambda)^{-r}] - \frac{1}{2\pi i}\int_{L_\xi} G(\zeta)\frac{\pi e^{-\pi i\zeta}}{\sin \pi\zeta}(t-\lambda)^{\zeta+\rho}d\zeta \quad (|t-\lambda|<R).$$

We here apply the results of B. L. J. Braaksma [2; p. 271–278; Lemma 6 and Lemma 6a] to the above integral: The integral

$$-\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} G(\zeta)\frac{\pi e^{-\pi i\zeta}}{\sin \pi\zeta}(t-\lambda)^{\zeta+\rho}d\zeta$$

is the analytic continuation of the integral in (4.4) for t which in view of the growth order of $G(\zeta)$, lies in the sector

$$(4.5)\qquad \arg(\lambda''-\lambda) + \varepsilon' \leqq \arg(t-\lambda) \leqq \arg(\lambda'-\lambda) + 2\pi - \varepsilon',$$

ε' being an arbitrary small positive number. Moreover, for t in the sector (4.5), we have

$$\left| -\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} G(\zeta)\frac{\pi e^{-\pi i\zeta}}{\sin \pi\zeta}(t-\lambda)^{\zeta+\rho}d\zeta \right| < K\left(\frac{|t-\lambda|}{M}\right)^{\xi+\mathrm{Re}\,\rho},$$

where $M=\max\{|\lambda'-\lambda|, |\lambda''-\lambda|\}$ and K is a constant independent of t (but depending on ε'). Hence $X(t)$ is analytically continued into the sector (4.5), and there we have

$$X(t) = \sum_{p=1}^{2} \sum_{q=1}^{2} T^{pq} Y^{pq}(t;\lambda)$$

for $|t-\lambda|>M$.

4.2. Results

The residues of $G_{0j}(m)$ $(j=1, 2)$ at $m=-\rho_j+\mu_p$ $(p=1, 2)$ et al. can be calculated by their explicit forms obtained in §2. The residues of $\hat{G}_{0j}(m)$ $(j=1, 2)$ at $m=-\rho_j+\mu_p$ $(p=1, 2)$, and those of $G_{0j}(m)$ $(j=1, 2)$ and $G_{2,1+j}(m)$ $(j=1, 2)$ at $m=\mu_p$ $(p=1, 2)$ are equal to the vectors $H_0^{pj}=[h_{0,k}^{pj}]_{k=1}^{4}$ $(j=1, 2;\ p=1, 2)$ defined by

$$(4.6)\qquad \begin{cases} h_{0,j}^{pj} = (\rho_j-\mu_{3-p})(\lambda_2-\lambda_0)^{\rho_j-\mu_p}{}_2F_1\left(\begin{matrix}\rho_j-\mu_p,\ \rho_j+\rho_4-\mu_1-\mu_2\\ 2\rho_j-\rho_1-\rho_2\end{matrix}\middle|\ \chi\right), \\ h_{0,3-j}^{pj} = \dfrac{a_{4j}a_{3-j,4}}{2\rho_j-\rho_1-\rho_2+1}(\lambda_2-\lambda_0)^{\rho_j-\mu_p}\chi \\ \qquad\qquad \times {}_2F_1\left(\begin{matrix}\rho_j-\mu_p+1,\ \rho_j+\rho_4-\mu_1-\mu_2+1\\ 2\rho_j-\rho_1-\rho_2+2\end{matrix}\middle|\ \chi\right), \end{cases}$$

$$\begin{cases} h_{0,3}^{pj} = a_{3j}(\lambda_2-\lambda_0)^{\rho_j-\mu_p}(1-\chi)\,{}_2F_1\left(\begin{matrix}\rho_j-\mu_p+1,\ \rho_j+\rho_4-\mu_1-\mu_2+1\\ 2\rho_j-\rho_1-\rho_2+1\end{matrix}\middle|\,\chi\right),\\[2ex] h_{0,4}^{pj} = a_{4j}(\lambda_2-\lambda_0)^{\rho_j-\mu_p}\,{}_2F_1\left(\begin{matrix}\rho_j-\mu_p+1,\ \rho_j+\rho_4-\mu_1-\mu_2\\ 2\rho_j-\rho_1-\rho_2+1\end{matrix}\middle|\,\chi\right)\end{cases}$$

