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Asymptotic behavior of solutions of a class of second order nonlinear differential equations

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1. Introduction

This paper is concerned with the asymptotic behavior of solutions of the second order nonlinear differential equation

$$(p(t)y')' + f(t, y, y') = 0, \tag{1}$$

for which the following conditions are assumed to hold:

(A_1) p: $[0, \infty) \to (0, \infty)$ is continuous, and

$$P(t) = \int_0^t \frac{ds}{p(s)} \to \infty \quad \text{as} \quad t \to \infty; \tag{2}$$

(A_2) f: $[0, \infty) \times \boldsymbol{R} \times \boldsymbol{R} \to (0, \infty)$ is continuous and nondecreasing in each of the last two variables.

A prototype of equation (1) satisfying (A_1) and (A_2) is

$$y'' + \varphi(t)e^y = 0, \tag{3}$$

or more generally

$$y'' + \varphi(t) \exp(|y|^{\gamma-1}y + |y'|^{\delta-1}y') = 0, \tag{4}$$

where φ: $[0, \infty) \to (0, \infty)$ is continuous and γ and δ are positive constants. It seems to us that no systematic study of the qualitative behavior of solutions has so far been attempted even for the simple equation (3) or (4), and this observation motivated the present work.

We begin by noticing that all solutions of (1) can be continued to infinity. In fact, let $y(t)$ be a solution of (1) with given initial values at $t=a$ ($a \geqq 0$) and let $[a, T)$ be its right maximal interval of existence. Suppose that $T < \infty$. From (1), $(p(t)y'(t))' = -f(t, y(t), y'(t)) < 0$ on $[a, T)$, so that $p(t)y'(t)$ is decreasing and tends to $-\infty$ as $t \to T^-$. Hence there exist constants $t_0 \in (a, T)$, k and l such that $y(t) \leqq k$ and $y'(t) \leqq l$ on $[t_0, T)$. Integrating (1) on $[t_0, t]$, we have

$$p(t_0)y'(t_0) - p(t)y'(t) = \int_{t_0}^t f(s, y(s), y'(s))ds \leqq \int_{t_0}^t f(s, k, l)ds,$$

which, in the limit as $t \to T^-$, gives

$$\infty = \int_{t_0}^{T} f(s, k, l)ds < \infty.$$

This contradiction shows that T must be ∞, that is, $y(t)$ does exist throughout $[a, \infty)$. On the basis of this remark we classify the set $\mathscr{S}$ of all solutions $y(t)$ of (1) existing on $[a, \infty)$ $(a \geqq 0)$ into the following four subsets according to the values of $\lim_{t\to\infty} p(t)y'(t)$:

$$\mathscr{A} = \{y \in \mathscr{S}: \lim_{t\to\infty} p(t)y'(t) = -\infty\},$$
$$\mathscr{B} = \{y \in \mathscr{S}: \lim_{t\to\infty} p(t)y'(t) = const < 0\},$$
$$\mathscr{C} = \{y \in \mathscr{S}: \lim_{t\to\infty} p(t)y'(t) = 0\},$$
$$\mathscr{D} = \{y \in \mathscr{S}: \lim_{t\to\infty} p(t)y'(t) = const > 0\}.$$

If $y \in \mathscr{A} \cup \mathscr{B}$, then $y(t)$ is eventually decreasing and satisfies $\lim_{t\to\infty} y(t)/P(t) = -\infty$ or $\lim_{t\to\infty} y(t)/P(t) = const < 0$ according as $y \in \mathscr{A}$ or $y \in \mathscr{B}$; if $y \in \mathscr{C} \cup \mathscr{D}$, then $y(t)$ is increasing on $[a, \infty)$ and satisfies $\lim_{t\to\infty} y(t)/P(t) = 0$ or $\lim_{t\to\infty} y(t)/P(t) = const > 0$ according as $y \in \mathscr{C}$ or $y \in \mathscr{D}$.

