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# Oscillatory properties of the solutions of hyperbolic differential equations with "maximum"

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## 1. Introduction

Recently there has been a growing interest towards qualitative research of partial differential equations with deviating arguments. However, two papers only have been published so far considering the oscillatory properties of the solutions of partial differential equations with deviating arguments. These are the contributions of D. Georgiou and K. Kreith [1] and of M. Tramov [2].

The present paper studies the oscillatory properties of the solutions of various classes of hyperbolic differential equations with "maximum". Note that the problems for ordinary differential equations with "maximum" find application in the theory for automatic control of various real systems [3], [4]. A. D. Mishkis also points out the necessity to study differential equations with "maximum" in his survey [5]. Oscillatory and asymptotic properties of a class of functional-differential equations with "maximum" have been investigated in the paper of A. Zahariev and D. Bainov [6]. Theorems for existence and uniqueness of the solution of ordinary differential equations with "maximum" have been obtained in [7], [8].

Note that the author is not aware of papers considering partial differential equations with "maximum".

## 2. Problem of Goursat

In this section we consider the oscillatory properties of the solutions of the problem of Goursat concerning hyperbolic differential equations with "maximum" of the form

$$u_{xy} + p(x, y) \max_{\theta_1 \in [0, \sigma], \theta_2 \in [0, \tau]} u(x-\theta_1, y-\theta_2) = 0 \tag{1}$$

where $\sigma$, $\tau = const > 0$. Consider the following problem:

To find a solution of equation (1) in the domain $\Pi = \{(x, y)\colon x > 0, y > 0\}$, satisfying the conditions

$$\begin{cases} u(x, y) = \varphi(x, y) & \text{for} \quad (x, y) \in [-\sigma, \infty) \times [-\tau, 0], \\ u(x, y) = \psi(x, y) & \text{for} \quad (x, y) \in [-\sigma, 0] \times [-\tau, \infty). \end{cases} \tag{2}$$

Besides, assume that conditions for smoothness

$$\varphi(x, y) \in C^1([-\sigma, \infty) \times [-\tau, 0]), \quad \psi(x, y) \in C^1([-\sigma, 0] \times [-\tau, \infty)) \tag{3}$$

are fulfilled, as well as conditions for agreement of the boundary conditions

$$\varphi(x, y) = \psi(x, y) \qquad \text{for} \quad (x, y) \in [-\sigma, 0] \times [-\tau, 0]. \tag{4}$$

We will assume that the following conditions (A) hold:

A1. $p(x, y) \in C(\bar{\Pi})$,

A2. $\varphi(x, y) > 0$, $\varphi_x(x, y) \leqq 0$ and $\varphi_y(x, y) \leqq 0$ for $(x, y) \in [-\sigma, \infty) \times [-\tau, 0]$,

A3. $\psi(x, y) > 0$, $\psi_x(x, y) \leqq 0$ and $\psi_y(x, y) \leqq 0$ for $(x, y) \in [-\sigma, 0] \times [-\tau, \infty)$.

LEMMA 1. *Let conditions* (3), (4), (A) *hold and let besides*

$$p(x, y) \geqq \kappa^2 \qquad \textit{for} \quad (x, y) \in \Pi, \tag{5}$$

*where* $\kappa = const \neq 0$ *and* $p(x, y) \not\equiv \kappa^2$. *Then, if* $u(x, y)$ *is a solution of the problem* (1), (2) *and* $\lambda$ *is an arbitrary positive number, then* $u(x, y)$ *has a zero in each of the domains*

$$\mathscr{D}_n(\lambda) = \{(x, y) \in \Pi : n\pi < \lambda x + \lambda^{-1}\kappa^2 y < (n+1)\pi\},$$

*where* $n = 0, 1, 2, \ldots$.

PROOF. Let $n$ be an arbitrary even number. The case when $n$ is odd is considered in an analogous way. It is easily verified that the equation $v_{xy} + \kappa^2 v = 0$ has a solution $v(x, y) = \sin(\lambda x + \lambda^{-1}\kappa^2 y)$, which is positive in the domain $\mathscr{D}_n(\lambda)$. Let $u(x, y)$ be a solution of problem (1), (2). Assume to the contrary that $u(x, y)$ has no zero in the domain $\mathscr{D}_n(\lambda)$. Then, conditions A2 and A3 imply that $u(x, y) > 0$ for $(x, y) \in \mathscr{D}_n(\lambda)$. Taking into account condition (5) and the inequality

