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On the growth of α-potentials in R^n and thinness of sets

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1. Introduction

In the n-dimensional euclidean space R^n, we define the α-potential of a nonnegative (Radon) measure μ by

$$R_\alpha\mu(x) = \int R_\alpha(x-y)d\mu(y),$$

where $R_\alpha(x)=|x|^{\alpha-n}$ if $0<\alpha<n$ and $R_n(x)=\log(1/|x|)$. Then it is easy to see that $|R_\alpha\mu|\not\equiv\infty$ if and only if

$$(1)\qquad \begin{aligned} &\int (1+|y|)^{\alpha-n}d\mu(y) < \infty \qquad \text{in case} \quad \alpha < n, \\ &\int \log(2+|y|)d\mu(y) < \infty \qquad \text{in case} \quad \alpha = n. \end{aligned}$$

Let h be a positive and nonincreasing function on the interval $(0, \infty)$ such that $h(r) \leqq$ const. $h(2r)$ for $r>0$. In this paper, we first discuss the behavior of $h(|x|)^{-1}R_\alpha\mu(x)$ at the origin, in connection with the growth of the mean value of $R_\alpha\mu$ over the open balls centered at the origin. In our discussions, the aim is to find a criterion of the exceptional set E for which $h(|x|)^{-1}R_\alpha\mu(x)$ has limit zero or remains bounded above as x tends to 0 outside E. Our results obtained below will be similar to the characterizations of minimal thinness ([4]), minimal semithinness ([5], [6]) and logarithmical thinness and semithinness ([7]).

The thinness can be defined in terms of the α-capacity, like the expression of Wiener's criterion (see e.g. Brelot [1] and Landkof [3]). In this paper, letting $B(x, r)$ denote the open ball with center at x and radius r, we define the α-capacity of a set E in $B(0, 2^{-1})$ by

$$C_\alpha(E) = \inf \mu(R^n),$$

where the infimum is taken over all nonnegative measures μ with support in $B(0, 1)$ such that $R_\alpha\mu(x) \geqq 1$ for every $x \in E$.

The exceptional set E appeared in the discussion will satisfy the condition that $h_i^{-1} \sum_{j=1}^{\infty} h_j \min\{a_i, a_j\} C_\alpha(E_j)$ is bounded or has limit zero as $i\to\infty$, where $h_j=h(2^{-j})$, $a_j=2^{j(n-\alpha)}$ if $\alpha<n$, $a_j=j$ if $\alpha=n$ and $E_j=E\cap B(0, 2^{-j})-B(0, 2^{-j-1})$. For particular choices of h, the condition means the α-thinness of E, the α-semi-

thinness of E and so on.

Further we discuss the best possibility of our results as to the size of the exceptional sets; that is, if E satisfies the above condition, then we find a nonnegative measure μ such that μ satisfies the required properties but $R_\alpha\mu$ behaves ill on E. When we want to find μ with finite energy, the above type condition only is not sufficient. To do so, we require an additional condition on E and show the existence of a nonnegative measure μ satisfying

(i) $\int R_\alpha\mu d\mu < \infty$,

(ii) $R_\alpha\mu(x) \leqq h(|x|)$ for any $x \in S_\mu$ (the support of μ)

and

(iii) $R_\alpha\mu(x) \geqq h(|x|)$ for any $x \in E$.

By considering the inversion with respect to $\partial B(0, 1)$, our results will give a generalization of the results in [8], which deal with the existence of equilibrium measure of a closed set in the plane R^2.

2. Behaviors at the origin of α-potentials

If u is a function integrable on $B(0, r)$, then we define

$$A(u, 0, r) = \frac{1}{|B(0, r)|} \int_{B(0,r)} u(y)\, dy,$$

where $|B(0, r)|$ denotes the n-dimensional Lebesgue measure of $B(0, r)$.

The following result can be easily proved.

LEMMA 1. *Let $\phi_\alpha(r) = R_\alpha(x)$ for $r = |x|$, and $R_{\alpha,y}(x) = R_\alpha(x-y)$. Then there exist positive constants c_1 and c_2 such that*

$$c_1 \min\{\phi_\alpha(r), R_\alpha(y)\} \leqq A(R_{\alpha,y}, 0, r) \leqq c_2 \min\{\phi_\alpha(r), R_\alpha(y)\}$$

whenever $r \leqq 1/2$ and $|y| \leqq 1/2$.

Throughout this paper, we write $a_j = \phi_\alpha(2^{-j})$ for each integer j. First we give the following result (cf. [5], [6], [7]).

THEOREM 1. *Let h be a positive and nonincreasing function on the interval $(0, \infty)$ such that $h(r) \leqq$ const. $h(2r)$ for $r > 0$, and let μ be a nonnegative measure on R^n satisfying* (1). *Then the following statements are equivalent:*

(i) $A(R_\alpha\mu, 0, r) \leqq$ const. $h(r)$ *for* $0 < r < 1$.

(ii) *If* $1 \leqq p < n/(n-\alpha)$, *then* $A(|R_\alpha\mu|^p, 0, r)^{1/p} \leqq$ const. $h(r)$ *for* $0 < r < 1$.

(iii) *There exists a sequence* $\{x^{(j)}\}$ *such that* $\lim_{j\to\infty} x^{(j)} = 0$, $|x^{(j)}| \leqq$ const. $|x^{(j+1)}|$ *and* $R_\alpha\mu(x^{(j)}) \leqq$ const. $h(|x^{(j)}|)$ *for each* j.