The objective of this paper is to establish criteria for the existence (or non-existence) of members of the classes $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$. The structure of the sets $\mathscr{A} \cup \mathscr{B}$ and $\mathscr{C} \cup \mathscr{D}$ are examined separately in Sections 2 and 3. It is shown in particular that while $\mathscr{A} \cup \mathscr{B}$ (more precisely, exactly one of $\mathscr{A}$ and $\mathscr{B}$) is always nonempty, $\mathscr{C} \cup \mathscr{D}$ may or may not be empty. Our main tool is the Schauder-Tychonoff fixed point theorem applied to nonlinear integro-differential operators acting on the Fréchet space of continuously differentiable functions on $[a, \infty)$. Examples illustrating the main results are provided in Section 4.

For related results concerning equations of the form (1) but with different nonlinearity the reader is referred to Belohorec [1], Kusano, Swanson and Usami [2], Liang [3], and Usami [4].

2. Decreasing solutions

Let us first study the structure of the classes $\mathscr{A}$ and $\mathscr{B}$. We observe that the set $\mathscr{A} \cup \mathscr{B}$ is always nonempty, since the solution $y(t)$ of (1) with the initial data $y(a) = \alpha$, $y'(a) = \beta \leqq 0$ is decreasing on $[a, \infty)$, and hence is a member of $\mathscr{A}$ or $\mathscr{B}$.

THEOREM 1. *Suppose that* (A_1) *and* (A_2) *hold. Then,* $\mathscr{A} \neq \phi$ *if and only if*

$$\int_a^\infty f\left(t, -kP(t), -\frac{k}{p(t)}\right)dt = \infty \quad \textit{for all} \quad k > 0. \tag{5}$$

THEOREM 2. *Suppose that* (A_1) *and* (A_2) *hold. Then,* $\mathscr{B} \neq \phi$ *if and only if*

$$\int_a^\infty f\left(t,\ -kP(t),\ -\frac{k}{p(t)}\right)dt < \infty \quad \textit{for some} \quad k > 0. \tag{6}$$

Proof of Theorem 2. Let $y \in \mathscr{B}$. An integration of (1) shows that

$$\int_a^\infty f(t,\ y(t),\ y'(t))dt < \infty. \tag{7}$$

Since $\lim_{t\to\infty} p(t)y'(t) = \lim_{t\to\infty} y(t)/P(t) = const < 0$, there exist constants $k > 0$ and $t_0 > a$ such that

$$y(t) \geqq -kP(t) \quad \text{and} \quad y'(t) \geqq -k/p(t) \qquad \text{for} \quad t \geqq t_0. \tag{8}$$

Combining (8) with (7) yields (6).

Conversely, suppose that (6) holds. Let $\alpha \in \boldsymbol{R}$ be fixed. In view of (A_2), $\int_a^\infty f(t,\ \alpha - \lambda P(t),\ -\lambda/p(t))dt$ is a nonincreasing function of λ for $\lambda > k$, and so one can choose an $l > k$ large enough so that

$$\int_a^\infty f\left(t,\ \alpha - lP(t),\ -\frac{l}{p(t)}\right)dt \leqq l.$$

Let $C^1[a, \infty)$ denote the Fréchet space of all continuously differentiable functions on $[a, \infty)$ with the usual metric topology, and let Y be the subset of $C^1[a, \infty)$ defined by

$$Y = \Big\{ y \in C^1[a, \infty) : \alpha - 2lP(t) \leqq y(t) \leqq \alpha - lP(t),$$
$$-\frac{2l}{p(t)} \leqq y'(t) \leqq -\frac{l}{p(t)},\ t \geqq a \Big\}.$$

Define the integro-differential operator $\mathscr{F}$: $Y \to C^1[a, \infty)$ by

$$\mathscr{F}y(t) = \alpha - 2lP(t) + \int_a^t \frac{1}{p(s)} \int_s^\infty f(r,\ y(r),\ y'(r))drds, \quad t \geqq a. \tag{9}$$

