HIROSHIMA MATH. J.
16 (1986), 51–75

Lie algebras satisfying the weak minimal condition on ideals

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(Received January 21, 1985)

Introduction

A group is said to satisfy the weak minimal condition on subgroups if it has no infinite descending chains of subgroups in which all neighbouring indices are infinite. Groups satisfying such a condition were first studied by D. I. Zaĭcev [17].

The purpose of this paper is to introduce analogously in Lie algebras the weak minimal conditions (wmin) relaxing the minimal conditions, and to investigate the properties of Lie algebras satisfying the weak minimal conditions on various subalgebras. The aspects of Lie algebras satisfying the weak minimal conditions are not similar to those of groups satisfying the corresponding conditions. One of the main reasons seems to be the following: In group theory every subgroup of finite index contains a normal subgroup of finite index, while in the theory of Lie algebras a subalgebra of finite codimension does not necessarily contain an ideal of finite codimension. Moreover, we define the weak maximal conditions (wmax) and develop the results on them in the course of the study of the weak minimal conditions.

The main results of this paper are as follows.

(1) An ideally soluble, hypoabelian Lie algebra satisfying the weak minimal condition on ideals is soluble (Theorem 2.4).

(2) A non-abelian ideally finite Lie algebra satisfying the weak minimal condition on non-abelian 2-step subideals satisfies the minimal condition on ideals (Theorem 3.6).

(3) If $\mathfrak{X}_i$ $(i=1, 2, 3)$ is one of the classes of abelian, nilpotent and soluble Lie algebras, then the following conditions are equivalent: (a) wmin on $\mathfrak{X}_1$-subideals (resp. ascendant $\mathfrak{X}_1$-subalgebras); (b) wmax on $\mathfrak{X}_2$-subideals (resp. ascendant $\mathfrak{X}_2$-subalgebras); (c) the minimal condition on $\mathfrak{X}_3$-subideals (resp. ascendant $\mathfrak{X}_3$-subalgebras) (Theorem 4.2).

(4) Each of the following Lie algebras is finite-dimensional: (a) a nilpotent algebra satisfying wmin or wmax on abelian ideals; (b) a supersoluble algebra satisfying wmin or wmax on abelian 2-step subideals; (c) a hyperabelian algebra satisfying wmin or wmax on abelian 3-step subideals; (d) an $\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$-algebra satisfying wmin or wmax on abelian subideals; (e) an $\acute{\mathrm{E}}\mathfrak{A}$-algebra satisfying wmin or wmax on abelian ascendant subalgebras; (f) an $\acute{\mathrm{E}}\mathrm{L}(\mathrm{wasc})(\mathrm{E}\mathfrak{A} \cup \mathfrak{F})$-algebra

satisfying wmin or wmax on abelian subalgebras (Theorem 5.8).

(5) There exist Lie algebras satisfying the following (a), (b) and (c) respectively: (a) wmin and wmax on ideals but neither the minimal nor the maximal condition on ideals, (b) wmin on ideals but neither the minimal condition nor wmax on ideals and (c) wmax on ideals but neither the maximal condition nor wmin on ideals (Theorem 6.5).

The author would like to express his hearty thanks to Professor S. Tôgô for his valuable comments and encouragement in preparing this paper.

1.

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [2] for the use of notations and terminology.

We begin with the definitions of ascendant subalgebras, weakly ascendant subalgebras ([15]), serial subalgebras and weakly serial subalgebras ([3]) of Lie algebras. Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For an ordinal σ, H is a σ-step ascendant (resp. weakly ascendant) subalgebra of L, denoted by $H \lhd^{\sigma} L$ (resp. $H \leq^{\sigma} L$), if there exists an ascending series (resp. chain) $(H_\alpha)_{\alpha \leq \sigma}$ of subalgebras (resp. subspaces) of L such that

(1) $H_0 = H$ and $H_\sigma = L$,

(2) $H_\alpha \lhd H_{\alpha+1}$ (resp. $[H_{\alpha+1}, H] \subseteq H_\alpha$) for any ordinal $\alpha < \sigma$,

(3) $H_\lambda = \cup_{\alpha<\lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is an ascendant (resp. a weakly ascendant) subalgebra of L, denoted by H asc L (resp. H wasc L), if $H \lhd^{\sigma} L$ (resp. $H \leq^{\sigma} L$) for some ordinal σ. When σ is finite, H is a subideal (resp. weak subideal) of L and denoted by H si L (resp. H wsi L). For a totally ordered set Σ, a series (resp. weak series) from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subalgebras (resp. subspaces) of L such that

(1) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,

(2) $L \backslash H = \cup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,

(3) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,

(4) $V_\sigma \lhd \Lambda_\sigma$ (resp. $[\Lambda_\sigma, H] \subseteq V_\sigma$) for all $\sigma \in \Sigma$.

H is a serial (resp. weakly serial) subalgebra of L, denoted by H ser L (resp. H wser L), if there exists a series (resp. weak series) from H to L of type Σ for some Σ. H is a local subideal (resp. locally ascendant subalgebra) of L, denoted by H lsi L (resp. H lasc L), if whenever X is a finite subset of L we have

$$H \operatorname{si} \langle H, X \rangle \quad (\text{resp. } H \operatorname{asc} \langle H, X \rangle).$$

For an ordinal α we denote by $L^{(\alpha)}$ (resp. L^{α}, $\zeta_\alpha(L)$) the α-th term of the transfinite derived (resp. lower central, upper central) series of L. For $H \leq L$ we put $H^L = \sum_{n \geq 0} [H, {}_n L]$.

Let $\mathfrak{X}$ be a class of Lie algebras and let Δ be any of the relations $\leq$, si, wsi, asc, wasc, $\lhd^{\sigma}$, $\leq^{\sigma}$, lsi, lasc, ser, wser. A Lie algebra L is said to lie in $\mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra K of L such that $X \subseteq K \Delta L$. In particular we write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\leq)\mathfrak{X}$. When $L \in \mathrm{L}(\lhd)\mathfrak{X}$ (resp. $\mathrm{L}\mathfrak{X}$), L is called an ideally (resp. a locally) $\mathfrak{X}$-algebra. $\mathfrak{F}_m$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{A}$, $\mathfrak{N}$, $\mathfrak{Z}$ and $\mathfrak{Z}_{\omega}$ are the classes of Lie algebras which are of dimension $\leq m$, finite-dimensional, finitely generated, abelian, nilpotent, hypercentral and hypercentral of central height $\leq \omega$ respectively. $\mathrm{R}\mathfrak{X}$ is the class of Lie algebras L having a collection $\{I_{\alpha}\}_{\alpha \in A}$ of ideals such that $\bigcap_{\alpha \in A} I_{\alpha} = 0$ and $L/I_{\alpha} \in \mathfrak{X}$ for any $\alpha \in A$. If $H \Delta L$ (resp. $H \in \mathfrak{X}$ and $H \Delta L$), we say that H is a Δ-subalgebra (resp. $\Delta\mathfrak{X}$-subalgebra) of L. $\acute{\mathrm{E}}_{\sigma}(\Delta)\mathfrak{X}$ (resp. $\grave{\mathrm{E}}_{\sigma}(\Delta)\mathfrak{X}$) is the class of Lie algebras L having an ascending (resp. a descending) series $(L_{\alpha})_{\alpha \leq \sigma}$ of Δ-subalgebras such that

(1) $L_0 = 0$ and $L_{\sigma} = L$ (resp. $L_0 = L$ and $L_{\sigma} = 0$),

(2) $L_{\alpha} \lhd L_{\alpha+1}$ and $L_{\alpha+1}/L_{\alpha} \in \mathfrak{X}$ (resp. $L_{\alpha+1} \lhd L_{\alpha}$ and $L_{\alpha}/L_{\alpha+1} \in \mathfrak{X}$) for any ordinal $\alpha < \sigma$,

(3) $L_{\lambda} = \bigcup_{\alpha < \lambda} L_{\alpha}$ (resp. $L_{\lambda} = \bigcap_{\alpha < \lambda} L_{\alpha}$) for any limit ordinal $\lambda \leq \sigma$.

We define $\acute{\mathrm{E}}(\Delta)\mathfrak{X} = \bigcup_{\sigma > 0} \acute{\mathrm{E}}_{\sigma}(\Delta)\mathfrak{X}$, $\grave{\mathrm{E}}(\Delta)\mathfrak{X} = \bigcup_{\sigma > 0} \grave{\mathrm{E}}_{\sigma}(\Delta)\mathfrak{X}$ and $\mathrm{E}(\Delta)\mathfrak{X} = \bigcup_{n < \omega} \acute{\mathrm{E}}_{n}(\Delta)\mathfrak{X}$. In particular we write $\acute{\mathrm{E}}_{\sigma}\mathfrak{X}$, $\grave{\mathrm{E}}_{\sigma}\mathfrak{X}$, $\acute{\mathrm{E}}\mathfrak{X}$, $\grave{\mathrm{E}}\mathfrak{X}$ and $\mathrm{E}\mathfrak{X}$ for $\acute{\mathrm{E}}_{\sigma}(\leq)\mathfrak{X}$, $\grave{\mathrm{E}}_{\sigma}(\leq)\mathfrak{X}$, $\acute{\mathrm{E}}(\leq)\mathfrak{X}$, $\grave{\mathrm{E}}(\leq)\mathfrak{X}$ and $\mathrm{E}(\leq)\mathfrak{X}$ respectively. When $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{F}_1$ (resp. $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$, $\grave{\mathrm{E}}\mathfrak{A}$, $\mathrm{E}\mathfrak{A}$), L is called a supersoluble (resp. hyperabelian, hypoabelian, soluble) Lie algebra. Min-Δ (resp. Min-$\Delta\mathfrak{X}$) is the class of Lie algebras satisfying the minimal condition on Δ-subalgebras (resp. $\Delta\mathfrak{X}$-subalgebras). In particular we write Min (resp. Min-$\mathfrak{X}$) for Min-$\leq$ (resp. Min-$\leq\mathfrak{X}$). Max-Δ and Max-$\Delta\mathfrak{X}$ are similarly defined. $\Delta\mathfrak{X}$-Fin is the class of Lie algebras in which every $\Delta\mathfrak{X}$-subalgebra is finite-dimensional.