(iv) *For* $\varepsilon > 0$, *there exists a set* $E \subset B(0, 2^{-1})$ *such that*

(a) $\sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} C_\alpha(E_j) \leqq \varepsilon h_k$ *for each* k;

(b) $\limsup_{x\to 0, x\notin E} h(|x|)^{-1} R_\alpha\mu(x) < \infty$,

where $h_j = h(2^{-j})$ *and* $E_j = E \cap B(0, 2^{-j}) - B(0, 2^{-j-1})$.

PROOF. We write $R_\alpha\mu = u + v$, where

$$u(x) = \int_{B(0,1/2)} R_\alpha(x-y)d\mu(y),$$

$$v(x) = \int_{R^n - B(0,1/2)} R_\alpha(x-y)d\mu(y).$$

Then it follows from (1) that $v(x)$ is continuous on $B(0, 1/4)$. Hence it suffices to prove the equivalence between (i)$\sim$(iv) with $R_\alpha\mu$ replaced by u. By Lemma 1, we have

$$c_1 \int_{B(0,1/2)} \min\{\phi_\alpha(r), R_\alpha(y)\} d\mu(y) \leqq A(u, 0, r)$$

$$\leqq c_2 \int_{B(0,1/2)} \min\{\phi_\alpha(r), R_\alpha(y)\} d\mu(y)$$

for $r < 1/2$.

By Hölder's inequality we have

$$A(u, 0, r) \leqq A(u^p, 0, r)^{1/p} \qquad \text{for} \quad r < 1/2,$$

if $p \geqq 1$. Conversely, we derive from Minkowski's inequality,

$$A(u^p, 0, r)^{1/p} \leqq \int_{B(0,1/2)} A(R_{\alpha,y}^p, 0, r)^{1/p} d\mu(y)$$

$$\leqq \text{const.} \int_{B(0,1/2)} \min\{\phi_\alpha(r), R_\alpha(y)\} d\mu(y)$$

for $r < 1/2$ and p, $1 \leqq p < n/(n-\alpha)$. Thus (i) and (ii) are equivalent.

Assume that (i) holds. Then

$$\sum_{j=1}^{\infty} \min\{a_j, a_k\} \mu(B_j) \leqq \text{const.}\, h_k \qquad \text{for each} \quad k, \tag{2}$$

where $B_j = B(0, 2^{-j}) - B(0, 2^{-j-1})$. Letting $\tilde{B}_j = B_{j-1} \cup B_j \cup B_{j+1}$ and $\varepsilon > 0$, we consider the sets

$$E_j = \left\{x \in B_j;\ \int_{\tilde{B}_j} R_\alpha(x-y)d\mu(y) \geqq \varepsilon^{-1} h_j\right\} \quad \text{and} \quad E = \cup_{j=2}^{\infty} E_j.$$

Then it follows from the definition of $C_\alpha(\cdot)$ that

$$C_\alpha(E_j) \leqq \varepsilon h_j^{-1} \mu(\tilde{B}_j).$$

In view of (2), E satisfies condition (a) of (iv). On the other hand,

$$\int_{B(0,1/2)-\tilde{B}_k} R_\alpha(x-y)d\mu(y) \leqq \text{const.} \sum_{j=1}^{\infty} \min\{a_j, a_k\}\mu(\tilde{B}_j) \leqq \text{const.}\, h_k$$

whenever $x \in B_k$, and

$$\int_{\tilde{B}_k} R_\alpha(x-y)d\mu(y) < \varepsilon^{-1}h_k$$

for $x \in B_k - E_k$, so that (b) of (iv) is fulfilled. Thus (i) implies (iv).

Assume that $\{x^{(j)}\}$ satisfies all the conditions in (iii), and define $r_j = |x^{(j)}|$. By Lemma 1, we have

$$A(u, 0, r_j) \leqq \text{const.}\, u(x^{(j)}) \leqq \text{const.}\, h(r_j) \qquad \text{for large} \quad j.$$

Take $M > 1$ such that $r_j \leqq M r_{j+1}$ for each j, and note that

$$(0, M^{-1}r_1] \subset \cup_{j=1}^{\infty} [M^{-1}r_j, Mr_j].$$

If $M^{-1}r_j \leqq r < Mr_j$, then Lemma 1 again gives

$$A(u, 0, r) \leqq \text{const.}\, A(u, 0, Mr_j) \leqq \text{const.}\, h(r_j) \leqq \text{const.}\, h(r).$$

Consequently, we have proved that (iii) implies (i).

Finally assume that (iv) is true. Note that

$$c^{-1}a_j^{-1} \leqq C_\alpha(B_j) \leqq ca_j^{-1}$$

for any j, where c is a positive constant. For $\varepsilon = c^{-1}$, take a set E satisfying property (b) and

$$\sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} C_\alpha(E_j) < c^{-1}h_k \qquad \text{for each} \quad k.$$

Then $B_j - E_j$ is not empty. Letting $x^{(j)} \in B_j - E_j$, we see easily that (iii) holds for $\{x^{(j)}\}$. Thus (iv) implies (iii), and hence the proof of the theorem is complete.