It is a matter of simple computation to show that $\mathscr{F}$ is a continuous operator which maps Y into a compact subset of Y. Therefore, from the Schauder-Tychonoff fixed point theorem it follows that $\mathscr{F}$ has a fixed point y in Y. This fixed point $y = y(t)$ satisfies the integro-differential equation

$$y(t) = \alpha - 2lP(t) + \int_a^t \frac{1}{p(s)} \int_s^\infty f(r,\ y(r),\ y'(r))drds, \quad t \geqq a,$$

from which one easily sees that $y(t)$ is a solution of (1) such that $\lim_{t\to\infty} p(t)y'(t) = -2l$, that is, $y(t)$ is a member of the class $\mathscr{B}$.

Proof of Theorem 1. Let $y \in \mathscr{A}$. Then, integrating (1) on $[a, \infty)$, we have

(10) $$\int_a^\infty f(t, y(t), y'(t))dt = \infty.$$

Since $\lim_{t\to\infty} p(t)y'(t) = \lim_{t\to\infty} y(t)/P(t) = -\infty$, for any $k>0$ there exists $t_0>a$ such that

(11) $$y(t) \leqq -kP(t) \quad \text{and} \quad y'(t) \leqq -k/p(t) \qquad \text{for} \quad t \geqq t_0.$$

The relation (5) follows from (10) and (11).

Suppose now that (5) holds. Then, Theorem 2 implies that $\mathscr{B} = \phi$, and so $\mathscr{A} \neq \phi$ by the remark made at the beginning of this section. This completes the proof.

REMARK 1. In view of Theorems 1 and 2 we see that $\mathscr{A} \neq \phi$ if and only if $\mathscr{B} = \phi$, that is, members of $\mathscr{A}$ and $\mathscr{B}$ cannot coexist.

3. Increasing solutions

We now turn to the study of the classes $\mathscr{C}$ and $\mathscr{D}$. In order to ensure the existence of members of $\mathscr{C} \cup \mathscr{D}$ we need the following additional hypothesis:

(A_3) $\lim_{u\to-\infty} f(t, u, v) = 0$ for any fixed $(t, v) \in [0, \infty) \times \boldsymbol{R}$.

Clearly, (A_3) is satisfied for (3) and (4). The structure of $\mathscr{D}$ is described in the next theorem.

THEOREM 3. *In addition to* (A_1) *and* (A_2) *suppose that* (A_3) *is satisfied. Then,* $\mathscr{D} \neq \phi$ *if and only if*

(12) $$\int_a^\infty f\left(t, kP(t), \frac{k}{p(t)}\right)dt < \infty \qquad \textit{for some} \quad k > 0.$$

PROOF. Let $y \in \mathscr{D}$. There exist constants $k>0$ and $t_0>a$ such that

(13) $$y(t) \geqq kP(t) \quad \text{and} \quad y'(t) \geqq k/p(t) \qquad \text{for} \quad t \geqq t_0.$$

Combining (13) with (7) which also holds for $y \in \mathscr{D}$, we see that (12) is satisfied.

Conversely, suppose that (12) holds. From (A_2), (A_3), (12) and the Lebesgue dominated convergence theorem it follows that

$$\lim_{\alpha\to-\infty} \int_a^\infty f\left(t, \alpha + kP(t), \frac{k}{p(t)}\right) dt = 0,$$

and so there exists an $\alpha < 0$ such that

$$\int_a^\infty f\left(t, \alpha + kP(t), \frac{k}{p(t)}\right) dt \leqq \frac{k}{2}.$$

Consider the set $Y \subset C^1[a, \infty)$ and the mapping $\mathcal{F}$: $Y \to C^1[a, \infty)$ defined by

$$Y = \Big\{ y \in C^1[a, \infty) : \alpha + \frac{k}{2} P(t) \leqq y(t) \leqq \alpha + kP(t),$$

$$\frac{k}{2p(t)} \leqq y'(t) \leqq \frac{k}{p(t)}, \quad t \geqq a \Big\}$$

and

$$(14) \qquad \mathcal{F}y(t) = \alpha + \frac{k}{2} P(t) + \int_a^t \frac{1}{p(s)} \int_s^\infty f(r, y(r), y'(r))\, drds, \quad t \geqq a.$$

Applying the Schauder-Tychonoff fixed point theorem, we conclude that there exists a fixed point $y \in Y$ of $\mathcal{F}$ which gives a solution of (1) belonging to class $\mathcal{D}$. This completes the proof.