$$(j=1,\,2;\ p=1,\,2)$$

except for constant factors. For example,

$$\lim_{\zeta\to-\rho_j+\mu_p}[(\zeta+\rho_j-\mu_p)\hat{G}_{0j}(\zeta)]$$

$$= -\frac{\Gamma(\rho_j+1)\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(\rho_j-\mu_1+1)\Gamma(\rho_j-\mu_2+1)\Gamma(\mu_p+1)\Gamma(-\rho_j+\mu_p)}H_0^{pj}$$

$$(j=1,\,2;\ p=1,\,2).$$

And the residues of $G_{1,1+j}(m)$ $(j=1, 2)$ at $m=\mu_p$ $(p=1, 2)$ are equal to the vectors $\hat{H}_0^{pj}=[\hat{h}_{0,k}^{pj}]_{k=1}^4$ $(j=1, 2;\ p=1, 2)$ defined by

$$\begin{cases} \hat{h}_{0,j}^{pj} = (\rho_j-\mu_{3-p})(\lambda_1-\lambda_0)^{\rho_j-\mu_p}\,{}_2F_1\left(\begin{matrix}\rho_j-\mu_p,\ \rho_j+\rho_3-\mu_1-\mu_2\\ 2\rho_j-\rho_1-\rho_2\end{matrix}\middle|\,\frac{\chi}{\chi-1}\right),\\[2ex] \hat{h}_{0,3-j}^{pj} = \dfrac{a_{3j}a_{3-j,3}}{2\rho_j-\rho_1-\rho_2+1}(\lambda_1-\lambda_0)^{\rho_j-\mu_p}\dfrac{\chi}{\chi-1}\cdot\\[1ex] \qquad\qquad\times\,{}_2F_1\left(\begin{matrix}\rho_j-\mu_p+1,\ \rho_j+\rho_3-\mu_1-\mu_2+1\\ 2\rho_j-\rho_1-\rho_2+2\end{matrix}\middle|\,\frac{\chi}{\chi-1}\right),\\[2ex] \hat{h}_{0,3}^{pj} = a_{3j}(\lambda_1-\lambda_0)^{\rho_j-\mu_p}\,{}_2F_1\left(\begin{matrix}\rho_j-\mu_p+1,\ \rho_j+\rho_3-\mu_1-\mu_2\\ 2\rho_j-\rho_1-\rho_2+1\end{matrix}\middle|\,\frac{\chi}{\chi-1}\right),\\[2ex] \hat{h}_{0,4}^{pj} = a_{4j}(\lambda_1-\lambda_0)^{\rho_j-\mu_p}\dfrac{1}{1-\chi}\,{}_2F_1\left(\begin{matrix}\rho_j-\mu_p+1,\ \rho_j+\rho_3-\mu_1-\mu_2+1\\ 2\rho_j-\rho_1-\rho_2+1\end{matrix}\middle|\,\frac{\chi}{\chi-1}\right)\end{cases}$$

$$(j=1,\,2;\ p=1,\,2)$$

except for constant factors. In fact, for instance,

$$\lim_{\zeta\to\mu_p}[(\zeta-\mu_p)G_{1,1+j}(\zeta)]$$

$$= \frac{\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(\mu_p+1)\Gamma(\rho_j-\mu_{3-p}+1)}\frac{(\lambda_0-\lambda_1)^{-\mu_p}}{(\lambda_1-\lambda_0)^{\rho_j-\mu_p}}\hat{H}_0^{pj}$$

$$(j=1,\,2;\ p=1,\,2).$$

Here one can easily verify that the vector $\hat{H}_0^{pj}$ is equal to H_0^{pj} $(j=1, 2;\ p=1, 2)$ by virtue of the well-known formula

$$ {}_2F_1\left(\begin{matrix}\beta, \alpha \\ \gamma\end{matrix}\middle|\, w\right) = (1-w)^{-\alpha}{}_2F_1\left(\begin{matrix}\alpha, \gamma-\beta \\ \gamma\end{matrix}\middle|\, \frac{w}{w-1}\right) \quad (|\arg(1-w)|<\pi). $$

On the other hand, the residues of $\hat{G}_l(m)$ $(l=1, 2)$ at $m=-\rho_{2+l}+\mu_p$ $(p=1, 2)$ and those of $G_{l1}(m)$ $(l=1, 2)$ and $\tilde{G}_{0l}(m)$ $(l=1, 2)$ at $m=\mu_p$ $(p=1, 2)$ are equal to the vector $H_l^p=[h_{l,k}^p]_{k=1}^4$ $(l=1, 2;\ p=1, 2)$ defined by