THEOREM 2. *Let h and μ be as in Theorem 1. Then the following statements are equivalent:*

(i) $\lim_{r \downarrow 0} h(r)^{-1} A(R_\alpha\mu, 0, r) = 0.$

(ii) *For* $1 \leqq p < n/(n-\alpha)$, $\lim_{r \downarrow 0} h(r)^{-1} A((R_\alpha\mu)^p, 0, r)^{1/p} = 0.$

(iii) *There exists a sequence* $\{x^{(j)}\}$ *such that* $\lim_{j\to\infty} x^{(j)} = 0$, $|x^{(j)}| \leqq$ const. $|x^{(j+1)}|$ *for each* j *and* $\lim_{j\to\infty} h(|x^{(j)}|)^{-1} R_\alpha\mu(x^{(j)}) = 0.$

(iv) *There exists a set* $E \subset B(0, 2^{-1})$ *such that*

(a) $\lim_{k\to\infty} h_k^{-1} \sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} C_\alpha(E_j) = 0;$

(b) $\lim_{x\to 0, x\notin E} h(|x|)^{-1} R_\alpha\mu(x) = 0.$

PROOF. Since the proof can be carried out in a way similar to that of Theorem 1, we shall give only a proof of the implication (i)→(iv). Assume that (i) holds.

Then, as in the proof of Theorem 1, we obtain

(3) $$\lim_{k\to\infty} h_k^{-1} \sum_{j=1}^{\infty} \min\{a_j, a_k\}\mu(\tilde{B}_j) = 0.$$

Set $\varepsilon_k = h_k^{-1} \sum_{j=1}^{\infty} \min\{a_j, a_k\}\mu(\tilde{B}_j)$, and find a sequence $\{b_j\}$ of positive numbers such that $b_j \leqq b_{j+1} \leqq 2b_j$, $b_j \leqq \varepsilon_j^{-1/2}$,

$$\sum_{j=k}^{\infty} b_j \min\{a_j, a_k\}\mu(\tilde{B}_j) \leqq 2b_k \sum_{j=k}^{\infty} \min\{a_j, a_k\}\mu(\tilde{B}_j)$$

for each k and $\lim_{k\to\infty} b_k = \infty$ (see [7; Lemma 6]). Then (3) is fulfilled with $\mu(\tilde{B}_j)$ replaced by $b_j\mu(\tilde{B}_j)$. As in the previous proof, define

$$E_j = \left\{x \in B_j;\ \int_{B_j} R_\alpha(x-y)\,d\mu(y) \geqq b_j^{-1} h_j\right\} \quad \text{and} \quad E = \cup_{j=1}^{\infty} E_j.$$

Then it is easy to see that (a) and (b) hold for this E, and hence (iv) holds. Thus the proof of Theorem 2 is established.

REMARK 1. Let $\alpha > 1$. Then $\limsup_{r\downarrow 0} h(r)^{-1} A(R_\alpha\mu, 0, r) < \infty$ (resp. $=0$) if and only if $\limsup_{r\downarrow 0} h(r)^{-1} S(R_\alpha\mu, 0, r) < \infty$ (resp. $=0$), where

$$S(u, 0, r) = \frac{1}{\sigma(\partial B(0, r))} \int_{\partial B(0,r)} u(y)\,d\sigma(y),$$

σ denoting the surface measure on the boundary $\partial B(0, r)$.

REMARK 2. If $h \equiv 1$ or if $h(r) = \max\{\phi_\alpha(r), 1\}$, then (a) of (iv) in Theorem 1 implies

$$\sum_{j=1}^{\infty} a_j C_\alpha(E_j) < \infty,$$

which means that E is α-thin at 0 (cf. [1], [3]).

REMARK 3. If h satisfies the additional conditions:

$$\int_0^r h(s) s^{n-\alpha-1}\,ds \leqq \text{const.}\ h(r) r^{n-\alpha} \quad \text{and} \quad \int_r^1 h(s) s^{-1} ds \leqq \text{const.}\ h(r)$$

for $r < 1$, then (a) of (iv) in Theorem 1 can be replaced by

(a′) $a_j C_\alpha(E_j) < \varepsilon$ for all j;

and (a) of (iv) in Theorem 2 is equivalent to

(a″) $\lim_{j\to\infty} a_j C_\alpha(E_j) = 0$.

If (a″) holds, then E is said to be α-semithin at 0 (cf. [6]). We note that

$$h(r) = \begin{cases} r^{-a}(\log(r+1))^b & \text{for } r < r_0, \\ r_0^{-a}(\log(r_0+1))^b & \text{for } r \geqq r_0, \end{cases}$$

satisfies all the conditions mentioned above if $0<a<n-\alpha$, $-\infty<b<\infty$ and r_0 is chosen so that h is nonincreasing.

3. Thinness of sets

The proof of the implication (i)→(iv) in Theorem 1 shows the following: We can find $c_1, c_2>0$ such that if $0<\varepsilon<c_1$ and E is a subset of $B(0, 2^{-1})$ for which there exists a nonnegative measure ν satisfying

(i) $A(R_\alpha \nu, 0, r) \leqq \varepsilon h(r)$ for $0<r<1$

and

(ii) $R_\alpha \nu(x) \geqq h(|x|)$ for any $x \in E$,

then

$$\sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} C_\alpha(E_j) \leqq c_2 \varepsilon h_k \quad \text{for any} \quad k.$$

Conversely we establish the following result, which serves as showing the best possibility of Theorem 1 as to the size of the exceptional sets.

PROPOSITION 1. *Let h be a positive and nonincreasing function on the interval $(0, \infty)$ such that $h(r) \leqq Mh(2r)$ and $\int_0^r h(s)s^{n-1}ds \leqq Mh(r)r^n$ for any $r>0$, where M is a positive constant. Let E be a subset of $B(0, 2^{-1})$ satisfying* (a) *of Theorem* 1, (iv) *for some $\varepsilon>0$. Then there exists a nonnegative measure ν with support in $B(0, 1)$ such that*

(i) $A(R_\alpha \nu, 0, r) \leqq c\varepsilon h(r)$ *for* $0<r<1/2$

and

(ii) $R_\alpha \nu(x) \geqq h(|x|)$ *for any* $x \in E$,

where c is a positive constant independent of ε and E.