$$\max_{\theta_1 \in [0, \sigma], \theta_2 \in [0, \tau]} u(x - \theta_1, y - \theta_2) \geqq u(x, y) \qquad \text{for} \quad (x, y) \in \mathscr{D}_n(\lambda)$$

we get

$$\begin{aligned} 0 &< \iint_{\mathscr{D}_n} uv[p(x, y) - \kappa^2]dxdy \\ &\leqq \iint_{\mathscr{D}_n} [p(x, y) \max_{\theta_1 \in [0, \sigma], \theta_2 \in [0, \tau]} u(x - \theta_1, y - \theta_2) \cdot v - \kappa^2 uv]dxdy \\ &= \iint_{\mathscr{D}_n} [-u_{xy}v + uv_{xy}]dxdy = \iint_{\mathscr{D}_n} [(v_y u)_x - (u_x v)_y]dxdy. \end{aligned}$$

Introduce the following notations:

$$\Gamma_n = \{(x, y) \in \Pi: \lambda x + \lambda^{-1}\kappa^2 y = n\pi\}, \quad x_n = n\pi\lambda^{-1}, \quad y_n = n\pi\lambda\kappa^{-2}.$$

Applying the Green's formula, we obtain

$$\begin{aligned} 0 &< \iint_{\mathscr{D}_n} [(v_y u)_x - (u_x v)_y] dx dy \\ &= \int_{x_n}^{x_{n+1}} u_x(x, 0) v(x, 0) dx + \int_{\Gamma_{n+1}} u v_y dy - \int_{y_n}^{y_{n+1}} u(0, y) v_y(0, y) dy \\ &\quad - \int_{\Gamma_n} u v_y dy \leqq \int_{x_n}^{x_{n+1}} u_x(x, 0) v(x, 0) dx + \int_{y_n}^{y_{n+1}} u_y(0, y) v(0, y) dy. \end{aligned}$$

The last inequality is implied by the fact that $v_y<0$ on $\Gamma_{n+1}$ and $v_y>0$ on $\Gamma_n$. Moreover, conditions A2 and A3 yield that the integrals in the right-hand side of the last inequality are non-positive. The contradiction we have obtained establishes the lemma.

DEFINITION 1. A curved line on which a continuous function $u(x, y)$ is zero is called a nodal line for $u(x, y)$.

THEOREM 1. *Let the conditions of Lemma* 1 *be fulfilled. Then, every solution $u(x, y)$ of the problem* (1), (2) *has a nodal line of the form $y=f(x)$ or $x=g(y)$, where $f(x)$ and $g(y)$ are smooth, strictly decreasing functions and*

$$\lim_{x\to\infty} f(x) = 0, \quad \lim_{y\to\infty} g(y) = 0.$$

PROOF. Let $u(x, y)$ be a solution of problem (1), (2). By $\Gamma$ denote the set of all points $(\xi, \eta) \in \Pi$, for which

$$u(\xi, \eta) = 0 \tag{6}$$

and

$$u(x, y) > 0 \qquad \text{for} \quad (x, y) \in [0, \xi] \times [0, \eta] \setminus \{(\xi, \eta)\}. \tag{7}$$

Integrating equation (1) with respect to $x$ over the interval $[0, \xi]$, we get

$$\begin{aligned} u_y(\xi, \eta) &= \psi_y(0, \eta) - \int_0^\xi p(x, \eta) \max_{\theta_1 \in [0,\sigma],\, \theta_2 \in [0,\tau]} u(x-\theta_1, \eta-\theta_2) dx \\ &\leqq \psi_y(0, \eta) - \int_0^\xi p(x, \eta) u(x, \eta) dx. \end{aligned}$$

Taking into account (5), (7) and A3, the last inequality yields that

$$u_y(\xi, \eta) < 0. \tag{8}$$

It can be analogously proved that

$$u_x(\xi, \eta) < 0. \tag{9}$$

Then, (6), (8) and the theorem for existence and differentiability of an implicit function imply that in a neighbourhood of the point $(\xi, \eta)$ the curve $\Gamma$ can be represented in the form $y=f(x)$ where $f(x)$ is a differentiable function. Therefore, a number $\varepsilon>0$ exists, so that

$$\Gamma\colon y=f(x) \qquad \text{for} \quad x\in[\xi-\varepsilon,\ \xi+\varepsilon].$$

Moreover, from (8) and (9) we obtain that $f'(x)<0$ for $x\in[\xi-\varepsilon, \xi+\varepsilon]$. Extend the function $y=f(x)$ repeating the above considerations for the points $\xi\pm\varepsilon$ and so on.