(4.7)

$$ \begin{cases} h_{l,\nu}^p = a_{\nu,2+l}\,{}_2F_1\left(\begin{matrix}\rho_\nu+\rho_{2+l}-\mu_1-\mu_2,\ \rho_{3-\nu}+\rho_{2+l}-\mu_1-\mu_2+1 \\ \rho_{2+l}-\mu_{3-p}+1\end{matrix}\middle|\, \chi_l\right) \quad (\nu=1, 2), \\ h_{l,2+l}^p = (\rho_{2+l}-\mu_{3-p})\,{}_2F_1\left(\begin{matrix}\rho_1+\rho_{2+l}-\mu_1-\mu_2,\ \rho_2+\rho_{2+l}-\mu_1-\mu_2 \\ \rho_{2+l}-\mu_{3-p}\end{matrix}\middle|\, \chi_l\right), \\ h_{l,5-l}^p = a_{5-l,2+l}(1-\chi_l)\,{}_2F_1\left(\begin{matrix}\rho_1+\rho_{2+l}-\mu_1-\mu_2+1,\ \rho_2+\rho_{2+l}-\mu_1-\mu_2+1 \\ \rho_{2+l}-\mu_{3-p}+1\end{matrix}\middle|\, \chi_l\right) \end{cases} $$

$$ (l=1, 2;\ p=1, 2) $$

except for constant factors. For example,

$$ (4.8) \qquad \lim_{\zeta\to-\rho_{2+l}+\mu_p} [(\zeta+\rho_{2+l}-\mu_p)\hat{G}_l(\zeta)] $$
$$ = -\frac{\Gamma(\rho_{2+l}+1)\Gamma(2\mu_p-\mu_1-\mu_2)(\lambda_0-\lambda_l)^{\rho_{2+l}-\mu_p}}{\Gamma(\rho_{2+l}-\mu_1+1)\Gamma(\rho_{2+l}-\mu_2+1)\Gamma(-\rho_{2+l}+\mu_p)\Gamma(\mu_p+1)} H_l^p $$
$$ (l=1, 2;\ p=1, 2). $$

Now we here apply the connection formula

$$ {}_2F_1\left(\begin{matrix}\alpha, \ \beta \\ \gamma\end{matrix}\middle|\, w^{-1}\right) = \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(we^{-\pi i})^\alpha{}_2F_1\left(\begin{matrix}\alpha, \alpha+1-\gamma \\ \alpha+1-\beta\end{matrix}\middle|\, w\right) $$
$$ + \frac{\Gamma(\alpha-\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(we^{-\pi i})^\beta{}_2F_1\left(\begin{matrix}\beta, \beta+1-\gamma \\ \beta+1-\alpha\end{matrix}\middle|\, w\right) $$
$$ (0<\arg w<2\pi) $$

to (4.7), obtaining

$$ (4.9) \qquad H_l^p = \sum_{j=1}^2 \frac{\Gamma(\rho_{2+l}-\mu_{3-p}+1)\Gamma(\rho_1+\rho_2-2\rho_j)}{\Gamma(\mu_p-\rho_j)\Gamma(\rho_{3-j}+\rho_{2+l}-\mu_1-\mu_2)} $$

$$\times \frac{(e^{(3-2l)\pi i}\chi_l)^{\mu_1+\mu_2-\rho_j-\rho_{2+l}}}{a_{2+l,j}(\lambda_l-\lambda_0)^{\rho_j-\mu_p}} H_0^{pj} \qquad (l=1,\ 2;\ p=1,\ 2),$$

where we take $\arg \chi_l = \arg \{(\lambda_0-\lambda_{3-l})/(\lambda_l-\lambda_{3-l})\}$ $(l=1, 2)$.

Taking account of the above facts, we shall define the initial values $H^{pq}(0; \lambda)$ of the solutions $Y^{pq}(t; \lambda)$ $(p, q=1, 2)$ of (1.1) near $t=\infty$, not depending on λ, by

$$H^{pq}(0; \lambda) = H_0^{pq} \quad (p,\ q=1,\ 2). \tag{4.10}$$

The linear independence of $H^{pq}(0; \lambda)$ $(q=1, 2)$ for each p $(p=1, 2)$ can be seen from the non-vanishing of

$$h_{0,3}^{p1}h_{0,4}^{p2} - h_{0,3}^{p2}h_{0,4}^{p1} = \frac{a_{41}a_{42}}{a_{43}}(1-\chi)\frac{(\lambda_2-\lambda_0)^{\rho_1+\rho_2-2\mu_p}}{\chi^{\rho_2-\rho_1-1}}[\omega_1\omega_2' - \omega_1'\omega_2](\chi),$$