A local system for a Lie algebra L is a collection $\{L_i\}_{i \in I}$ of subalgebras of L which generate L and have the property that whenever $i, j \in I$ there exists $k \in I$ such that $\langle L_i, L_j \rangle \leq L_k$.

We shall now introduce the new concept in the following

Definition. Let $\mathfrak{X}$ be a class of Lie algebras and let Δ be any of the relations $\leq$, $\lhd^{\sigma}$, si, asc, ser. A Lie algebra L is said to satisfy the weak minimal condition on $\Delta\mathfrak{X}$-subalgebras if it does not possess an infinite descending chain

$$H_1 \supseteq H_2 \supseteq \cdots \supseteq H_i \supseteq H_{i+1} \supseteq \cdots \qquad (*)$$

of $\Delta\mathfrak{X}$-subalgebras satisfying the condition that the codimension of the subspace H_{i+1} in the space H_i is infinite, or equivalently if for every descending chain $(*)$ of $\Delta\mathfrak{X}$-subalgebras of L there exists $r \in \boldsymbol{N}$ such that the dimension of the vector space H_i/H_{i+1} is finite for any $i \geq r$. The weak maximal condition on $\Delta\mathfrak{X}$-subalgebras is similarly defined. We denote by wmin-$\Delta\mathfrak{X}$ (resp. wmax-$\Delta\mathfrak{X}$) the class of Lie algebras satisfying the weak minimal (resp. weak maximal) condition on $\Delta\mathfrak{X}$-subalgebras. When $\mathfrak{X}$ is the class of all Lie algebras, we simply write wmin-Δ

(resp. wmax-Δ) instead of wmin-$\Delta\mathfrak{X}$ (resp. wmax-$\Delta\mathfrak{X}$). Moreover we write wmin (resp. wmax) for wmin-$\leq$ (resp. wmax-$\leq$).

Now as elementary properties of the classes wmin-Δ and wmax-Δ we have

LEMMA 1.1. *Let Δ be any of the relations $\leq$, $\lhd^{\sigma}$, si, asc. Then* wmin-Δ *and* wmax-Δ *are* {E, Q}*-closed.*

PROOF. Let I be an ideal of a Lie algebra L and suppose that I and L/I satisfy the weak minimal condition on Δ-subalgebras. For any descending chain $H_1 \supseteq H_2 \supseteq \cdots$ of Δ-subalgebras of L, we have two descending chains $H_1 \cap I \supseteq H_2 \cap I \supseteq \cdots$ of Δ-subalgebras of I and $(H_1+I)/I \supseteq (H_2+I)/I \supseteq \cdots$ of Δ-subalgebras of L/I. Then there exists $n \in \boldsymbol{N}$ such that the dimensions of $(H_k \cap I)/(H_{k+1} \cap I)$ and $(H_k+I)/(H_{k+1}+I)$ are finite for any $k \geq n$. Since

$$\dim H_k/H_{k+1} = \dim (H_k \cap I)/(H_{k+1} \cap I) + \dim (H_k+I)/(H_{k+1}+I),$$

the dimension of H_k/H_{k+1} is also finite for any $k \geq n$. Hence $L \in$ wmin-Δ. This implies that wmin-Δ is E-closed. Similarly we can prove that wmax-Δ is E-closed. Q-closedness of wmin-Δ and wmax-Δ is trivial.

In group theory, there exists an infinite abelian group satisfying the weak minimal condition on subgroups (e.g. an infinite cyclic group). For Lie algebras, however, we have

LEMMA 1.2. *An abelian Lie algebra satisfying the weak minimal or the weak maximal condition on subalgebras is finite-dimensional.*

PROOF. Let L be an infinite-dimensional abelian Lie algebra. Then there exists a linearly independent subset $\{e_{ij}: i, j \in \boldsymbol{N}\}$ of L. Now we define

$$I_n = \oplus_{i \geq n}(\oplus_{j \in \boldsymbol{N}} \langle e_{ij} \rangle), \quad J_n = \oplus_{i \leq n}(\oplus_{j \in \boldsymbol{N}} \langle e_{ij} \rangle) \qquad \text{for any } n \in \boldsymbol{N}.$$

Evidently $I_0 > I_1 > \cdots$ and $J_0 < J_1 < \cdots$, and furthermore I_n/I_{n+1} and J_{n+1}/J_n are infinite-dimensional. Therefore L satisfies neither the weak minimal nor the weak maximal condition on subalgebras.

From Lemma 1.2 we deduce

PROPOSITION 1.3. (1) *Let L be a Lie algebra satisfying the weak minimal or the weak maximal condition on ideals. Then $\zeta_n(L)$ is finite-dimensional for any $n \in \boldsymbol{N}$.*

(2) *A nilpotent Lie algebra satisfying the weak minimal or the weak maximal condition on ideals is finite-dimensional.*

(3) *A soluble Lie algebra satisfying the weak minimal or the weak maximal condition on 2-step subideals is finite-dimensional.*

PROOF. (1) We use induction on n. Since $\zeta_1(L) \in \text{wmin} \cup \text{wmax}$ it follows from Lemma 1.2 that $\zeta_1(L) \in \mathfrak{F}$. Let $n>1$ and suppose that $\zeta_{n-1}(L) \in \mathfrak{F}$. By Lemma 1.1 $L/\zeta_{n-1}(L) \in \text{wmin-}\lhd \cup \text{wmax-}\lhd$. As $\zeta_n(L)/\zeta_{n-1}(L) = \zeta_1(L/\zeta_{n-1}(L)) \in \mathfrak{F}$ we have $\zeta_n(L) \in \mathfrak{F}$.

(2) is a direct consequence of (1).

(3) Let $L \in \mathrm{E}\mathfrak{A} \cap (\text{wmin-}\lhd^2 \cup \text{wmax-}\lhd^2)$. Then $L^{(n)}=0$ for some $n \in N$. Since $L^{(i)} \in \text{wmin-}\lhd \cup \text{wmax-}\lhd$ it follows from Lemmas 1.1 and 1.2 that

$$L^{(i)}/L^{(i+1)} \in (\text{wmin-}\lhd \cup \text{wmax-}\lhd) \cap \mathfrak{A} \leq \mathfrak{F}$$

for $0 \leq i < n$. Therefore we have $L \in \mathfrak{F}$.

Hypercentral Lie algebras satisfying the weak minimal and the weak maximal conditions on ideals need not be finite-dimensional (see Example 6.2). On the other hand for hypercentral Lie algebras of central height $\leq \omega$ we have the following

COROLLARY 1.4. *Let L be a Lie algebra satisfying the weak minimal or the weak maximal condition on ideals. Then L is hypercentral of central height $\leq \omega$ if and only if L is locally nilpotent and ideally finite.*

PROOF. The 'if' part is clear since $\mathrm{L}(\lhd)(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{L}\mathfrak{N} \cap \mathrm{L}(\lhd)\mathfrak{F} \leq \mathfrak{Z}_\omega$ by [13, Theorem 3.6]. Let $L \in \mathfrak{Z}_\omega$. Then $L = \bigcup_{n<\omega} \zeta_n(L)$. Using Proposition 1.3(1) we have $\zeta_n(L) \in \mathfrak{N} \cap \mathfrak{F}$ for any $n \in N$. Therefore $L \in \mathrm{L}(\lhd)(\mathfrak{N} \cap \mathfrak{F})$.

2.

In this section we shall consider several conditions under which Lie algebras satisfying the weak minimal condition on ideals are soluble. To do this we need the following key lemma.

LEMMA 2.1. *Let L be a Lie algebra belonging to $\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A}$ and let $\{A_i\}_{i=0}^{\infty}$ be a descending chain of ideals of L such that $\bigcap_{i=0}^{\infty} A_i = 0$ and that A_{i+1} is of finite codimension in A_i for any $i \geq 0$. If A_0 is non-soluble, then L has a non-soluble ideal of infinite codimension in A_0.*

PROOF. Assume that $A_0 \notin \mathrm{E}\mathfrak{A}$. We put $k(0)=1$. Since $A_i/A_{i+1} \in \mathfrak{F}$ there exist finitely many elements $a_1^{(i)}, a_2^{(i)}, \dots, a_{n(i)}^{(i)}$ of A_i such that

$$A_i/A_{i+1} = \langle a_1^{(i)} + A_{i+1}, \dots, a_{n(i)}^{(i)} + A_{i+1} \rangle .$$

Put $X_i = \{a_1^{(i)}, \dots, a_{n(i)}^{(i)}\}$. Owing to $L \in \mathrm{L}(\lhd)\mathrm{E}\mathfrak{A}$ we can choose an ideal S_1 of L such that $X_0 \subseteq S_1 \subseteq A_0$ and that S_1 is soluble of derived length $s(1)$. Now there exists an integer $k(1)$ such that $k(1) > k(0)$ and $A_{k(1)} \nsubseteq A_{k(1)+1} + S_1$. Indeed, if such an integer does not exist then for any $j > k(0)$ we have $A_j \subseteq A_{j+1} + S_1$ and so