PROOF. By [3; Theorem 2.7], for each positive integer j we can find a nonnegative measure ν_j such that $S_{\nu_j} \subset \tilde{B}_j$, $\nu_j(\tilde{B}_j) < C_\alpha(E_j) + \delta_j$ and $R_\alpha \nu_j(x) \geqq 1$ for every $x \in E_j$, where $\{\delta_j\}$ is a sequence of positive numbers such that

$$\sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} [C_\alpha(E_j) + \delta_j] \leqq 2\varepsilon h_k \quad \text{for each} \quad k.$$

Define

$$\nu = \sum_{j=1}^{\infty} h_{j+1} \nu_j.$$

Then ν is a nonnegative measure with support in $B(0, 1)$ and

$$\int_{B_k} R_\alpha \nu(x)dx = \int R_\alpha \chi_{B_k} d\nu \leqq \text{const. } 2^{-kn} \sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} \nu_j(R^n)$$
$$\leqq \text{const. } \varepsilon\, 2^{-kn} h_k,$$

where χ_A denotes the characteristic function of a measurable set A. Since $\int_0^r h(s)s^{n-1}ds \leqq Mh(r)r^n$, $\sum_{k=\ell}^{\infty} 2^{-kn}h_k \leqq \text{const. } 2^{-\ell n}h_\ell$, so that (i) holds. Clearly, $R_\alpha\nu(x) \geqq h(|x|)$ for every $x \in E$. Thus ν satisfies all assertions in the proposition.

Theorem 2, (iv) is also best possible as to the size of the exceptional set.

PROPOSITION 2. *Let h be as in Proposition* 1. *If E satisfies* (a) *of Theorem* 2, (iv), *then there exists a nonnegative measure ν with support in* $B(0, 1)$ *such that*

(i) $\lim_{r\downarrow 0} h(r)^{-1}A(R_\alpha\nu, 0, r) = 0$;

(ii) $\lim_{x\to 0, x\in E} h(|x|)^{-1}R_\alpha\nu(x) = \infty$.

PROOF. Let $\varepsilon_k = h_k^{-1} \sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} C_\alpha(E_j)$, and take a sequence $\{b_j\}$ of positive numbers such that $\lim_{j\to\infty} b_j = \infty$, $b_j \leqq b_{j+1} \leqq 2b_j$, $b_j \leqq \varepsilon_j^{-1/2}$ and

$$\sum_{k=j}^{\infty} b_k h_k C_\alpha(E_k) \leqq 2b_j \sum_{k=j}^{\infty} h_k C_\alpha(E_k)$$

for any positive integer j (cf. [7; Lemma 6]). Then

$$\lim_{k\to\infty} h_k^{-1} \sum_{j=1}^{\infty} b_j h_j \min\{a_j, a_k\} C_\alpha(E_j) = 0. \tag{4}$$

As in the proof of Proposition 1, for each j take a nonnegative measure ν_j such that $S_{\nu_j} \subset \tilde{B}_j$, $\nu_j(\tilde{B}_j) < C_\alpha(E_j) + \delta_j$ and $R_\alpha\nu_j(x) \geqq 1$ for any $x \in E_j$, where $\{\delta_j\}$ is a sequence of positive numbers satisfying (4) with $C_\alpha(E_j)$ replaced by $C_\alpha(E_j) + \delta_j$. Define

$$\nu = \sum_{j=1}^{\infty} b_j h_j \nu_j.$$

Then $R_\alpha\nu(x) \geqq b_j h_j R_\alpha \nu_j(x) \geqq b_j h_j$ for $x \in E_j$, and

$$\int_{B_k} R_\alpha\nu(x)dx \leqq \text{const. } 2^{-kn} \sum_{j=1}^{\infty} b_j h_j \min\{a_j, a_k\} \nu_j(\tilde{B}_j),$$

from which it follows that ν satisfies (i) and (ii). Thus the proof of Proposition 2 is complete.

We here give several properties which are equivalent to the α-thinness. For this purpose, denote by $\mathscr{H}$ the family of functions h *on* $(0, \infty)$ which is positive and nonincreasing on $(0, \infty)$ such that $h(r)\phi_\alpha(r)^{-1}$ is nondecreasing on $(0, \infty)$ and $\lim_{r\downarrow 0} h(r) = \infty$.

PROPOSITION 3. *Let $E \subset R^n$. Then the following statements are equivalent.*

(i) *E is α-thin at* 0.

(ii) $\lim_{k\to\infty} h_k^{-1} \sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} C_\alpha(E_j) = 0$ *for any* $h \in \mathscr{H}$.

(iii) $\sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} C_\alpha(E_j) \leqq \text{const.}\, h_k$ *for any positive integer* k *whenever* $h \in \mathscr{H}$.

(iv) *For any* $h \in \mathscr{H}$, *there exists a nonnegative measure* ν *with compact support such that*

(a) $\lim_{r\downarrow 0} h(r)^{-1} A(R_\alpha \nu, 0, r) = 0$;

(b) $R_\alpha \nu(x) \geqq h(|x|)$ *for any* $x \in E \cap B(0, 1)$.

(v) *For any* $h \in \mathscr{H}$, *there exists a nonnegative measure* ν *with compact support for which* $\lim_{x\to 0, x\in E} h(|x|)^{-1} R_\alpha \nu(x) = \infty$.