Next we examine the class $\mathcal{C}$. This class consists of bounded solutions and unbounded solutions: $\mathcal{C} = \mathcal{C}_b \cup \mathcal{C}_u$, where

$$\mathcal{C}_b = \{ y \in \mathcal{C} : \lim_{t \to \infty} y(t) = const \}, \quad \mathcal{C}_u = \{ y \in \mathcal{C} : \lim_{t \to \infty} y(t) = \infty \}.$$

THEOREM 4. (i) *Suppose that* (A_1) *and* (A_2) *hold. If* $\mathcal{C}_b \neq \phi$, *then there is a constant* k *such that*

$$(15) \qquad \int_a^\infty P(t) f(t, k, 0)dt < \infty.$$

(ii) *Suppose that* (A_1), (A_2) *and* (A_3) *hold. Then,* $\mathcal{C}_b \neq \phi$ *if there are constants* k *and* $l > 0$ *such that*

$$(16) \qquad \int_a^\infty P(t)\, f\Big(t, k, \frac{l}{p(t)}\Big) dt < \infty.$$

PROOF. (i) Let $y \in \mathcal{C}_b$. Integrating (1) from t to ∞ yields

$$p(t)y'(t) = \int_t^\infty f(s, y(s), y'(s))ds, \quad t \geqq a.$$

Dividing the above by $p(t)$ and integrating on $[a, t]$, we have

$$y(t) = y(a) + \int_a^t \frac{1}{p(s)} \int_s^\infty f(r, y(r), y'(r))drds, \quad t \geqq a,$$

which implies that

$$\int_a^\infty \frac{1}{p(s)} \int_s^\infty f(r, y(r), y'(r))\, drds = \int_a^\infty \Big(\int_a^r \frac{ds}{p(s)} \Big) f(r, y(r), y'(r))\, dr < \infty,$$

that is,

(17) $$\int_a^\infty P(t)f(t, y(t), y'(t))dt < \infty.$$

Noting that $y'(t)>0$ and $y(t)\geqq y(a)$ on $[a, \infty)$, we see from (17) that (15) holds as desired.

(ii) Suppose that (16) holds for some k and $l>0$. Choose α so that $\alpha\leqq\min\{k, -l\}$ and

$$\int_a^\infty \max\{1, P(t)\} f\left(t, \alpha, \frac{l}{p(t)}\right) dt \leqq l,$$

let Y denote the set

$$Y=\left\{y\in C^1[a, \infty): 2\alpha\leqq y(t)\leqq\alpha,\ 0\leqq y'(t)\leqq \frac{l}{p(t)},\ t\geqq a\right\},$$

and define the mapping $\mathscr{F}$ by

(18) $$\mathscr{F}y(t) = 2\alpha + \int_a^t \frac{1}{p(s)}\int_s^\infty f(r, y(r), y'(r))drds, \quad t\geqq a.$$

It is easy to show that $\mathscr{F}$ maps Y continuously into a compact subset of Y. Consequently, $\mathscr{F}$ has a fixed point y in Y, which is a solution of (1) satisfying $\lim_{t\to\infty} p(t)\cdot y'(t)=0$ and $\lim_{t\to\infty} y(t)=const\in[2\alpha, \alpha]$. This establishes the existence of a member of $\mathscr{C}_b$. The proof is thus complete.