$A_j+S_1=A_{j+1}+S_1$. In particular $A_j+S_1=A_{k(0)+1}+S_1$. Consequently,

$$A_{k(0)+1}/A_j \subseteq (A_j+S_1)/A_j \cong S_1/(S_1 \cap A_j).$$

This means that $A_{k(0)+1}/A_j$ is soluble of derived length $\leq s(1)$ for any $j>k(0)$. As $\bigcap_{j>k(0)} A_j=0$, $A_{k(0)+1}$ is imbeddable into the direct sum $\bigoplus_{j>k(0)} A_{k(0)+1}/A_j$. It follows that $A_{k(0)+1}$ is soluble of derived length $\leq s(1)$. On the other hand $A_0/A_{k(0)+1}$ is soluble since $\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A}$ is $\{\mathrm{S}, \mathrm{Q}\}$-closed. Therefore A_0 is soluble. This is contrary to the assumption. Thus there exists an integer $k(1)$ as desired. We select an element h_1 such that

$$h_1 \in A_{k(1)} \backslash (A_{k(1)+1}+S_1).$$

Furthermore there exists an integer $l(1)>k(1)$ such that $\langle X_{k(1)+1},\dots, X_{l(1)}\rangle$ is soluble of derived length $>s(1)$. Indeed, if such an integer does not exist, then for any $l>k(1)$ we see that $\langle X_{k(1)+1},\dots, X_l\rangle$ is soluble of derived length $\leq s(1)$. Since $A_{k(1)+1}=A_{l+1}+\langle X_{k(1)+1},\dots, X_l\rangle$ we have

$$A_{k(1)+1}/A_{l+1} \cong \langle X_{k(1)+1},\dots, X_l\rangle/(A_{l+1} \cap \langle X_{k(1)+1},\dots, X_l\rangle).$$

This implies that $A_{k(1)+1}/A_{l+1}$ is soluble of derived length $\leq s(1)$ for any $l>k(1)$. It is impossible for the same reason above. Thus there exists an integer $l(1)$ as desired. Then we can choose an ideal S_2 of L such that

$$\langle X_{k(1)+1},\dots, X_{l(1)}\rangle \subseteq S_2 \subseteq A_{k(1)+1}$$

and that S_2 is soluble of derived length $s(2)>s(1)$. Furthermore we can find an integer $k(2)>k(1)$ such that

$$A_{k(2)} \nsubseteq A_{k(2)+1} + S_1 + S_2.$$

The existence of $k(2)$ is shown similarly to that of the integer $k(1)$. We select an element h_2 such that

$$h_2 \in A_{k(2)} \backslash (A_{k(2)+1}+S_1+S_2).$$

Furthermore we can find an integer $l(2)>k(2)$ such that $\langle X_{k(2)+1},\dots, X_{l(2)}\rangle$ is soluble of derived length $>s(2)$. The existence of $l(2)$ is proved in the same way as that of the integer $l(1)$. Then we choose an ideal S_3 of L such that

$$\langle X_{k(2)+1},\dots, X_{l(2)}\rangle \subseteq S_3 \subseteq A_{k(2)+1}$$

and that S_3 is soluble of derived length $s(3)>s(2)$.

Continuing this procedure, we obtain two sequences of positive integers

$$k(0) < k(1) < \cdots < k(i) < \cdots,$$

$$s(1) < s(2) < \cdots < s(i) < \cdots,$$

a sequence of ideals

$$S_1,\ S_2,\ \cdots,\ S_i,\ \cdots$$

of L, and a sequence of elements

$$h_1,\ h_2,\ \cdots,\ h_i,\ \cdots$$

of L, satisfying the following requirements:

(1) $h_i \in A_{k(i)} \backslash (A_{k(i)+1} + S_1 + \cdots + S_i)$,

(2) $S_i \subseteq A_{k(i-1)+1}$ for $i \geq 2$ and $S_1 \subseteq A_0$,

(3) S_i is soluble of derived length $s(i)$.

From the requirement (2) we deduce that

$$S_j \subseteq A_{k(i)+1} \qquad \text{for} \quad j > i \geq 1. \tag{$*$}$$

Now we define

$$A = \textstyle\sum_{i=1}^{\infty} S_i .$$

Then it follows from the requirements (2) and (3) that A is a non-soluble ideal of L contained in A_0. We shall show that A is of infinite codimension in A_0. First we see that each h_i does not belong to A. In fact, if $h_i \in A$ then using $(*)$ we have

$$h_i \in A_{k(i)+1} + A = A_{k(i)+1} + S_1 + \cdots + S_i,$$

which is contrary to the requirement (1). Now it is enough for our purpose to prove that $\{h_i + A\colon i = 1, 2, \ldots\}$ is linearly independent in A_0/A. Furthermore to do this it is enough to prove that $h_{m(1)} + A$, $h_{m(2)} + A, \ldots, h_{m(i)} + A$ $(m(1) < m(2) < \cdots < m(i))$ are linearly independent in A_0/A. We use induction on i. It is clear for $i = 1$. Let $i > 1$ and suppose that the result holds for $i - 1$. Assume that $\sum_{j=1}^{i} \alpha_j h_{m(j)} \in A$ with $\alpha_j \in \mathfrak{k}$. Since $A_{k(m(i))} \subseteq \cdots \subseteq A_{k(m(2))} \subseteq A_{k(m(1))+1}$ we have

$$\alpha_1 h_{m(1)} \in A_{k(m(1))+1} + A = A_{k(m(1))+1} + S_1 + \cdots + S_{m(1)},$$

using $(*)$. Therefore $\alpha_1 = 0$ from the requirement (1). By the inductive hypothesis $\alpha_2 = \cdots = \alpha_i = 0$. This completes the proof of the lemma.

From Lemma 2.1 we deduce

LEMMA 2.2. *Let* $L \in \mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \mathrm{wmin}\text{-}\lhd$ *and let* $\{A_i\}_{i=0}^{\infty}$ *be a descending chain of ideals of* L *with* $\bigcap_{i=0}^{\infty} A_i = 0$. *Then* A_n *is soluble for some* $n \in \boldsymbol{N}$.

PROOF. Since L satisfies the weak minimal condition on ideals, there is $n \in \boldsymbol{N}$ such that A_i is of finite codimension in A_n for any $i \geq n$. Suppose that A_n is non-soluble. By Lemma 2.1 there exists an ideal L_1 of L such that L_1 is non-soluble and of infinite codimension in A_n. Now $\{L_1 \cap A_i\}_{i=n}^{\infty}$ is a chain of ideals of L such that $\bigcap_{i=n}^{\infty}(L_1 \cap A_i)=0$ and that $L_1 \cap A_i$ is of finite codimension in L_1 for any $i \geq n$. By Lemma 2.1 again there exists an ideal L_2 of L such that L_2 is non-soluble and of infinite codimension in L_1. By continuing this procedure, we obtain a strictly descending chain

$$L_1 > L_2 > \cdots > L_i > \cdots$$

of ideals of L such that L_{i+1} is of infinite codimension in L_i for any $i \geq 1$. This contradicts $L \in \mathrm{wmin}\text{-}\lhd$. Thus A_n is soluble.

Let L be a Lie algebra and let σ be an ordinal. It is easy to see that $L \in \grave{\mathrm{E}}_\sigma\mathfrak{A}$ if and only if $L^{(\sigma)}=0$. Defining $L^{(*)}=\bigcap_{\alpha\geq 0} L^{(\alpha)}$ we have the following

COROLLARY 2.3. *If* $L \in \mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \mathrm{wmin}\text{-}\lhd$ *then* $L/L^{(*)} \in \mathrm{E}\mathfrak{A}$.

PROOF. In the quotient algebra $\bar{L}=L/L^{(\omega)}$ we see that

$$\bar{L} \geq \bar{L}^{(1)} \geq \bar{L}^{(2)} \geq \cdots \geq \bar{L}^{(i)} \geq \cdots \text{ and } \bigcap_{i=0}^{\infty} \bar{L}^{(i)} = 0.$$

Since $\bar{L} \in \mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \mathrm{wmin}\text{-}\lhd$, in virtue of Lemma 2.2 there is $n \in N$ such that $\bar{L}^{(n)}$ is soluble. Hence $(\bar{L}^{(n)})^{(m)}=0$ for a suitable $m \in N$, i.e., $L^{(n+m)}=L^{(\omega)}=L^{(*)}$. Thus $L/L^{(*)}$ is soluble.

Obviously if $L \in \grave{\mathrm{E}}\mathfrak{A}$ then $L^{(*)}=0$. So as an immediate consequence of Proposition 1.3(3) and Corollary 2.3 we have the following main theorem in this section.