where

$$\begin{cases} \omega_1(z) = {}_2F_1\left(\begin{matrix}\rho_1-\mu_p+1,\ \rho_1+\rho_4-\mu_1-\mu_2 \\ \rho_1-\rho_2+1\end{matrix}\middle| z\right), \\ \omega_2(z) = z^{\rho_2-\rho_1}{}_2F_1\left(\begin{matrix}\rho_2-\mu_p+1,\ \rho_2+\rho_4-\mu_1-\mu_2 \\ \rho_2-\rho_1+1\end{matrix}\middle| z\right). \end{cases}$$

Now we are in a position to evaluate the connection coefficients. For example, we consider the relation

$$\hat{X}_1(t) = \textstyle\sum_{p,q=1}^2 \hat{T}_1^{pq} Y^{pq}(t; \lambda_1). \tag{4.11}$$

According to the consideration in §4.1, for each p $(p=1, 2)$ we have from (4.3)

$$\lim_{\zeta\to-\rho_3+\mu_p} [(\zeta+\rho_3-\mu_p)\hat{G}_1(\zeta)] = \frac{-\sin\pi(\mu_p-\rho_3)}{\pi e^{-\pi i(\mu_p-\rho_3)}} \textstyle\sum_{q=1}^2 \hat{T}_1^{pq}H^{pq}(0; \lambda_1)$$

and replace the left hand side by (4.8) together with (4.9). Then the definition (4.10) and the linear independence of H_0^{pq} $(q=1, 2)$ lead to

$$\begin{aligned} \hat{T}_1^{pq} = {} & \frac{-\pi e^{-\pi i(\mu_p-\rho_3)}}{\sin\pi(\mu_p-\rho_3)}(-1)\frac{\Gamma(\rho_3+1)\Gamma(2\mu_p-\mu_1-\mu_2)(\lambda_0-\lambda_1)^{\rho_3-\mu_p}}{\Gamma(\rho_3-\mu_1+1)\Gamma(\rho_3-\mu_2+1)\Gamma(-\rho_3+\mu_p)\Gamma(\mu_p+1)} \\ & \times \frac{\Gamma(\rho_3-\mu_{3-p}+1)\Gamma(\rho_1+\rho_2-2\rho_q)}{\Gamma(\mu_p-\rho_q)\Gamma(\rho_{3-q}+\rho_{2+l}-\mu_1-\mu_2)}\frac{(e^{\pi i}\chi_1)^{\mu_1+\mu_2-\rho_q-\rho_3}}{a_{3q}(\lambda_1-\lambda_0)^{\rho_q-\mu_p}} \\ & - \frac{1}{a_{3q}}\frac{\Gamma(\rho_3+1)\Gamma(\rho_1+\rho_2-2\rho_q)\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(\mu_p-\rho_q)\Gamma(\rho_{3-q}+\rho_3-\mu_1-\mu_2)\Gamma\mu_p+1)} \\ & \times \frac{(e^{\pi i}(\lambda_0-\lambda_1))^{\rho_3-\mu_p}}{(\lambda_1-\lambda_0)^{\rho_q-\mu_p}}\left(e^{\pi i}\left(1-\frac{1}{\chi}\right)\right)^{\mu_1+\mu_2-\rho_q-\rho_3} \qquad (p,\ q=1,\ 2). \end{aligned}$$

We here observe that the relation (4.11) holds for

$$\arg(\lambda_0-\lambda_1) < \arg(t-\lambda_1) < \arg(\lambda_2-\lambda_1) + 2\pi.$$

This follows from the growth order of $\hat{G}_1(m)$ obtained in §2.

By the similar calculations for the other solutions, we obtain the following

THEOREM. *The solutions* $\hat{X}_{0j}(t)$ $(j=1, 2)$, $X_{0j}(t)$ $(j=1, 2)$, $\tilde{X}_{0l}(t)$ $(l=1, 2)$, $\hat{X}_l(t)$ $(l=1, 2)$, $X_{l1}(t)$ $(l=1, 2)$ and $X_{l,1+j}$ $(l=1, 2; j=1, 2)$ *subject to the initial values* (2.4), (2.5), (2.6), (2.18), (2.19) *and* (2.20), *respectively, are represented by* $Y^{pq}(t; \lambda)$ $(p, q=1, 2; \lambda=\lambda_0, \lambda_1$ *or* $\lambda_2)$ *subject to the initial values* (4.10) $(=(4.6))$ *as follows*:

$$\hat{X}_{0j}(t) = \sum_{p,q=1}^{2} \hat{T}_{0j}^{pq} Y^{pq}(t; \lambda_0), \quad X_{0j}(t) = \sum_{p,q=1}^{2} T_{0j}^{pq} Y^{pq}(t; \lambda_0)$$

$$\textit{for} \quad \arg(\lambda_2-\lambda_0) < \arg(t-\lambda_0) < \arg(\lambda_1-\lambda_0) + 2\pi \quad (j=1, 2),$$

$$\tilde{X}_{0l}(t) = \sum_{p,q=1}^{2} \tilde{T}_{0l}^{pq} Y^{pq}(t; \lambda_0)$$

$$\textit{for} \quad \arg(\lambda_l-\lambda_0) < \arg(t-\lambda_0) < \arg(\lambda_l-\lambda_0) + 2\pi \quad (l=1, 2).$$

$$\hat{X}_1(t) = \sum_{p,q=1}^{2} \hat{T}_1^{pq} Y^{pq}(t; \lambda_1), \quad X_{11}(t) = \sum_{p,q=1}^{2} T_{11}^{pq} Y^{pq}(t; \lambda_1)$$

$$\textit{for} \quad \arg(\lambda_0-\lambda_1) < \arg(t-\lambda_1) < \arg(\lambda_2-\lambda_1) + 2\pi,$$

$$\hat{X}_2(t) = \sum_{p,q=1}^{2} \hat{T}_2^{pq} Y^{pq}(t; \lambda_2), \quad X_{21}(t) = \sum_{p,q=1}^{2} T_{21}^{pq} Y^{pq}(t; \lambda_2)$$

$$\textit{for} \quad \arg(\lambda_1-\lambda_2) < \arg(t-\lambda_2) < \arg(\lambda_0-\lambda_2) + 2\pi,$$

and

$$X_{l,1+j}(t) = \sum_{p,q=1}^{2} T_{l,1+j}^{pq} Y^{pq}(t; \lambda_l)$$

$$\textit{for} \quad \arg(\lambda_0-\lambda_l) < \arg(t-\lambda_l) < \arg(\lambda_0-\lambda_l) + 2\pi \quad (l=1, 2; j=1, 2),$$

where the connection coefficients are given by

$$\hat{T}_{0j}^{p,3-j} = T_{0j}^{p,3-j} = 0 \quad (p, j=1, 2)$$

$$\hat{T}_{0j}^{pj} = \frac{\Gamma(\rho_j+1)\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(\rho_j-\mu_{3-p}+1)\Gamma(\mu_p+1)} e^{-\pi i(\mu_p-\rho_j)} \quad (p, j=1, 2)$$

$$T_{0j}^{pj} = -\frac{\Gamma(-\rho_j+1)\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(-\rho_j+\mu_p)\Gamma(-\mu_{3-p})} e^{-\pi i\mu_p} \quad (p, j=1, 2)$$

$$\tilde{T}_{0l}^{pq} = \frac{1}{a_{2+l,q}} \frac{\Gamma(\rho_1+\rho_2-2\rho_q)\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(\mu_p-\rho_q)\Gamma(\rho_{3-q}+\rho_{2+l}-\mu_1-\mu_2)\Gamma(-\mu_{3-p})}$$

$$\times \frac{e^{-\pi i\mu_p}}{(\lambda_l-\lambda_0)^{\rho_q}} (e^{(3-2l)\pi i}\chi_l)^{\mu_1+\mu_2-\rho_q-\rho_{2+l}} \quad (p, q, l=1, 2)$$

$$\hat{T}_l^{pq} = \frac{1}{a_{2+l,q}} \frac{\Gamma(\rho_{2+l}+1)\Gamma(\rho_1+\rho_2-2\rho_q)\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(\mu_p-\rho_q)\Gamma(\rho_{3-q}+\rho_{2+l}-\mu_1-\mu_2)\Gamma(\mu_p+1)}$$

$$\times \frac{(e^{\pi i}(\lambda_0-\lambda_l))^{\rho_{2+l}-\mu_p}}{(\lambda_l-\lambda_0)^{\rho_q-\mu_p}}(e^{(3-2l)\pi i})^{\mu_1+\mu_2-\rho_q-\rho_{2+l}} \quad (p,\ q,\ l=1,\ 2)$$