PROOF. First assume that E is α-thin at 0. For $\varepsilon > 0$, take j_0 such that $\sum_{j=j_0}^{\infty} a_j C_\alpha(E_j) < \varepsilon$. Since h_k increases to infinity,

$$\limsup_{k\to\infty} h_k^{-1} \sum_{j=1}^{k} h_j a_j C_\alpha(E_j) = \limsup_{k\to\infty} h_k^{-1} \sum_{j=j_0}^{k} h_j a_j C_\alpha(E_j) \leqq \limsup_{k\to\infty} \sum_{j=j_0}^{k} a_j C_\alpha(E_j) < \varepsilon.$$

On the other hand, since $h_j a_j^{-1}$ is nonincreasing, we have

$$\limsup_{k\to\infty} h_k^{-1} \sum_{j=k}^{\infty} h_j a_k C_\alpha(E_j) = \limsup_{k\to\infty} a_k h_k^{-1} \sum_{j=k}^{\infty} (h_j a_j^{-1}) a_j C_\alpha(E_j) = 0.$$

Thus (i) implies (ii). Clearly (ii) implies (iii). Since (iii) implies (i) by Remark 2 after Theorem 2, (i), (ii) and (iii) are equivalent to each other.

In view of Proposition 2, we infer that (ii) implies (iv) and (v). It follows from Theorem 2 that (iv) implies (ii).

From [1; Theorem IX, 7] we see that E is α-thin at 0 if and only if there exists a nonnegative measure ν satisfying (1) and

$$\lim_{x\to 0, x\in E} R_\alpha(x)^{-1} R_\alpha \nu(x) > \nu(\{0\}).$$

Since $\phi_\alpha \in \mathscr{H}$, (v) implies (i), and hence the proof of Proposition 3 is complete.

REMARK. Let E be a closed set in $B(0, 2^{-1})$. Then the following statements are equivalent (cf. Wu [9; Theorems 1 and 2]):

(i) E is α-thin at 0.

(ii) For any $h \in \mathscr{H}$, there eixsts a nonnegative measure ν with support in E such that

(a) $\lim_{r\downarrow 0} h(r)^{-1} A(R_\alpha \nu, 0, r) = 0$;

(b) $R_\alpha \nu(x) \geqq h(|x|)$ for any $x \in E$ except those in a set with vanishing α-capacity.

(iii) For any $h \in \mathscr{H}$, there exists a nonnegative measure ν with support in E such that $\lim_{x\to 0, x\in E-A} h(|x|)^{-1} R_\alpha \nu(x) = \infty$, where $C_\alpha(A) = 0$.

Denote by $\mathscr{H}^*$ the family of all positive and nonincreasing functions h on

$(0, \infty)$ satisfying the following conditions:

(a) $h(r) \leqq Mh(2r)$ for $r > 0$;

(b) $\int_r^\infty h(s)s^{-1}ds \leqq Mh(r)$ for $r > 0$;

(c) $\int_0^r h(s)s^{n-\alpha-1}ds \leqq Mh(r)r^{n-\alpha}$ for $r > 0$,

where M is a positive constant.

PROPOSITION 4 (cf. [6; Theorem 2]). *Let $E \subset R^n$. Then the following statements are equivalent*:

(i) *E is α-semithin at* 0.

(ii) $\lim_{k\to\infty} h_k^{-1} \sum_{j=1}^\infty h_j \min\{a_j, a_k\} C_\alpha(E_j) = 0$ *for any* $h \in \mathscr{H}^*$.

(iii) *For any $h \in \mathscr{H}^*$, there exists a nonnegative measure ν with compact support such that*

(a) $\lim_{r\downarrow 0} h(r)^{-1} A(R_\alpha \nu, 0, r) = 0$;

(b) $\lim_{x\to 0, x\in E} h(|x|)^{-1} R_\alpha \nu(x) = \infty$.

This proposition can be proved in a way similar to the proof of Proposition 3, so we omit its proof (cf. Remark 3 after Theorem 2).

4. α-potentials with finite energy

We say that a nonnegative measure μ has finite α-energy if

$$\langle \mu, \mu \rangle_\alpha \equiv \int R_\alpha \mu d\mu < \infty;$$

in case $n=2$, μ is assumed to have compact support.

THEOREM 3. *Let μ be a nonnegative measure with support in* $B(0, 1)$ *such that* $\langle \mu, \mu \rangle_\alpha < \infty$ *and*

$$A(R_\alpha \mu, 0, r) \leqq h(r) \qquad \textit{for any} \quad r > 0,$$

where h is a function on $(0, \infty)$ *as in Theorem* 1. *Then for any $\varepsilon > 0$, there exists a set $E \subset B(0, 2^{-1})$ possessing the following properties*:

(a) $\sum_{j=1}^\infty h_j \min\{a_j, a_k\} C_\alpha(E_j) \leqq \varepsilon h_k$ *for any* k.

(b) $\sum_{j,k=1}^\infty h_j h_k \min\{a_j, a_k\} C_\alpha(E_j) C_\alpha(E_k) < \infty$.

(c) $\limsup_{x\to 0, x\notin E} h(|x|)^{-1} R_\alpha \mu(x) < \infty$.

This theorem can be proved in the same way as the implication (i)$\to$(iv) of Theorem 1; so we omit its proof.