THEOREM 2.4. *An ideally soluble, hypoabelian Lie algebra satisfying the weak minimal condition on ideals is soluble, that is to say,*

$$\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \grave{\mathrm{E}}\mathfrak{A} \cap \mathrm{wmin}\text{-}\lhd \leq \mathrm{E}\mathfrak{A}.$$

Moreover

$$\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \grave{\mathrm{E}}\mathfrak{A} \cap \mathrm{wmin}\text{-}\lhd^2 \leq \mathrm{E}\mathfrak{A} \cap \mathfrak{F}.$$

As a special case of Theorem 2.4 we have

COROLLARY 2.5. *Let* $\mathfrak{X}$ *be one of the following classes*:

$$\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \mathrm{R}\mathfrak{F},\quad \mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \mathrm{Max}\text{-}\lhd,\quad \mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}),$$

$$\acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A} \cap \grave{\mathrm{E}}\mathfrak{A},\quad \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A},$$

$$\mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}),\quad \acute{\mathrm{E}}_\omega(\lhd)(\mathfrak{A} \cap \mathfrak{F}).$$

Then

$$\mathfrak{X} \cap \mathrm{wmin}\text{-}\lhd \leq \mathrm{E}\mathfrak{A} \quad \textit{and} \quad \mathfrak{X} \cap \mathrm{wmin}\text{-}\lhd^2 \leq \mathrm{E}\mathfrak{A} \cap \mathfrak{F}.$$

PROOF. The results follow from the facts that $\mathrm{LE}\mathfrak{A} \cap \mathrm{R}\mathfrak{F} \leq \mathrm{RE}\mathfrak{A} \leq \grave{\mathrm{E}}\mathfrak{A}$, $\mathrm{LE}\mathfrak{A} \cap \text{Max-}\lhd \leq \grave{\mathrm{E}}\mathfrak{A}$ (using [2, Lemma 8.6.2]), $\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \leq \grave{\mathrm{E}}\mathfrak{A}$ ([6, Corollary 3.8]) and $\acute{\mathrm{E}}_\omega(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \leq \mathrm{L}(\lhd)(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \leq \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A}$ ([16, Lemma 4.2]).

REMARK. The classes

$$\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \grave{\mathrm{E}}\mathfrak{A} \cap \text{wmin-}\lhd, \quad \mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \mathrm{R}\mathfrak{F} \cap \text{wmin-}\lhd,$$

$$\mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \text{Max-}\lhd \cap \text{wmin-}\lhd, \quad \mathrm{L}(\lhd)\mathrm{E}\mathfrak{A} \cap \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \text{wmin-}\lhd,$$

$$\acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A} \cap \grave{\mathrm{E}}\mathfrak{A} \cap \text{wmin-}\lhd, \quad \acute{\mathrm{E}}_\omega(\lhd)\mathfrak{A} \cap \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \cap \text{wmin-}\lhd,$$

are not subclasses of $\mathrm{E}\mathfrak{A} \cap \mathfrak{F}$ (Examples 6.1 and 6.2).

3.

In this section we shall consider several conditions under which Lie algebras satisfying the weak minimal conditions on various subalgebras satisfy the minimal condition on ideals or non-abelian ideals.

DEFINITION. Let H be a proper subalgebra of a Lie algebra L. As in group theory we say that H is finitely separable from an element x of L, not belonging to H, if there exists a homomorphism φ of L to a finite-dimensional Lie algebra such that $\varphi(x) \notin \varphi(H)$. We say that H is finitely separable from L if H is finitely separable from at least one element of L.

Now we require the following result for our aim.

LEMMA 3.1. *Let a Lie algebra L have a local system $\{L_\alpha\}_{\alpha \in A}$ consisting of ideals (resp. subalgebras) which are finitely separable from L. Then there exists in L a family $\{x_i\}_{i \in N_+}$ ($N_+ = \{1, 2, \dots\}$) of elements such that for $N_1 \subseteq N_2 \subseteq N_+$*

$$|N_2 \backslash N_1| \leq \dim H_2^L / H_1^L \quad (\text{resp.} \dim H_2/H_1),$$

with $H_j = \langle L_*, x_i \colon i \in N_j \rangle$ $(j = 1, 2)$, where L_* is 0 or any member of $\{L_\alpha\}$.

PROOF. First we shall show the existence in L of three infinite sequences: a sequence

$$x_1, \quad x_2, \quad \cdots, \quad x_n, \quad x_{n+1}, \quad \cdots \tag{1}$$

of elements of L, an ascending sequence

$$L_1 < L_2 < \cdots < L_n < L_{n+1} < \cdots \tag{2}$$

of ideals (resp. subalgebras) in the local system $\{L_\alpha\}$, and a descending sequence

$$A_1 > A_2 > \cdots > A_n > A_{n+1} > \cdots \tag{3}$$

of ideals of finite codimension, where the sequences must be connected by the relation

$$L = A_{n+1} + L_{n+1}, \quad x_n \in (A_n \backslash (A_{n+1} + L_n)) \cap (L_{n+1} \backslash L_n) \quad (n \geq 1). \tag{4}$$

We put $A_1 = L$ and choose an arbitrary subalgebra L_1 in the local system $\{L_\alpha\}$. Then there exists in L an ideal A_2 of finite codimension such that $L \neq A_2 + L_1$. In fact, since L_1 is finitely separable from L there exist an element x of L, not belonging to L_1, and a homomorphism φ of L to a finite-dimensional Lie algebra such that $\varphi(x) \notin \varphi(L_1)$. Hence $x \notin \operatorname{Ker} \varphi + L_1$. We may put $A_2 = \operatorname{Ker} \varphi$. Therefore we can choose an element $x_1 \in A_1 \backslash (A_2 + L_1)$. Furthermore, as A_2 is of finite codimension in L we can find finitely many elements $y_1, \ldots, y_k$ of L such that $L = A_2 + \langle y_1, \ldots, y_k \rangle$. Since $\{L_\alpha\}$ is a local system there exists a subalgebra L_2 in $\{L_\alpha\}$ such that

$$\langle x_1, L_1 \rangle \leq \langle x_1, y_1, \ldots, y_k, L_1 \rangle \leq L_2.$$

Thus we deduce that $L = A_2 + L_2$, $L_1 < L_2$ and $A_1 > A_2$.

Suppose that for $n \geq 1$ we have constructed the initial segments of the sequence (1) up to the n-th term and of the sequences (2), (3) up to the $(n+1)$-th term, and that they are connected by the relation (4). Since L_{n+1} is finitely separable from L, as above there exists in L an ideal I of finite codimension such that $L \neq I + L_{n+1}$. Put $A_{n+2} = A_{n+1} \cap I$. Then $A_{n+1} \nsubseteq A_{n+2} + L_{n+1}$. For, if not, we have

$$L = A_{n+1} + L_{n+1} \subseteq A_{n+2} + L_{n+1} \subseteq I + L_{n+1} \neq L,$$

which is impossible. Consequently we can choose an element

$$x_{n+1} \in A_{n+1} \backslash (A_{n+2} + L_{n+1}).$$

Furthermore, taking into account that $\{L_\alpha\}$ is a local system and that L/A_{n+2} is finite-dimensional, as above we can find in the system $\{L_\alpha\}$ a subalgebra L_{n+2} such that

$$\langle x_{n+1}, L_{n+1} \rangle \leq L_{n+2} \quad \text{and} \quad L = A_{n+2} + L_{n+2}.$$

Thus we have shown the existence in L of the infinite sequences (1)–(3) connected by the relation (4).

We shall show that $\{x_i\}_{i \in N_+}$ is a desired sequence. Here we prove this for the case where L_α's are ideals of L. Another case is similarly proved. Let $N_1 \subseteq N_2 \subseteq N_+$ and $H_j = \langle L_*, x_i \colon i \in N_j \rangle$ $(j = 1, 2)$, where L_* is 0 or L_1. To verify the inequality $|N_2 \backslash N_1| \leq \dim H_2^L / H_1^L$ it is enough to show that for $n_1, n_2, \ldots, n_k \in N_2 \backslash N_1$ $(n = n_1 < n_2 < \cdots < n_k)$

$$x_{n_1} + H_1^L, \quad \cdots, \quad x_{n_k} + H_1^L$$

are linearly independent in H_2^L/H_1^L. We use induction on k. For $k=0$ there is nothing to prove. Let $k>0$ and assume that the result holds for $k-1$. Suppose that

$$\textstyle\sum_{i=1}^k \alpha_i x_{n_i} \in H_1^L,$$

where $\alpha_i \in \mathfrak{k}$ $(1 \leq i \leq k)$. Let $j_1, \ldots, j_l$ be all the positive integers which are contained in N_1 and do not exceed $n=n_1$. Since $n \notin N_1$, we have $j_1, \ldots, j_l < n$. From the sequence (3) and the relation (4) it follows that $x_s \in A_s \leq A_{n+1}$ for all $s>n$. Then we have

$$\begin{aligned} A_{n+1} + H_1^L &= A_{n+1} + \langle L_*, x_i \colon i \in N_1 \rangle^L \\ &= A_{n+1} + \langle L_*, x_{j_1}, \ldots, x_{j_l} \rangle^L \subseteq A_{n+1} + L_n, \end{aligned}$$

since $\langle x_{j_1}, \ldots, x_{j_l} \rangle \leq L_n$ using (2) and (4). Hence we have

$$\textstyle\sum_{i=1}^k \alpha_i x_{n_i} \in A_{n+1} + L_n.$$

But as $n < n_2 < \cdots < n_k$ we see that $x_{n_i} \in A_{n_i} \leq A_{n+1}$ for all $i \geq 2$. Therefore

$$\alpha_1 x_n \in A_{n+1} + L_n,$$

which implies in virtue of (4) that $\alpha_1 = 0$. By inductive hypothesis we have $\alpha_2 = \cdots = \alpha_k = 0$. This completes the proof of the lemma.