REMARK 2. In the case where the nonlinear term of (1) satisfying (A_1)–(A_3) does not depend on y', that is, $f(t, y, y')=f(t, y)$, a necessary and sufficient condition for the existence of a member of $\mathscr{C}_b$ is that

(19) $$\int_a^\infty P(t)f(t, k)dt < \infty \quad \text{for some constant } k.$$

REMARK 3. Consider the particular equation

(20) $$y'' + \varphi(t)f(y)g(y') = 0,$$

where φ: $[0, \infty)\to(0, \infty)$ is continuous, f, g: $\boldsymbol{R}\to(0, \infty)$ are continuous and nondecreasing, and $\lim_{u\to-\infty} f(u)=0$. As easily checked, conditions (15) and (16) for this equation are equivalent and reduce to

(21) $$\int_a^\infty t\varphi(t)dt < \infty.$$

It follows that (20) has a solution of class $\mathscr{C}_b$ (i.e., a bounded increasing solution on $[a, \infty)$) if and only if (21) holds.

It is very difficult to find sufficient conditions which ensure that $\mathscr{C}_u\neq\phi$. A simple necessary condition for $\mathscr{C}_u\neq\phi$ is given in the following theorem.

THEOREM 5. *Suppose that* (A_1) *and* (A_2) *hold. If* $\mathscr{C}_u \neq \phi$ *for* (1), *then*

$$\int_a^\infty f(t, k, 0)dt < \infty \qquad \textit{for all} \quad k > 0. \tag{22}$$

PROOF. If $y \in \mathscr{C}_u$, then clearly (7) holds. On the other hand, since $\lim_{t\to\infty} p(t)y'(t)=0$ and $\lim_{t\to\infty} y(t)=\infty$, for any $k>0$ there is $t_0>a$ such that $y(t)\geqq k$ and $y'(t)>0$ for $t\geqq t_0$. Using these inequalities in (7) leads to (22).

COROLLARY 1. *Suppose that* (A_1), (A_2) *are satisfied. Then,* $\mathscr{C} \cup \mathscr{D}=\phi$ *if*

$$\int_a^\infty f(t, k, 0)dt = \infty \quad \textit{for every constant } k. \tag{23}$$

PROOF. That (23) ensures $\mathscr{C}_u=\phi$ follows from Theorem 5. Since (23) implies

$$\int_a^\infty P(t)f(t, k, 0)dt = \infty \qquad \text{for all} \quad k,$$

(i) of Theorem 4 shows that $\mathscr{C}_b=\phi$. Noting that (23) also implies

$$\int_a^\infty f\left(t, kP(t), \frac{k}{p(t)}\right)dt=\infty \qquad \text{for all} \quad k > 0,$$

we see from Theorem 3 that $\mathscr{D}=\phi$. This completes the proof.

The condition (23) ensuring that $\mathscr{C} \cup \mathscr{D}=\phi$ can be strengthened under more restrictive conditions on the nonlinearity of equation (1).

THEOREM 6. *Consider the equation*

$$(p(t)y')' + \varphi(t)f(y, y') = 0, \tag{24}$$

where $p(t)$ *is as before, and* φ: $[0, \infty)\to(0, \infty)$ *and* f: $\boldsymbol{R}\times\boldsymbol{R}\to(0, \infty)$ *are continuous. Suppose that* $f(u, v)$ *is nondecreasing in* u *and* v, *and*

$$\int_\delta^\infty \frac{du}{f(u, 0)} < \infty \qquad \textit{for every} \quad \delta. \tag{25}$$

If

$$\int_a^\infty P(t)\varphi(t)dt = \infty, \tag{26}$$

then $\mathscr{C} \cup \mathscr{D}=\phi$ *for* (24), *that is,* (24) *has no increasing solution.*

PROOF. Suppose that (24) has a solution $y \in \mathscr{C} \cup \mathscr{D}$. Integrating (24) from t to ∞ and noting that $y(t)$ is increasing, we have

$$p(t)y'(t) \geqq \int_t^\infty \varphi(s)f(y(s), y'(s))ds$$

$$\geqq f(y(t), 0)\int_t^\infty \varphi(s)ds, \quad t \geqq a,$$

whence it follows that

$$\int_a^t \frac{1}{p(s)}\int_s^\infty \varphi(r)\, drds \leqq \int_a^t \frac{y'(s)}{f(y(s), 0)}\, ds = \int_{y(a)}^{y(t)} \frac{du}{f(u, 0)}.$$

Letting $t\to\infty$ in the above, we obtain

$$\int_a^\infty \left(\int_a^t \frac{ds}{p(s)}\right)\varphi(t)dt \leqq \int_{y(a)}^\infty \frac{du}{f(u, 0)} < \infty,$$

which concludes the proof.