REMARK. By Theorem 3 one can find $c_1, c_2 > 0$ such that if $0 < \varepsilon < c_1$ and E is a subset of $B(0, 2^{-1})$ for which there exists a nonnegative measure ν in $B(0, 1)$ satisfying

(i) $\langle \nu, \nu \rangle_\alpha < \infty$,

(ii) $A(R_\alpha \nu, 0, r) \leqq \varepsilon h(r)$ for any $r > 0$

and

(iii) $R_\alpha \nu(x) \geqq h(|x|)$ for any $x \in E$,

then E satisfies

(a) $\sum_{j=1}^\infty h_j \min\{a_j, a_k\} C_\alpha(E_j) \leqq c_2 \varepsilon h_k$ for any k

and

(b) $\sum_{j,k=1}^\infty h_j h_k \min\{a_j, a_k\} C_\alpha(E_j) C_\alpha(E_k) < \infty$.

We do not know whether Theorem 3 is best possible as to the size of the exceptional set or not. We shall prove only the following result.

PROPOSITION 5. *Let E be a subset of $B(0, 2^{-1})$ satisfying* (a) *in Theorem* 3 *for some $\varepsilon > 0$ and*

(b′) $\sum_{j=1}^\infty h_j^2 C_\alpha(E_j) < \infty$.

Then there exists a nonnegative measure ν with support in $B(0, 1)$ such that $\langle \nu, \nu \rangle_\alpha < \infty$ and

$$\lim_{x \to 0, x \in E} h(|x|)^{-1} R_\alpha \nu(x) = \infty.$$

REMARK 1. If E satisfies (a) and (b′), then it also satisfies (b). In case $\{a_j/h_j\}$ is bounded above, (b′) implies (a) for any $\varepsilon < 0$; but, in general, the converse is not true.

REMARK 2. Let $h_j = a_j^\beta$ for $\beta > 0$. If $\beta < 1$, then we can find a positive constant c such that

$$\sum_{j=1}^\infty h_j \min\{a_j, a_k\} C_\alpha(E_j) \leqq c h_k \quad \text{for any} \quad k,$$

whenever $E \subset B(0, 2^{-1})$; if $\beta < 1/2$, then (b′) holds for any set $E \subset B(0, 2^{-1})$.

PROOF OF PROPOSITION 5. Let E be as in the proposition. Then, in view of [7; Lemma 6], we can construct a sequence $\{b_j\}$ of positive numbers such that $\lim_{j \to \infty} b_j = \infty$, $b_k \leqq b_{k+1} \leqq 2b_k$,

$$\sum_{j=1}^\infty b_j h_j \min\{a_j, a_k\} C_\alpha(E_j) \leqq \text{const.}\ b_k h_k \tag{5}$$

and

$$\sum_{j=1}^\infty b_j^2 h_j^2 C_\alpha(E_j) < \infty. \tag{6}$$

Take a sequence $\{\delta_j\}$ of positive numbers which satisfies (5) and (6) with $C_\alpha(E_j)$ replaced by $C_\alpha(E_j) + \delta_j$. By [3; Theorem 2.7], for each j we can find a nonnegative measure ν_j such that $S_{\nu_j} \subset \tilde{B}_j$, $\nu_j(\tilde{B}_j) < C_\alpha(E_j) + \delta_j$, $R_\alpha \nu_j(x) \geqq 1$ for any $x \in E_j$ and $R_\alpha \nu_j(x) \leqq 2^{n-\alpha}$ for every $x \in R^n$. Define

$$\nu = \sum_{j=1}^{\infty} b_j h_j \nu_j .$$

Then $R_\alpha \nu(x) \geqq b_k h_k R_\alpha \nu_k(x) \geqq b_k h_k$ for $x \in E_k$ and

$$R_\alpha \nu(x) = \sum_{j=1}^{\infty} b_j h_j R_\alpha \nu_j(x) \leqq \text{const.} \{\sum_{j=1}^{k-2} b_j h_j a_j \nu_j(\tilde{B}_j) + \sum_{j=k-1}^{k+1} b_j h_j + \sum_{j=k+2}^{\infty} b_j h_j a_k \nu_j(\tilde{B}_j)\} \leqq \text{const.}\ b_k h_k$$

for $x \in B_k$. Hence it follows that $\lim_{x \to 0, x \in E} h(|x|)^{-1} R_\alpha \nu(x) = \infty$ and

$$\langle \nu, \nu \rangle_\alpha = \sum_{k=1}^{\infty} b_k h_k \int R_\alpha \nu d\nu_k \leqq \text{const.} \sum_{k=1}^{\infty} b_k^2 h_k^2 \nu_k(\tilde{B}_k) < \infty.$$

Thus ν satisfies all the conditions required in the proposition, and the proposition is proved.

5. Gauss variation

Throughout this section, let f be a continuous function in $R^n - \{0\}$ such that

(7) $$\sup_{B_j} |f| \leqq \text{const.} \inf_{B_{j-1}} |f|,$$

where $B_j = B(0, 2^{-j}) - B(0, 2^{-j-1})$ as before. Define

$$f_j = \sup_{B_j} |f|$$

and

$$h_j = \max\{f_1, \ldots, f_j\}.$$

By (7), $h_j \leqq h_{j+1} \leqq \text{const.}\ h_j$ for any positive integer j.

Our main result in this section is the following.

THEOREM 4. *Let E be a subset of $B(0, 2^{-1})$ possessing the following properties*:

(a) $\sum_{j=1}^{\infty} h_j \min\{a_j, a_k\} C_\alpha(E_j) \leqq \text{const.} h_k$ *for each* k;

(b) $\sum_{j=1}^{\infty} h_j^2 C_\alpha(E_j) < \infty$,

where $a_j = \phi(2^{-j})$ and $E_j = E \cap B_j$ as before. Then there exists a nonnegative measure μ with support in $B(0, 1)$ such that

(i) $R_\alpha \mu(x) \geqq f(x)$ *for any* $x \in E - \{0\}$;

(ii) $R_\alpha \mu(x) \leqq f(x)$ *for any* $x \in S_\mu - \{0\}$;

(iii) $\langle \mu, \mu \rangle_\alpha \equiv \int R_\alpha \mu d\mu < \infty$.