Now Lemma 3.1 yields the following

LEMMA 3.2. (1) *Let $L \in \mathrm{L}(\lhd)\mathfrak{G}$. If either* (a) *$L$ satisfies the weak minimal condition on ideals or* (b) *L is non-abelian and satisfies the weak minimal condition on non-abelian ideals, then there exists a finitely generated ideal of L which is finitely inseparable from L.*

(2) *If either* (a) *L satisfies the weak minimal condition on subalgebras or* (b) *L is non-abelian and satisfies the weak minimal condition on non-abelian subalgebras, then there exists a finitely generated subalgebra of L which is finitely inseparable from L.*

PROOF. (1) Let $\{L_\alpha\}_{\alpha \in A}$ be the set of all finitely generated ideals of L. Then $\{L_\alpha\}$ becomes a local system for L. If all ideals L_α are finitely separable from L, then by Lemma 3.1 there exists a family of elements $\{x_i\}_{i \in \boldsymbol{N}_+}$. Put $N_i = 2^i \boldsymbol{N}_+$ for all $i \geq 1$. Then

$$N_1 \supset N_2 \supset \cdots \supset N_i \supset N_{i+1} \supset \cdots \quad \text{and} \quad |N_i \backslash N_{i+1}| = \infty.$$

We define $H_j = \langle x_i \colon i \in N_j \rangle^L$ for the case (a) and $H_j = \langle L_\beta, x_i \colon i \in N_j \rangle^L$, where

L_β is a non-abelian subalgebra in $\{L_\alpha\}$, for the case (b). Again by using Lemma 3.1 we have

$$H_1 > H_2 > \cdots > H_i > H_{i+1} > \cdots \quad \text{and} \quad \dim H_i/H_{i+1} = \infty,$$

which contradicts the weak minimal conditions. Thus there exists an ideal L_α finitely inseparable from L.

(2) is similarly proved by letting $\{L_\alpha\}_{\alpha\in A}$ be the set of all finitely generated subalgebras of L and defining $H_j=\langle x_i\colon i\in N_j\rangle$ for the case (a) and $H_j=\langle L_\beta, x_i\colon i\in N_j\rangle$ for the case (b).

By making use of Lemma 3.2 we have the following

PROPOSITION 3.3. (1) *Let L be an ideally finite Lie algebra.*

a) *If L satisfies the weak minimal condition on 2-step subideals then L satisfies the minimal condition on ideals.*

b) *If L satisfies the weak minimal condition on non-abelian 2-step subideals then L satisfies the minimal condition on non-abelian ideals.*

(2) *Let L be a locally finite Lie algebra.*

a) *If L satisfies the weak minimal condition on subalgebras then L satisfies the minimal condition on ideals.*

b) *If L satisfies the weak minimal condition on non-abelian subalgebras then L satisfies the minimal condition on non-abelian ideals.*

PROOF. (1) Let $L\in \mathrm{L}(\lhd)\mathfrak{F}$ and let $I_1\supseteq I_2\supseteq\cdots$ be a descending chain of ideals (resp. non-abelian ideals) of L. Since L satisfies the weak minimal condition on ideals (resp. non-abelian ideals) there is a positive integer n such that $I_n/I_{n+k}\in\mathfrak{F}$ for all $k\geq 0$. Now $I_n\in \mathrm{L}(\lhd)\mathfrak{F}$ and I_n satisfies the weak minimal condition on ideals (resp. non-abelian ideals). So from Lemma 3.2 it follows that there exists a finite-dimensional subalgebra K of I_n which is finitely inseparable from I_n. Considering the natural homomorphism of I_n onto the finite-dimensional quotient algebra I_n/I_{n+k} we have

$$I_n/I_{n+k} = (K+I_{n+k})/I_{n+k} \cong K/(K\cap I_{n+k}) \qquad \text{for any} \quad k\geq 0.$$

Thus we conclude that $I_{n+m}=I_{n+m+1}=\cdots$ for a suitable $m\in \boldsymbol{N}$.

(2) is similarly proved.

REMARK. In general Lie algebras satisfying the weak minimal condition on 2-step subideals (resp. non-abelian 2-step subideals) do not necessarily satisfy the minimal condition on ideals (resp. non-abelian ideals) (Example 6.6). However, we do not know a Lie algebra satisfying the weak minimal condition on subalgebras (resp. non-abelian subalgebras) but not the minimal condition on ideals (resp. non-abelian ideals), still less an infinite-dimensional Lie algebra satisfying the weak minimal condition on subalgebras.

Let Δ be any of the relations $\leq$, si, asc, $\lhd^{\alpha}$ (α an ordinal ≥ 1). Next we shall consider a condition under which Lie algebras satisfying the minimal (resp. weak minimal) condition on non-abelian Δ-subalgebras satisfy the minimal (resp. weak minimal) condition on ideals.

LEMMA 3.4. *Let Δ be one of the relations $\leq$,* si, asc, $\lhd^{\alpha}$ *(α an ordinal ≥ 1). If L satisfies the minimal (resp. weak minimal) condition on non-abelian Δ-subalgebras and contains a non-abelian Δ-subalgebra H satisfying the minimal (resp. weak minimal) condition on ideals, then L satisfies the minimal (resp. weak minimal) condition on ideals.*

PROOF. Let $I_1 \supseteq I_2 \supseteq \cdots$ denote an arbitrary descending chain of ideals of L. Then it is clear that

$$H \cap I_i \lhd H, \ H + I_i \Delta L \quad \text{and} \quad H + I_i \notin \mathfrak{A} \quad \text{for all } i \geq 1.$$

By the hypothesis we can find a positive integer n such that

$$\begin{aligned} &\dim (H \cap I_i)/(H \cap I_{i+1}) = 0 \quad (\text{resp.} < \infty) \quad \text{and} \\ &\dim (H+I_i)/(H+I_{i+1}) = 0 \quad (\text{resp.} < \infty) \end{aligned}$$

for all $i \geq n$. Therefore

$$\dim I_i/I_{i+1} = \dim (H \cap I_i)/(H \cap I_{i+1}) + \dim (H+I_i)/(H+I_{i+1}) = 0 \quad (\text{resp.} < \infty)$$

for all $i \geq n$. This implies that $L \in \text{Min-}\lhd$ (resp. $\text{wmin-}\lhd$).

PROPOSITION 3.5. *Let Δ be one of the relations $\leq$,* si, asc, $\lhd^{\alpha}$ *(α an ordinal ≥ 1) and let L be a non-abelian Lie algebra belonging to $\text{L}(\Delta)\mathfrak{F}$. If L satisfies the minimal (resp. weak minimal) condition on non-abelian Δ-subalgebras, then L satisfies the minimal (resp. weak minimal) condition on ideals.*

PROOF. Since L is not abelian we can choose two elements x, y of L such that $[x, y] \neq 0$. By $L \in \text{L}(\Delta)\mathfrak{F}$ there exists a finite-dimensional subalgebra F of L such that $\langle x, y \rangle \leq F \Delta L$. As F is not abelian we deduce from Lemma 3.4 that L satisfies the minimal (resp. weak minimal) condition on ideals.

Finally we extend Proposition 3.3(1) in the following main theorem in this section, which corresponds to Zaĭcev's result [18, Corollary 1] that a non-abelian locally finite group satisfying the weak minimal condition on non-abelian subgroups satisfies the minimal condition on subgroups.

THEOREM 3.6. *Let L be a non-abelian ideally finite Lie algebra. If L satisfies the weak minimal condition on non-abelian 2-step subideals, then*

L satisfies the minimal condition on ideals.

PROOF. Employing Proposition 3.3(1) we see that L satisfies the minimal condition on non-abelian ideals. Furthermore, by using Proposition 3.5 we conclude that L satisfies the minimal condition on ideals.

4.

Let Δ be any of the relations si, asc, $\lhd^{\alpha}$ (α an infinite ordinal) and let $\mathfrak{X}$ be any of the classes $\mathfrak{A}$, $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$. In this section we shall study Lie algebras satisfying the weak minimal or the weak maximal condition on $\Delta\mathfrak{X}$-subalgebras. We first prove

LEMMA 4.1. *Let Δ be one of the relations* si, asc, $\lhd^{\alpha}$ (*α an infinite ordinal*) *and let $\mathfrak{X}$ be an* I-*closed subclass of* $\mathrm{E}\mathfrak{A}$. *Then*

$$\text{wmin-}\Delta\mathfrak{X} = \text{wmax-}\Delta\mathfrak{X} = \text{Min-}\Delta\mathfrak{X} = \Delta\mathfrak{X}\text{-Fin}.$$

PROOF. It is trivial that

$$\Delta\mathfrak{X}\text{-Fin} \leq \text{Min-}\Delta\mathfrak{X} \leq \text{wmin-}\Delta\mathfrak{X} \quad \text{and} \quad \Delta\mathfrak{X}\text{-Fin} \leq \text{wmax-}\Delta\mathfrak{X}.$$

Therefore we show that $\text{wmin-}\Delta\mathfrak{X} \cup \text{wmax-}\Delta\mathfrak{X} \leq \Delta\mathfrak{X}\text{-Fin}$. Let $L \in \text{wmin-}\Delta\mathfrak{X} \cup \text{wmax-}\Delta\mathfrak{X}$ and let H be any $\Delta\mathfrak{X}$-subalgebra of L. If A is a subideal of H then A is a $\Delta\mathfrak{X}$-subalgebra of L by I-closedness of $\mathfrak{X}$. Hence $H \in (\text{wmin-si} \cup \text{wmax-si}) \cap \mathrm{E}\mathfrak{A}$ and so in view of Proposition 1.3(3) H is finite-dimensional. Thus $L \in \Delta\mathfrak{X}\text{-Fin}$.