$$T_{l1}^{pq} = \frac{-1}{a_{2+l,q}} \frac{\Gamma(-\rho_{2+l}+1)\Gamma(\rho_1+\rho_2-2\rho_q)\Gamma(\rho_{2+l}-\mu_{3-p}+1)\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(\mu_p-\rho_q)\Gamma(\mu_p-\rho_{2+l})\Gamma(\rho_{3-q}+\rho_{2+l}-\mu_1-\mu_2)\Gamma(-\mu_{3-p})}$$

$$\times \frac{(e^{\pi i}(\lambda_0-\lambda_l))^{-\mu_p}}{(\lambda_l-\lambda_0)^{\rho_q-\mu_p}}(e^{(3-2l)\pi i}\chi_l)^{\mu_1+\mu_2-\rho_p-\rho_{2+l}} \ (p,\ q,\ l=1,\ 2)$$

$$T_{l,1+j}^{p,3-j} = 0 \quad (p,\ l,\ j=1,\ 2)$$

and

$$T_{l,1+j}^{pj} = \frac{\Gamma(2\mu_p-\mu_1-\mu_2)}{\Gamma(\rho_j-\mu_{3-p}+1)\Gamma(-\mu_{3-p})} \frac{(e^{\pi i}(\lambda_0-\lambda_l))^{-\mu_p}}{(\lambda_l-\lambda_0)^{\rho_j-\mu_p}} \quad (p,\ l,\ j=1,\ 2),$$

where we take $\arg\chi_l=\arg\{(\lambda_0-\lambda_{3-l})/(\lambda_l-\lambda_{3-l})\}$ $(l=1,\ 2)$.

REMARK. $\tilde{X}_{0l}(t)$ $(l=1, 2)$ and $X_{l,1+j}(t)$ $(j=1, 2;\ l=1, 2)$ are holomorphic at $t=\lambda_{3-l}$. This follows from the fact that the integrals in (4.4) for $\tilde{X}_{0l}(t)$ and $X_{l,1+j}(t)$ $(j=1, 2)$ are analytically continued in the domain which contains the point λ_{3-l} from the growth order of $\tilde{G}_{0l}(m)$ and $G_{l,1+j}(m)$, respectively.

From the above Theorem, we immediately obtain the following result.

COROLLARY. *Each set of* $[\hat{X}_{01}(t), \hat{X}_{02}(t), X_{01}(t), X_{02}(t)]$, $[\hat{X}_{01}(t), \hat{X}_{02}(t), \tilde{X}_{01}(t), \tilde{X}_{02}(t)]$ *and* $[\hat{X}_l(t), X_{l1}(t), X_{l2}(t), X_{l3}(t)]$ $(l=1, 2)$ *forms a fundamental set of solutions of* (1.1).

PROOF. As to $[\hat{X}_{01}(t), \hat{X}_{02}(t), X_{01}(t), X_{02}(t)]$ and $[\hat{X}_{01}(t), \hat{X}_{02}(t), \tilde{X}_{01}(t), \tilde{X}_{02}(t)]$, it is sufficient to show that the linear independence of $X_{01}(t)$ and $X_{02}(t)$, and $\tilde{X}_{01}(t)$ and $\tilde{X}_{02}(t)$, respectively.
The linear independence of $X_{01}(t)$ and $X_{02}(t)$ immediately follows from the explicit values of T_{0j}^{pq} $(p, q, j=1, 2)$ in Theorem. On the other hand, if $\tilde{X}_{01}(t)$ and $\tilde{X}_{02}(t)$ are not linearly independent, then there exists a non-trivial entrie solution of (1.1). This contradicts the assumption $[A_1]$.

As to $[\hat{X}_l(t), X_{l1}(t), X_{l2}(t), X_{l3}(t)]$, it is sufficient to show that $X_{l1}(t)$, $X_{l2}(t)$ and $X_{l3}(t)$ are linearly independent. This follows from the non-vanishing of

$$T_{l3}^{12}(T_{l1}^{11}T_{l2}^{21}-T_{l2}^{11}T_{l1}^{21}),$$

which immediately follows from the explicit values of T_{li}^{pq} $(p, q, l=1, 2;\ i=1, 2, 3)$ in Theorem. Thus Corollary is proved.

By Proposition 2, taking suitable values for $\arg(\lambda_0-\lambda_1)$ and $\arg(\lambda_0-\lambda_2)$, the connection matrices between fundamental sets of solutions of (1.1) near $t=\lambda_l$ and $t=\lambda_k$ $(l, k=0, 1, 2;\ l\neq k)$ are calculated by means of connection coefficients in Theorem.

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