COROLLARY 2. *Consider the equation*

$$(27) \qquad (p(t)y')' + \varphi(t)f(y) = 0,$$

where $p(t)$ *and* $\varphi(t)$ *are as in Theorem* 6, *and* f: $\boldsymbol{R}\to(0, \infty)$ *is a continuous non-decreasing function such that* $\lim_{u\to-\infty}f(u)=0$ *and*

$$(28) \qquad \int_\delta^\infty \frac{du}{f(u)} < \infty \qquad \text{for every} \quad \delta.$$

Then, (27) *has no increasing solution* ($\mathscr{C} \cup \mathscr{D}=\phi$) *if and only if* (26) *holds.*

COROLLARY 3. *Consider equation* (20) *satisfying the conditions as mentioned in Remark* 3. *Suppose moreover that* (28) *is satisfied. Then,* (20) *has no increasing solution* ($\mathscr{C} \cup \mathscr{D}=\phi$) *if and only if*

$$(29) \qquad \int_a^\infty t\varphi(t)dt = \infty.$$

REMARK 4. From the proof of Theorem 6 we find that under the condition

$$(30) \qquad \int_a^\infty P(t)\varphi(t)dt < \infty,$$

if α is large enough so that

$$\int_a^\infty\left(\int_a^t \frac{ds}{p(s)}\right)\varphi(t)dt > \int_\alpha^\infty \frac{du}{f(u, 0)},$$

then none of the solutions of (24) starting from the point (a, α) belongs to $\mathscr{C} \cup \mathscr{D}$, that is, every solution $y(t)$ of (24) with $y(a)=\alpha$ is eventually decreasing regardless of the values of $y'(a)$. This observation justifies the situation encountered in the proof of (ii) of Theorem 4, where in order to ensure the existence of a solution in $\mathscr{C}_b$ of (1) its initial value at $t=a$ had to be taken to be a sufficiently small negative number.

4. Examples

We give two examples which illustrate the results developed in the preceding sections.

EXAMPLE 1. Consider the equation

$$y'' + \varphi(t)\exp(|y|^{\gamma-1}y + |y'|^{\delta-1}y') = 0, \tag{4}$$

where γ and δ are positive constants. Our results specialized to (4) yield the following statements:

(i) Equation (4) has a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/t = const > 0$ (i.e. $\mathscr{D} \neq \phi$) if and only if

$$\int_a^\infty \exp(kt^\gamma)\varphi(t)dt < \infty \qquad \text{for some} \quad k > 0. \tag{31}$$

(ii) Equation (4) has a solution $y(t)$ such that $\lim_{t\to\infty} y(t) = const$ (i.e. $\mathscr{C}_b \neq \phi$) if and only if

$$\int_a^\infty t\varphi(t)dt < \infty. \tag{32}$$

(iii) Equation (4) has a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/t = const < 0$ (i.e. $\mathscr{B} \neq \phi$) if and only if

$$\int_a^\infty \exp(-kt^\gamma)\varphi(t)dt < \infty \qquad \text{for some} \quad k > 0. \tag{33}$$

(iv) Equation (4) has a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/t = -\infty$ (i.e. $\mathscr{A} \neq \phi$) if and only if

$$\int_a^\infty \exp(-kt^\gamma)\varphi(t)dt = \infty \qquad \text{for all} \quad k > 0. \tag{34}$$

(v) If (31) holds (e.g. if $\varphi(t) = \exp(-mt^n)$ with $m > 0$ and $n \geqq \gamma$), then each of $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ has a member, and we have $\mathscr{S} = \mathscr{B} \cup \mathscr{C} \cup \mathscr{D}$ for (4).