From this we deduce the following

THEOREM 4.2. (1) *Let Δ be one of the relations* si, asc, $\lhd^{\alpha}$ (*α an infinite ordinal*). *Then the classes*

$$\begin{array}{lll} \text{wmin-}\Delta\mathfrak{A}, & \text{wmin-}\Delta\mathfrak{N}, & \text{wmin-}\Delta\mathrm{E}\mathfrak{A}, \\ \text{wmax-}\Delta\mathfrak{A}, & \text{wmax-}\Delta\mathfrak{N}, & \text{wmax-}\Delta\mathrm{E}\mathfrak{A}, \\ \text{Min-}\Delta\mathfrak{A}, & \text{Min-}\Delta\mathfrak{N}, & \text{Min-}\Delta\mathrm{E}\mathfrak{A}, \\ \Delta\mathfrak{A}\text{-Fin}, & \Delta\mathfrak{N}\text{-Fin}, & \Delta\mathrm{E}\mathfrak{A}\text{-Fin}, \end{array}$$

coincide with each other. Furthermore, over fields of characteristic zero the classes

$$\text{Max-}\Delta\mathfrak{A}, \quad \text{Max-}\Delta\mathfrak{N}, \quad \text{Max-}\Delta\mathrm{E}\mathfrak{A}$$

coincide with the classes above.

(2) *Let $\mathfrak{X}$ be any class of Lie algebras such that $\mathfrak{A} \leq \mathfrak{X} \leq \mathrm{E}\mathfrak{A}$. Then the classes*

$$\text{wmin-si}\mathfrak{X},\quad \bigcap_{n=1}^{\infty}\text{wmin-}\triangleleft^n\mathfrak{X},\quad \text{Min-si}\mathfrak{X},\quad \bigcap_{n=1}^{\infty}\text{Min-}\triangleleft^n\mathfrak{X},$$

$$\text{wmax-si}\mathfrak{X},\quad \bigcap_{n=1}^{\infty}\text{wmax-}\triangleleft^n\mathfrak{X},\quad \text{si}\mathfrak{A}\text{-Fin},\quad \bigcap_{n=1}^{\infty}\triangleleft^n\mathfrak{A}\text{-Fin},$$

coincide with each other. Furthermore, over fields of characteristic zero the classes

$$\text{Max-si}\mathfrak{X},\quad \bigcap_{n=1}^{\infty}\text{Max-}\triangleleft^n\mathfrak{X}$$

coincide with the classes above.

PROOF. (1) The first half of the assertion comes from Lemma 4.1 and [2, Corollary 9.2.2]. The second half comes from [2, Corollary 9.1.10].

(2) By (1) we have $\text{wmin-si}\mathfrak{X}=\text{wmax-si}\mathfrak{X}=\text{Min-si}\mathfrak{X}=\text{si}\mathfrak{A}\text{-Fin}$. Let $L\in \text{wmin-}\triangleleft^{n+1}\mathfrak{X}\cup\text{wmax-}\triangleleft^{n+1}\mathfrak{X}$ for any $n\geq 1$ and let A be any $\triangleleft^n\mathfrak{A}$-subalgebra of L. As $\mathfrak{A}\leq\mathfrak{X}$ we have $A\in(\text{wmin}\cup\text{wmax})\cap\mathfrak{A}\leq\mathfrak{F}$. Hence $L\in\triangleleft^n\mathfrak{A}\text{-Fin}$. Therefore

$$\text{si}\mathfrak{A}\text{-Fin}\leq\left(\bigcap_{n=1}^{\infty}\text{wmin-}\triangleleft^n\mathfrak{X}\right)\cap\left(\bigcap_{n=1}^{\infty}\text{wmax-}\triangleleft^n\mathfrak{X}\right)\leq\bigcap_{n=1}^{\infty}\triangleleft^n\mathfrak{A}\text{-Fin}.$$

Thus the result follows from [2, Proposition 9.2.3(b) and Corollary 9.1.11].

REMARK. Corresponding to the result [2, Proposition 9.1.12] we have

$$\text{wmin-asc}\mathfrak{X}=\bigcap_{\alpha>0}\text{wmin-}\triangleleft^{\alpha}\mathfrak{X}\quad\text{and}\quad\text{wmax-asc}\mathfrak{X}=\bigcap_{\alpha>0}\text{wmax-}\triangleleft^{\alpha}\mathfrak{X}$$

for any class $\mathfrak{X}$ of Lie algebras.

5.

In the first section we observed that every infinite-dimensional Lie algebra satisfying the weak minimal or the weak maximal condition on subalgebras possesses no infinite-dimensional abelian subalgebras. On the other hand, it is well known that every infinite-dimensional locally nilpotent Lie algebra possesses an infinite-dimensional abelian subalgebra. Concerning these facts above we raise the following question: In what classes containing that of locally nilpotent Lie algebras does an infinite-dimensional Lie algebra possess an infinite-dimensional abelian subalgebra? Recently Kashiwagi [7] showed that every infinite-dimensional Lie algebra belonging to the class of locally supersoluble Lie algebras, which is larger than that of locally nilpotent Lie algebras, possesses an infinite-dimensional abelian subalgebra. Thus in this section we shall show that every infinite-dimensional Lie algebra which belongs to a class $\mathfrak{X}_0$ containing that of locally supersoluble Lie algebras possesses an infinite-dimensional abelian subalgebra, and that $\mathfrak{X}_0$-algebras satisfying the weak minimal or the weak maximal condition on abelian subalgebras are finite-dimensional.

First we shall give several characterizations of Baer algebras, Gruenberg algebras and locally nilpotent Lie algebras. A Lie algebra L is called a Baer (resp. Gruenberg) algebra if $\langle x\rangle$ is a subideal (resp. an ascendant subalgebra) of L for any element x in L. $\mathfrak{B}$ (resp. $\mathfrak{Gr}$) is the class of Baer (resp. Gruenberg) algebras. We need a characterization of Gruenberg algebras, due to Amayo.

LEMMA 5.1 ([1, Theorem 4.6]). $\mathfrak{Gr}=\acute{\mathrm{E}}\mathfrak{A}\cap \mathrm{L}\mathfrak{N}=\acute{\mathrm{E}}\mathfrak{A}\cap\mathfrak{E}$.

REMARK. We can show that $\mathfrak{B}\leq\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$ by following the proof of [1, Theorem 4.6]. Furthermore from [2, Theorem 7.1.5(b)] and Examples 6.2 and 6.7 we have

$$\mathfrak{B}<\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}\cap \mathrm{L}\mathfrak{N}=\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}\cap\mathfrak{E}<\mathfrak{Gr}.$$

Now we state a characterization of locally nilpotent Lie algebras.

LEMMA 5.2. *Let L be a Lie algebra. Then the following conditions are equivalent*:

(1) *L is locally nilpotent.*

(2) *$\langle x\rangle$ is a local subideal of L for any element x in L.*

(3) *$\langle x\rangle$ is a locally ascendant subalgebra of L for any element x in L.*

PROOF. (1)$\Rightarrow$(2): For any element x of L and any finite subset X of L we have $\langle x, X\rangle\in\mathfrak{N}$, as $L\in \mathrm{L}\mathfrak{N}$. So $\langle x\rangle \,\mathrm{si}\, \langle x, X\rangle$. Therefore we have $\langle x\rangle \,\mathrm{lsi}\, L$.

(2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(1): Let X be any finite subset of L. For any element x of $\langle X\rangle$ we see that $\langle x\rangle \,\mathrm{asc}\, \langle x, X\rangle=\langle X\rangle$ since $\langle x\rangle$ is a locally ascendant subalgebra of L. Hence $\langle X\rangle$ is a Gruenberg algebra. From Lemma 5.1 we derive $\langle X\rangle\in \mathrm{L}\mathfrak{N}\cap\mathfrak{G}\leq\mathfrak{N}$. Therefore we have $L\in \mathrm{L}\mathfrak{N}$.

Let Δ and Δ' be any of the relations $\leq$, si, asc, lsi, lasc, etc. and let $\mathfrak{X}$ be a class of Lie algebras. We introduce a new class $\mathrm{F}(\Delta)\mathfrak{X}$ of Lie algebras as follows: A Lie algebra L belongs to $\mathrm{F}(\Delta)\mathfrak{X}$ if for any finite-dimensional subalgebra F of L there exists an $\mathfrak{X}$-subalgebra H of L such that $F\leq H\Delta L$. In particular we write $\mathrm{F}\mathfrak{X}$ for $\mathrm{F}(\leq)\mathfrak{X}$. Clearly $\mathrm{L}(\Delta)\mathfrak{X}\leq\mathrm{F}(\Delta)\mathfrak{X}$. We also write $\Delta\leq\Delta'$ if $H\Delta L$ implies $H\Delta' L$ for any Lie algebra L.

As a first step in this section we have the following proposition which includes [16, Theorem 5.3].

PROPOSITION 5.3. (1) *For any class $\mathfrak{X}$ of Lie algebras such that $\mathfrak{N}\cap\mathfrak{F}\leq\mathfrak{X}\leq\mathfrak{B}$, we have*

$$\mathfrak{E}\cap \mathrm{L}(\mathrm{si})\mathfrak{F}=\mathrm{L}(\mathrm{si})\mathfrak{X}=\mathrm{F}(\mathrm{si})\mathfrak{X}=\mathfrak{B}.$$

(2) *For any class $\mathfrak{X}$ of Lie algebras such that $\mathfrak{N}\cap\mathfrak{F}\leq\mathfrak{X}\leq\mathfrak{Gr}$, we have*

$$\mathfrak{E} \cap \mathrm{L}(\mathrm{asc})\mathfrak{F} = \mathrm{L}(\mathrm{asc})\mathfrak{X} = \mathrm{F}(\mathrm{asc})\mathfrak{X} = \mathfrak{Gr}.$$

(3) *Let Δ and Δ' be any of the relations such that* $\mathrm{lsi} \leq \Delta$ *and* $\mathrm{lsi} \leq \Delta' \leq \mathrm{lasc}$. *For all classes $\mathfrak{X}$ and $\mathfrak{X}'$ of Lie algebras such that $\mathfrak{N} \cap \mathfrak{F} \leq \mathfrak{X} \leq \mathrm{L}\mathfrak{N}$ and $\mathfrak{N} \cap \mathfrak{F} \leq \mathfrak{X}' \leq \mathfrak{Gr}$, we have*

$$\mathfrak{E} \cap \mathrm{L}(\Delta)\mathfrak{F} = \mathrm{L}(\Delta)\mathfrak{X} = \mathrm{F}(\Delta')\mathfrak{X}' = \mathrm{L}\mathfrak{N}.$$

PROOF. Let Δ be any of the relations $\leq$, si, asc, lsi, lasc, etc. By Engel's theorem we easily see that $\mathfrak{E} \cap \mathrm{L}(\Delta)\mathfrak{F} = \mathrm{L}(\Delta)(\mathfrak{N} \cap \mathfrak{F})$.