To complete the proof of Theorem 1 it is sufficient to prove that if $\Gamma$ does not cross one of the axes $Ox$ or $Oy$, then this axis is an asymptote for $\Gamma$. Assume to the contrary that the straight line $x=a$, $a=const>0$ is an asymptote for $\Gamma$. Hence,

$$u(x, y)>0 \qquad \text{for} \quad (x, y)\in \bar{\Pi}_a=\{(x, y)\colon 0\leqq x\leqq a,\ y\geqq 0\}. \tag{10}$$

Choose a positive number $\lambda=\pi/a$. Then the straight line $\lambda x+\lambda^{-1}\kappa^2 y=\pi$ passes through the point $(a, 0)$ and therefore the domain $\mathscr{D}_0(\lambda)$ lies entirely in the semi-strip $\Pi_a$. Lemma 1 implies that $u(x, y)$ has a zero in the domain $\mathscr{D}_0(\lambda)\subset\Pi_a$. This contradicts (10).

It is analogously proved that the straight line $y=b$, $b=const>0$ cannot be an asymptote for $\Gamma$ either.

This completes the proof of Theorem 1.

REMARK 1. The proof of Theorem 1 is analogous to the proof of Lemma 1 of Pagan [9].

## 3. Problem of Goursat for nonlinear hyperbolic equations with "maximum"

Here we find sufficient conditions for oscillation of the solutions of nonlinear hyperbolic equations of the form:

$$u_{xy}+c(x, y, \max_{\theta_1\in[0,\sigma],\theta_2\in[0,\tau]} u(x-\theta_1, y-\theta_2))=f(x, y) \tag{11}$$

where $\sigma$, $\tau=const>0$.

Note that an analogous problem for nonlinear hyperbolic equations without "maximum" has been considered in the contribution of K. Kreith, T. Kusano and N. Yoshida [10].

Consider the following problem:

To find a solution of equation (11) in the domain $\Pi=\{(x, y)\colon x>0,\ y>0\}$, satisfying the conditions (2).

We will assume that the following conditions (B) hold:

B1. $c(x, y, \xi) \in C(\Pi \times R)$

B2. $c(x, y, \xi) \geqq p(x+y)\cdot\varphi(\xi)$ where $p(t), \varphi(\xi) \in C((0, \infty); (0, \infty))$ and $\varphi(\xi)$ is a monotonely increasing and convex function.

B3. $\varphi(x, y) > 0$ for $(x, y) \in [-\sigma, \infty) \times [-\tau, 0]$

$\psi(x, y) > 0$ for $(x, y) \in [-\sigma, 0] \times [-\tau, \infty)$

Let $u(x, y)$ be a solution of the problem (11), (2). Introduce the following function:

$$U(t) = \frac{1}{t}\int_0^t u(t-\xi, \xi)d\xi, \quad t > 0.$$

LEMMA 2. *Let the conditions* (3), (4), B1 *and* B2 *be fulfilled. Then, if $u(x, y)$ is a positive solution of the problem* (11), (2) *in the domain $\Pi_{t_1} = \{(x, y) \in \Pi: x+y > t_1\}$, then the function $U(t)$ satisfies for $t > t_1$ the differential inequality*

$$(tU(t))'' + tp(t)\cdot\varphi(U(t)) \leqq \varphi_x(t, 0) + \psi_y(0, t) + \int_0^t f(t-\xi, \xi)d\xi. \tag{12}$$

PROOF. Let $u(x, y)$ be a positive solution of the problem (11), (2) in the domain $\Pi_{t_1}$. Employing Lemma 1 of the paper of N. Yoshida [11] we get

$$\begin{aligned}(tU(t))'' &= u_x(t, 0) + u_y(0, t) + \int_0^t u_{xy}(t-\xi, \xi)d\xi \\ &= \varphi_x(t, 0) + \psi_y(0, t) + \int_0^t u_{xy}(t-\xi, \xi)d\xi.\end{aligned} \tag{13}$$

Equation (11) implies that

$$\int_0^t u_{xy}(t-\xi, \xi)d\xi = -\int_0^t c(t-\xi, \xi, \max_{\theta_1\in[0,\sigma],\theta_2\in[0,\tau]} u(t-\xi-\theta_1, \xi-\theta_2))d\xi + \int_0^t f(t-\xi, \xi)d\xi. \tag{14}$$