(vi) If (32) holds but (31) does not (e.g. if $\varphi(t) = mt^{-n}$ with $m > 0$ and $n > 2$), then both $\mathscr{B}$ and $\mathscr{C}$ have members, and we have $\mathscr{S} = \mathscr{B} \cup \mathscr{C}$.

(vii) If (33) holds but (32) does not (e.g. if $\varphi(t) = \exp(mt^n)$ with $m > 0$, $n \leqq \gamma$), then all solutions of (4) are memberss of $\mathscr{B}$, that is, $\mathscr{S} = \mathscr{B}$.

(viii) If (33) does not hold (e.g. if $\varphi(t) = \exp(mt^n)$ with $m > 0$ and $n > \gamma$), then all solutions of (4) are members of $\mathscr{A}$, that is, $\mathscr{S}=\mathscr{A}$.

EXAMPLE 2. Consider the elliptic equation

$$\Delta u + e^u = 0 \quad \text{in} \quad \Omega_a, \tag{35}$$

where Δ is the Laplace operator in $\boldsymbol{R}^N$, $N \geqq 2$, and $\Omega_a = \{x \in \boldsymbol{R}^N \colon |x| > a\}$, $a > 0$. It can be shown that equation (35) possesses infinitely many solutions defined on Ω_a for any $a > 0$. Notice that a radially symmetric function $u(x) = y(|x|)$ is a solution of (35) on Ω_a if and only if $y(t)$ satisfies the ordinary differential equation

$$(t^{N-1}y')' + t^{N-1}e^y = 0, \quad t > a. \tag{36}$$

Let $N=2$. Then, (36) is a special case of (1) with $p(t)=t$ and $f(t, u, v)=te^u$. The function $P(t)$ defined by (2) can be taken to be $P(t)=\log t$. Since

$$\int_a^\infty f\left(t, -kP(t), -\frac{k}{p(t)}\right) dt = \int_a^\infty te^{-k\log t} dt$$
$$= \int_a^\infty t^{1-k}\, dt < \infty \qquad \text{if} \quad k > 2,$$

Theorem 2 applies to (36) and ensures the existence of a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/\log t = const < 0$. All solutions of (36) have the same logarithmic order of decrease as $t \to \infty$, since there is no increasing solution of (36) by Corollary 1. It thus follows that equation (35) with $N=2$ possesses a solution $u(x)$ on Ω_a with logarithmic decrease at infinity:

$$\lim_{|x|\to\infty} \frac{u(x)}{\log |x|} = const < 0, \tag{37}$$

and that all radially symmetric solutions of (35) have the same asymptotic behavior (37).

Let $N \geqq 3$. We transform (36) into

$$(t^{3-N}z')' + t\exp(t^{2-N}z) = 0, \quad t > a, \tag{38}$$

where $z = t^{N-2}y$, which is a special case of (1) with $p(t) = t^{3-N}$ and $f(t, u, v) = t\exp(t^{2-N}u)$. We can take $P(t) = t^{N-2}$ and verify that (5) is satisfied for (38):

$$\int_a^\infty f\left(t, -kP(t), -\frac{k}{p(t)}\right) dt = e^{-k}\int_a^\infty t\,dt = \infty.$$

According to Theorem 1 and Corollary 1 all solutions $z(t)$ of (38) have the asymptotic behavior

$$\lim_{t\to\infty} \frac{z(t)}{t^{N-2}} = -\infty.$$

This shows that equation (36) with $N \geqq 3$ has solutions on (a, ∞), all of which tend to $-\infty$ as $t \to \infty$. Therefore, in the case $N \geqq 3$ equation (35) has a solution $u(x)$ on Ω_a such that

$$\lim_{|x| \to \infty} u(x) = -\infty, \tag{39}$$

and all radially symmetric solutions of (35) eventually decrease to $-\infty$ as $|x| \to \infty$.

References

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