Without loss of generality, we may assume that $h_j > 0$ for any j. Let h be a nonincreasing and continuous function on $(0, \infty)$ such that $h(2^{-j}) = h_j$ for each j.

PROOF OF THEOREM 4. Since $C_\alpha(\cdot)$ is an outer capacity, there exists an open set G such that $E-\{0\}\subset G\subset B(0, 2^{-1})-\{0\}$ and (a), (b) in Theorem 4 hold for $E=G$. Denote by $U(G)$ the family of all nonnegative measures μ such that $S_\mu\subset G$ and $\langle\mu, \mu\rangle_\alpha<\infty$. Define

$$V(\mu) = \langle\mu, \mu\rangle_\alpha - 2\int f d\mu,$$

and consider

$$a = \inf\{V(\mu);\ \mu\in U(G)\}.$$

Take a nonnegative measure ν as in Proposition 5 with E replaced by G. Here we may assume that $S_\nu\subset B(0, 4^{-1})$. Then we obtain for $\mu\in U(G)$,

$$\int h(|x|)d\mu(x) \leqq M\int R_\alpha\nu d\mu \leqq 2^{-1}(\langle\mu, \mu\rangle_\alpha + M^2\langle\nu, \nu\rangle_\alpha),$$

which implies that

$$V(\mu) \geqq -M^2\langle\nu, \nu\rangle_\alpha,$$

where M is a positive constant. Hence the quantity a is finite. Take a sequence $\{\mu_j\}$ of nonnegative measures in $U(G)$ such that $\lim_{j\to\infty} V(\mu_j)=a$. Then it is easy to see that $\{\langle\mu_j, \mu_j\rangle_\alpha\}$, and hence $\{\int h(|x|)d\mu_j(x)\}$, is bounded. It follows that $\{\mu_j(G)\}$ is bounded, and hence we may assume that $\{\mu_j\}$ converges vaguely to a nonnegative measure μ_0. Note here that $\langle\mu_0, \mu_0\rangle_\alpha<\infty$, and hence $\mu_0(\{0\})=0$.

For $r>0$, define

$$A(r) = \inf\{h(|x|)^{-1}R_\alpha\nu(x);\ x\in G\cap B(0, r)\}.$$

By assumption, $\lim_{r\downarrow 0} A(r)=\infty$. Let ψ_r be a continuous function on R^n such that $\psi_r=1$ on $B(0, r/2)$, $\psi_r=0$ outside $B(0, r)$ and $0\leqq\psi_r\leqq 1$ on R^n. Then we have

$$\left|\int h(|x|)d\mu_j(x) - \int[1-\psi_r(x)]h(|x|)d\mu_j(x)\right| \leqq A(r)^{-1}\int_{B(0,r)} R_\alpha\nu d\mu_j$$

for r sufficiently small. Since $\lim_{r\downarrow 0} A(r)=\infty$ and $\left\{\int_{B(0,2^{-1})} R_\alpha\nu d\mu_j\right\}$ is bounded, it follows that

$$\lim_{j\to\infty}\int h(|x|)d\mu_j(x) = \int h(|x|)d\mu_0(x) < \infty.$$

In a similar manner, noting that $|f|\leqq h$, we obtain

$$\lim_{j\to\infty}\int f d\mu_j = \int f d\mu_0.$$

On the other hand,

$$a \leqq V((\mu_j+\mu_k)/2) = [V(\mu_j)+V(\mu_k)]/2 - \langle \mu_j-\mu_k, \mu_j-\mu_k \rangle_\alpha/4$$

for any positive integers j and k, so that

$$\lim_{j\to\infty} \langle \mu_j - \mu_0, \mu_j - \mu_0 \rangle_\alpha = 0.$$

Moreover it follows that $V(\mu_0)=a$.

If $\mu \in U(G)$ and $t>0$, then $\mu_j + t\mu \in U(G)$, which yields

$$\int R_\alpha \mu_0 d\mu \geqq \int f d\mu. \tag{8}$$

Similarly, since $(1-t)\mu_j \in U(G)$ for $0<t<1$, we establish

$$\int R_\alpha \mu_0 d\mu_0 = \int f d\mu_0 .$$

Let $x^0 \in G$. By taking as μ the unit uniform surface measure on the boundary $\partial B(x^0, r)$ and letting $r \downarrow 0$ in (8), we derive

$$R_\alpha \mu_0(x^0) \geqq f(x^0).$$

Thus it follows that $R_\alpha \mu_0 \geqq f$ on G. We next let $x^0 \in S_{\mu_0} - \{0\}$ and suppose

$$R_\alpha \mu_0(x^0) > f(x^0).$$

Since $R_\alpha \mu_0$ is lower semicontinuous, there exists $r>0$ such that

$$R_\alpha \mu_0(x) > f(x) \qquad \text{for any} \quad x \in B(x^0, r).$$

Let ψ be a continuous function on R^n such that $\psi=1$ on $B(x^0, r/2)$, $\psi=0$ outside $B(x^0, r)$ and $0 \leqq \psi \leqq 1$ on R^n. Then, since $\mu_j + t\psi\mu_j \in U(G)$ for $-1<t<1$, we obtain

$$\int (R_\alpha \mu_0 - f)\psi d\mu_0 = 0.$$

Thus a contradiction follows, and hence $R_\alpha \mu_0(x^0) \leqq f(x^0)$. The proof of the theorem is now complete.