Since for $t > t_1$ the inequality

$$\max_{\theta_1\in[0,\sigma],\theta_2\in[0,\tau]} u(t-\xi-\theta_1, \xi-\theta_2) \geqq u(t-\xi, \xi) > 0$$

holds, then by virtue of condition B2 we obtain

$$\begin{aligned}&\int_0^t c(t-\xi, \xi, \max_{\theta_1\in[0,\sigma],\theta_2\in[0,\tau]} u(t-\xi-\theta_1, \xi-\theta_2))d\xi \\ &\geqq p(t)\int_0^t \varphi(\max_{\theta_1\in[0,\sigma],\theta_2\in[0,\tau]} u(t-\xi-\theta_1, \xi-\theta_2))d\xi \\ &\geqq p(t)\int_0^t \varphi(u(t-\xi, \xi))d\xi.\end{aligned} \tag{15}$$

The inequality of Jensen implies that

$$\int_0^t \varphi(u(t-\xi,\ \xi))d\xi \geqq t\varphi(U(t)). \tag{16}$$

Then, (13), (14), (15) and (16) yield that

$$\begin{aligned}(tU(t))'' &= \varphi_x(t,\ 0) + \psi_y(0,\ t) \\ &- \int_0^t c(t-\xi,\ \xi,\ \max_{\theta_1\in[0,\sigma],\,\theta_2\in[0,\tau]} u(t-\xi-\theta_1,\ \xi-\theta_2))d\xi \\ &+ \int_0^t f(t-\xi,\ \xi)d\xi \leqq \varphi_x(t,\ 0) + \psi_y(0,\ t) - p(t)t\varphi(U(t)) \\ &+ \int_0^t f(t-\xi,\ \xi)d\xi.\end{aligned}$$

This completes the proof of Lemma 2.

DEFINITION 2. Inequality (12) is called oscillatory at $t=\infty$, if it does not have a solution that would be positive in the interval $[t_0,\ \infty)$ for any $t_0>0$.

THEOREM 2. *Let conditions* (3), (4), (B) *hold and let the differential inequality* (12) *be oscillatory at* $t=\infty$. *Then every solution* $u(x,\ y)$ *of the problem* (11), (2) *has a zero in the domain* $\Pi_\rho=\{(x,\ y)\in\Pi\colon x+y>\rho\}$, *where* $\rho\geqq 0$ *is an arbitrary number.*

PROOF. Let $\rho\geqq 0$ be an arbitrary number. Assume to the contrary that there is a solution $u(x,\ y)$ of the problem (11), (2) that has no zero in the domain $\Pi_\rho$. Condition B3 implies that $u(x,\ y)>0$ for $(x,\ y)\in\Pi_\rho$. Then, by virtue of Lemma 2 we obtain that $U(t)$ is a positive solution of inequality (12) for $t>\rho$ which contradicts the assumption of the theorem. Thus, Theorem 2 is proved.

In this way the study of the oscillatory properties of the solutions of the problem (11), (2) is reduced to the study of the oscillatory properties of the solutions of differential inequalities of the form

$$(q(t)(p(t)y)')' + h(t,\ y) \leqq r(t). \tag{17}$$

Employing Theorems 2 and 3 from the paper of T. Kusano and M. Naito [12], we obtain the following propositions:

THEOREM 3. *Let conditions* (3), (4), (B) *be fulfilled and let*

$$\liminf_{t\to\infty}\int_T^t\left(1-\frac{s}{t}\right)\left(\varphi_x(s,\ 0)+\psi_y(0,\ s)+\int_0^s f(s-\xi,\ \xi)d\xi\right)ds = -\infty$$

*for any sufficiently large T. Then, every solution* $u(x,\ y)$ *of the problem* (11),

(2) *has a zero in the domain* $\Pi_\rho$ *where* $\rho \geqq 0$ *is an arbitrary number.*

THEOREM 4. *Let conditions* (3), (4), (B) *be fulfilled and let the differential inequality* $(tz)'' + tp(t)\varphi(z) \leqq 0$ *be oscillatory at* $t = \infty$. *Moreover, let a function* $\theta(t) \in C^2([\rho, \infty); R)$ *exist, possessing the following properties*:

(i) $\theta(t)$ *assumes both positive and negative values for arbitrarily large t,*

(ii) $(t\theta(t))'' = \varphi_x(t, 0) + \psi_y(0, t) + \int_0^t f(t-\xi, \xi)d\xi, \quad t > \rho,$

(iii) $\liminf_{t\to\infty} t\theta(t) = 0$.

*Then every solution* $u(x, y)$ *of the problem* (11), (2) *has a zero in the domain* $\Pi_\rho$, *where* $\rho \geqq 0$ *is an arbitrary number.*

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