Suppose that $L \in \mathrm{F}(\mathrm{si})\mathfrak{B}$ (resp. $\mathrm{F}(\mathrm{asc})\mathfrak{Gr}$). Then for any element x of L there exists a subalgebra H of L such that

$$H \in \mathfrak{B} \quad (\text{resp. } \mathfrak{Gr}) \text{ and } \langle x \rangle \leq H \operatorname{si} L \quad (\text{resp. } H \operatorname{asc} L).$$

By the definition of Baer (resp. Gruenberg) algebras we have $\langle x \rangle$ si H (resp. $\langle x \rangle$ asc H). Hence $\langle x \rangle$ si L (resp. $\langle x \rangle$ asc L). So $L \in \mathfrak{B}$ (resp. $\mathfrak{Gr}$). Thus we observe that $\mathrm{F}(\mathrm{si})\mathfrak{B} = \mathfrak{B}$ and $\mathrm{F}(\mathrm{asc})\mathfrak{Gr} = \mathfrak{Gr}$.

(1) Obviously we have

$$\begin{array}{ccccc} \mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F}) & \leq & \mathrm{L}(\mathrm{si})\mathfrak{X} & \leq & \mathrm{L}(\mathrm{si})\mathfrak{B} \\ & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ \mathrm{F}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F}) & \leq & \mathrm{F}(\mathrm{si})\mathfrak{X} & \leq & \mathrm{F}(\mathrm{si})\mathfrak{B} = \mathfrak{B}. \end{array}$$

The result follows from the fact that $\mathfrak{B} = \mathrm{L}(\mathrm{si})(\mathfrak{N} \cap \mathfrak{F})$ by [16, Theorem 5.3(1)].

(2) Similarly we have

$$\begin{array}{ccccc} \mathrm{L}(\mathrm{asc})(\mathfrak{N} \cap \mathfrak{F}) & \leq & \mathrm{L}(\mathrm{asc})\mathfrak{X} & \leq & \mathrm{L}(\mathrm{asc})\mathfrak{Gr} \\ & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ \mathrm{F}(\mathrm{asc})(\mathfrak{N} \cap \mathfrak{F}) & \leq & \mathrm{F}(\mathrm{asc})\mathfrak{X} & \leq & \mathrm{F}(\mathrm{asc})\mathfrak{Gr} = \mathfrak{Gr}. \end{array}$$

It is enough to show that $\mathfrak{Gr} \leq \mathrm{L}(\mathrm{asc})(\mathfrak{N} \cap \mathfrak{F})$. Assume that $L \in \mathfrak{Gr}$ and let X be any finite subset of L. By [1, Corollary 4.7] we have $\langle X \rangle$ asc L. Furthermore by Lemma 5.1 we see that $\langle X \rangle \in \mathfrak{N} \cap \mathfrak{F}$. Therefore $L \in \mathrm{L}(\mathrm{asc})(\mathfrak{N} \cap \mathfrak{F})$.

(3) Assume that $L \in \mathrm{L}\mathfrak{N}$ and let X be a finite subset of L. For any finite subset Y of L we have $\langle X, Y \rangle \in \mathfrak{N}$. Hence $\langle X \rangle$ si $\langle X, Y \rangle$. So $\langle X \rangle$ lsi L and $\langle X \rangle \in \mathfrak{N} \cap \mathfrak{F}$. Therefore $L \in \mathrm{L}(\mathrm{lsi})(\mathfrak{N} \cap \mathfrak{F})$. From this it follows that

$$\mathrm{L}\mathfrak{N} \leq \mathrm{L}(\mathrm{lsi})(\mathfrak{N} \cap \mathfrak{F}) \leq \mathrm{L}(\Delta)\mathfrak{X} \leq \mathrm{L}(\Delta)\mathrm{L}\mathfrak{N} = \mathrm{L}\mathfrak{N},$$

$$\mathrm{L}\mathfrak{N} \leq \mathrm{F}(\mathrm{lsi})(\mathfrak{N} \cap \mathfrak{F}) \leq \mathrm{F}(\Delta')\mathfrak{X}' \leq \mathrm{F}(\mathrm{lasc})\mathfrak{Gr}.$$

Finally we show that $\mathrm{F}(\mathrm{lasc})\mathfrak{Gr} \leq \mathrm{L}\mathfrak{N}$. Assume that $L \in \mathrm{F}(\mathrm{lasc})\mathfrak{Gr}$. For any element x in L there exists a Gruenberg subalgebra F of L such that $\langle x \rangle \leq F$ lasc L. Hence for any finite subset X of L we have F asc $\langle F, X \rangle$. As $\langle x \rangle$ asc F we have $\langle x \rangle$ asc $\langle F, X \rangle$ and so $\langle x \rangle$ asc $\langle x, X \rangle$. Therefore $\langle x \rangle$ lasc L. By using Lemma

5.2 we conclude that $L \in \mathrm{L}\mathfrak{N}$.

This completes the proof of the proposition.

Next we shall search for several classes of Lie algebras which have an infinite-dimensional abelian subalgebra whenever the whole algebras are infinite-dimensional. We recall the two classes $\mathfrak{Q}$ and $\mathfrak{R}$ of Lie algebras:

$L \in \mathfrak{Q}$ if $L \in \mathfrak{F}$ or if L has an infinite-dimensional abelian subalgebra.

$L \in \mathfrak{R}$ if $L \in \mathfrak{F}$ or if there exists a non-zero element x of L with $C_L(x) \notin \mathfrak{F}$.

It is trivial that $\mathfrak{Q} \leq \mathfrak{R}$. Now we have a result which is useful in this section.

LEMMA 5.4. *Let $\mathfrak{X} = \{\mathrm{s}, \mathrm{Q}\}\mathfrak{X}$ be a class of Lie algebras. Then $\mathfrak{X} \leq \mathfrak{Q}$ if and only if $\mathfrak{X} \leq \mathfrak{R}$.*

PROOF. See [2, Lemma 10.1.2].

Let H be a subalgebra of a Lie algebra L. We note that if $H \,\mathrm{wsi}\, \langle H, x\rangle$ for each $x \in L$ then $H \leq^{\omega} L$. Using this and [3, Corollary 2.4] we see that in locally finite Lie algebras every finite-dimensional weakly serial subalgebra is an ω-step weakly ascendant subalgebra. Hence by Proposition 5.3(3) we obtain

$$\mathrm{L}\mathfrak{N} = \mathrm{L}(\leq^{\omega})(\mathfrak{N} \cap \mathfrak{F}) < \mathrm{L}(\leq^{\omega})\mathfrak{F} = \mathrm{L}(\mathrm{wser})\mathfrak{F}.$$

The fact that there exists a locally finite Lie algebra not belonging to $\mathrm{L}(\mathrm{wser})\mathfrak{F}$ is shown by Honda ([4, Example 4.3]).

We can now find a subclass of $\mathfrak{Q}$ which is larger than $\mathrm{L}\mathfrak{N}$.

LEMMA 5.5. *Every infinite-dimensional $\mathrm{L}(\mathrm{wser})\mathfrak{F}$-algebra has an infinite-dimensional abelian subalgebra.*

PROOF. First we note that $\mathrm{L}(\mathrm{wser})\mathfrak{F} = \mathrm{L}(\leq^{\omega})\mathfrak{F}$ is $\{\mathrm{s}, \mathrm{Q}\}$-closed by [15, Lemma 2]. Let $L \in \mathrm{L}(\leq^{\omega})\mathfrak{F}$ and assume that L is infinite-dimensional. Then for any finite subset X of L there exists a finite-dimensional subalgebra F_X of L such that $X \subseteq F_X \leq^{\omega} L$. By [3, Lemma 2.10] we have $F_X^{\omega} \lhd L$. If $F_X^{\omega} = 0$ for any finite subset X of L, then $F_X \in \mathfrak{N}$ and so $L \in \mathrm{L}\mathfrak{N}$. Since $\mathrm{L}\mathfrak{N} \leq \mathfrak{Q} \leq \mathfrak{R}$ by [2, Theorem 10.1.3], we have $L \in \mathfrak{R}$. If $F_X^{\omega} \neq 0$ for some finite subset X of L, then we can pick $0 \neq x \in F_X^{\omega}$. As F_X^{ω} is a finite-dimensional ideal of L we have $L/C_L(F_X^{\omega}) \in \mathfrak{F}$ by [2, Corollary 1.4.3]. It follows that $C_L(F_X^{\omega}) \notin \mathfrak{F}$ and so $C_L(x) \notin \mathfrak{F}$. Hence $L \in \mathfrak{R}$. This implies that $\mathrm{L}(\leq^{\omega})\mathfrak{F} \leq \mathfrak{R}$. Thus we can use Lemma 5.4 to conclude that $\mathrm{L}(\leq^{\omega})\mathfrak{F} \leq \mathfrak{Q}$.