In the same way we can prove the next theorem.

THEOREM 5. *If E is as in Theorem 4, then there exist a number γ and a nonnegative measure μ with support in $B(0, 1)$ such that $\mu(R^n)=1$, $R_\alpha \mu \leqq f+\gamma$ on $S_\mu - \{0\}$, $R_\alpha \mu \geqq f+\gamma$ on $E-\{0\}$ and $\langle \mu, \mu \rangle_\alpha < \infty$.*

We also establish the following results with a slight modification of the proof of Theorem 4.

THEOREM 4′. *Let K be a compact set in R^n containing the origin and* *satisfying* (a), (b) *in Theorem* 4 *with E replaced by K. Then there exists a nonnegative measure* μ *supported by K such that* $R_\alpha\mu \leqq f$ *on* $S_\mu - \{0\}$, $R_\alpha\mu \geqq f$ *on K except for a set of vanishing α-capacity and* $\langle\mu, \mu\rangle_\alpha < \infty$.

THEOREM 5′. *If K is as in Theorem* 4′, *then there exist a number* γ *and a nonnegative measure* μ *supported by K such that* $\mu(K)=1$, $\langle\mu, \mu\rangle_\alpha < \infty$, $R_\alpha\mu \leqq f+\gamma$ *on* $S_\mu - \{0\}$ *and* $R_\alpha\mu \geqq f+\gamma$ *on K except for a set of vanishing α-capacity.*

REMARK 1. If $\limsup_{x\to 0} R_\alpha(x)^{-\beta}|f(x)| < \infty$ for some β with $0<\beta<1/2$, then the conclusions of Theorems 4, 5, 4′ and 5′ remain true in view of Proposition 5 and its Remark 2.

REMARK 2. Let h be as in Theorem 1 and μ be a nonnegative measure on $B(0, 1)$. If $R_\alpha\mu \leqq H$ on S_μ, then $R_\alpha\mu \leqq MH$ on R^n, where M is a positive constant independent of μ and $H(x)=h(r)$ for $|x|=r$.

For a proof of this fact, let $h_j = h(2^{-j})$ and $\mu_j|_{\bar{B}_j}$, where $\bar{B}_j = \{x \in R^n;\, 2^{-j-1} \leqq |x| \leqq 2^{-j}\}$. Since $R_\alpha\mu_j \leqq M_1 h_j$ on S_{μ_j}, we see that

$$R_\alpha\mu_j \leqq 2^{n-\alpha} M_1 h_j \qquad \text{on} \quad R^n,$$

where M_1 is a positive constant so chosen that $H(x) \leqq M_1 h_j$ for $x \in \bar{B}_j$. If $x \in \bar{B}_j$, then

$$R_\alpha\mu(x) \geqq R_\alpha\mu_j(x) + M_2 \textstyle\sum_{k\neq j} \min\{a_j, a_k\}\mu(B_k),$$

so that

$$R_\alpha\mu_j \leqq 2^{n-\alpha}\{M_1 h_j - M_2 \textstyle\sum_{k\neq j} \min\{a_j, a_k\}\mu(B_k)\},$$

where M_2 is a positive constant and $B_j = B(0, 2^{-j}) - B(0, 2^{-j-1})$. Hence it follows that

$$\begin{aligned} R_\alpha\mu(x) &\leqq R_\alpha\mu_{j-1}(x) + R_\alpha\mu_{j+1}(x) + R_\alpha\mu_j(x) + M_3 \textstyle\sum_{k\neq j} \min\{a_j, a_k\}\mu(B_k) \\ &\leqq 2^{n-\alpha} M_1(h_{j-1} + h_{j+1}) + M_4 h_j \leqq M_5 H(x) \end{aligned}$$

for $x \in B_j$, where M_3, M_4 and M_5 are positive constants. Thus the conclusion of the remark follows by noting that the constants $M_1 \sim M_5$ are determined independently of μ.

Finally it is noted that the next statements are equivalent:

(i) $\sum_{j=1}^{\infty} a_j^2 C_\alpha(E_j) < \infty$.

(ii) There exists a nonnegative measure μ with support in $B(0, 1)$ such that $R_\alpha\mu(0) < \infty$ and

$$\lim_{x\to 0, x\in E} R_\alpha(x)^{-1} R_\alpha\mu(x) = \infty.$$

(iii) There exists a nonnegative measure μ with support in $B(0, 1)$ such that $R_\alpha\mu(0) < \infty$, $R_\alpha\mu(x) \geqq R_\alpha(x)$ on $E \cap B(0, 2^{-1})$ and $R_\alpha\mu(x) \leqq R_\alpha(x)$ on S_μ.

In fact, (i) implies (ii) by Lemma 3; (iii) follows from (ii) in view of the proof of Theorem 3; (iii) implies (i) by Proposition 6.

Further (i), (ii) and (iii) are equivalent to

(iv) There exist a nonnegative measure μ with support in $B(0, 1)$ and unit mass and a number γ such that $R_\alpha\mu(0) < \infty$, $R_\alpha\mu(x) \geqq R_\alpha(x) + \gamma$ on $E \cap B(0, 2^{-1})$ and $R_\alpha\mu(x) \leqq R_\alpha(x) + \gamma$ on S_μ.

In case $\alpha = n = 2$, this result gives Theorem 3 in [8] by considering the inversion with respect to the surface $\partial B(0, 1)$.

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