Concerning the class $\mathfrak{Q}$ we have the following

PROPOSITION 5.6. (1) $\mathrm{L}(\mathrm{wser})\mathfrak{Q} = \mathfrak{Q}$.

(2) *Let* $\mathfrak{X}=\{\mathrm{s}, \mathrm{q}\}\mathfrak{X}$ *be a class of Lie algebras. If* $\mathfrak{X}\leq\mathfrak{Q}$ *then* $\acute{\mathrm{E}}\mathfrak{X}\leq\mathfrak{Q}$.

PROOF. (1) Let $L\in \mathrm{L}(\mathrm{wser})\mathfrak{Q}$. Then for any finite subset X of L there exists a subalgebra H_X of L such that

$$X \subseteq H_X \operatorname{wser} L \quad \text{and} \quad H_X\in\mathfrak{Q}.$$

If $H_X\in\mathfrak{F}$ for any finite subset X of L then $L\in \mathrm{L}(\mathrm{wser})\mathfrak{F}$. So by Lemma 5.5 we have $L\in\mathfrak{Q}$. If $H_X\notin\mathfrak{F}$ for some finite subset X of L, then by the definition of $\mathfrak{Q}$ there exists an infinite-dimensional abelian subalgebra of H_X. Therefore $L\in\mathfrak{Q}$.

(2) Let $L\in\acute{\mathrm{E}}\mathfrak{X}$ and suppose that L is infinite-dimensional. Then there exists a strictly ascending series $\{L_\alpha\}_{\alpha\leq\sigma}$ of L such that

$$0 = L_0 \lhd L_1 \lhd\cdots L_\alpha \lhd L_{\alpha+1} \lhd\cdots L_\sigma = L \text{ and } L_{\alpha+1}/L_\alpha\in\mathfrak{X} \text{ for any } \alpha<\sigma.$$

By transfinite induction on σ we shall show that $L_\sigma\in\mathfrak{Q}$. It is trivial for $\sigma\leq 1$. Let $\sigma>1$ and assume that the result holds for all ordinals $\alpha<\sigma$. Now there exists the minimal ordinal $\beta\leq\sigma$ with respect to $L_\beta\notin\mathfrak{F}$. If $\beta<\sigma$ then by the inductive hypothesis L_β has an infinite-dimensional abelian subalgebra, and so $L_\sigma\in\mathfrak{Q}$. Hence we suppose that $\beta=\sigma$. If σ is a limit ordinal then

$$L=\bigcup_{\alpha<\sigma} L_\alpha, \qquad L_\alpha\in\mathfrak{F} \quad\text{and}\quad L_\alpha \operatorname{asc} L \quad \text{for any } \alpha<\sigma.$$

This means that $L\in \mathrm{L}(\mathrm{asc})\mathfrak{F}$, and so $L\in\mathfrak{Q}$ by Lemma 5.5. If σ is a non-limit ordinal then we have

$$0 \neq L_{\sigma-1} \lhd L \quad\text{and}\quad L_{\sigma-1}\in\mathfrak{F}.$$

Hence $L/C_L(L_{\sigma-1})\in\mathfrak{F}$ and so $C_L(L_{\sigma-1})\notin\mathfrak{F}$. Since there is a non-zero element x of $L_{\sigma-1}$ with $C_L(x)\notin\mathfrak{F}$ we see that $L\in\mathfrak{R}$. Therefore we conclude that $\acute{\mathrm{E}}_\sigma\mathfrak{X}\leq\mathfrak{R}$. Since $\acute{\mathrm{E}}_\sigma\mathfrak{X}$ is $\{\mathrm{s}, \mathrm{q}\}$-closed it follows from Lemma 5.4 that $\acute{\mathrm{E}}_\sigma\mathfrak{X}\leq\mathfrak{Q}$. Thus our induction has been completed.

Over fields of characteristic zero from [2, Theorems 10.1.1 and 10.2.1] we see that $\mathrm{L}(\mathrm{E}\mathfrak{A}\cup\mathfrak{F})\leq\mathfrak{Q}$. So as a consequence of Proposition 5.6 we have

COROLLARY 5.7. (1) *Every infinite-dimensional* $\acute{\mathrm{E}}\mathrm{L}(\mathrm{wasc})(\mathrm{E}\mathfrak{A}\cup\mathfrak{F})$*-algebra has an infinite-dimensional abelian subalgebra.*

(2) *Every infinite-dimensional* $\acute{\mathrm{E}}\mathrm{L}(\mathrm{E}\mathfrak{A}\cup\mathfrak{F})$*-algebra over a field of characteristic zero has an infinite-dimensional abelian subalgebra.*

REMARK. By [7, Lemma 5.4] and Example 6.1 we have $\mathrm{LE}(\lhd)\mathfrak{F}_1<(\mathrm{L}\mathfrak{N})\mathfrak{A}$. Since $(\mathrm{L}\mathfrak{N})\mathfrak{A}\leq\mathrm{EL}\mathfrak{N}\leq\acute{\mathrm{E}}\mathrm{L}(\mathrm{wasc})\mathfrak{F}$ we obtain $\mathrm{LE}(\lhd)\mathfrak{F}_1<\acute{\mathrm{E}}\mathrm{L}(\mathrm{wasc})(\mathrm{E}\mathfrak{A}\cup\mathfrak{F})$.

Now we shall give several conditions under which Lie algebras satisfying the weak minimal or the weak maximal conditions on various subalgebras are finite-dimensional.

THEOREM 5.8. *The following Lie algebras are finite-dimensional*:

(1) *nilpotent algebras satisfying the weak minimal or the weak maximal condition on abelian ideals,*

(2) *supersoluble algebras satisfying the weak minimal or the weak maximal condition on abelian* 2-*step subideals,*

(3) *hyperabelian algebras satisfying the weak minimal or the weak maximal condition on abelian* 3-*step subideals,*

(4) $\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$-*algebras satisfying the weak minimal or the weak maximal condition on abelian subideals,*

(5) $\acute{\mathrm{E}}\mathfrak{A}$-*algebras satisfying the weak minimal or the weak maximal condition on abelian ascendant subalgebras,*

(6) $\acute{\mathrm{E}}\mathrm{L}(\mathrm{wasc})(\mathrm{E}\mathfrak{A}\cup\mathfrak{F})$-*algebras satisfying the weak minimal or the weak maximal condition on abelian subalgebras.*

PROOF. (1) Let $L\in\mathfrak{N}\cap(\text{wmin-}\lhd\mathfrak{A}\cup\text{wmax-}\lhd\mathfrak{A})$ and let A be a maximal abelian ideal of L. Then $A=A\cap\zeta_n(L)$ for some $n\in N$. Let $X/(A\cap\zeta_i(L))\leq(A\cap\zeta_{i+1}(L))/(A\cap\zeta_i(L))$. Then X is an abelian ideal of L, since $[X, L]\subseteq A\cap\zeta_i(L)\subseteq X$. Thus $(A\cap\zeta_{i+1}(L))/(A\cap\zeta_i(L))\in(\text{wmin}\cup\text{wmax})\cap\mathfrak{A}\leq\mathfrak{F}$, whence $A=A\cap\zeta_n(L)\in\mathfrak{F}$. Therefore $L/C_L(A)\in\mathfrak{F}$ and since $C_L(A)=A$ by [2, Lemma 9.1.2(a)], we have $L\in\mathfrak{F}$.

(2) Let $L\in\acute{\mathrm{E}}(\lhd)\mathfrak{F}_1\cap(\text{wmin-}\lhd^2\mathfrak{A}\cup\text{wmax-}\lhd^2\mathfrak{A})$. If $L\notin\mathfrak{F}$ then by [7, Corollary 5.3] there exists an infinite-dimensional abelian ideal A of L. On the other hand we have $A\in(\text{wmin}\cup\text{wmax})\cap\mathfrak{A}\leq\mathfrak{F}$, a contradiction. Therefore we have $L\in\mathfrak{F}$.

(3)–(6) can be shown by using [2, Theorem 10.1.1] and Corollary 5.7(1) as in the proof of (2).

As a special case of Theorem 5.8 we have the following

COROLLARY 5.9. *The following Lie algebras are finite-dimensional*:

(1) *hypercentral algebras satisfying the weak minimal or the weak maximal condition on* 2-*step subideals,*

(2) *ideally soluble algebras satisfying the weak minimal or the weak maximal condition on* 3-*step subideals,*

(3) *Baer algebras satisfying the weak minimal or the weak maximal condition on subideals,*

(4) *Gruenberg algebras satisfying the weak minimal or the weak maximal condition on ascendant subalgebras,*

(5) *locally nilpotent algebras satisfying the weak minimal or the weak maximal condition on subalgebras.*

6.

In this section we shall observe several examples concerning the weak minimal conditions and the weak maximal conditions on various subalgebras.

EXAMPLE 6.1. Let X be a vector space with basis $\{x_0, x_1, x_2, \dots\}$ and let σ be the upward shift on X, that is, $x_i\sigma = x_{i+1}$ for all $i \geq 0$. Think of X as an abelian Lie algebra and form the split extension $L = X \dotplus \langle\sigma\rangle$. Then it is well known (e.g. see [9, Theorem 3]) that every non-zero ideal of L is of finite co-dimension. Hence $L \in \text{Max-}\lhd \cap \text{wmin-}\lhd$. Put $I_n = \langle x_n, x_{n+1}, \dots\rangle$ for all $n \geq 1$. Then I_n is an abelian ideal of L. Now since $I_1 > I_2 > \cdots$, $L/I_n \in \mathfrak{F}$ and $\bigcap_{n=1}^{\infty} I_n = 0$, we see that $L \in \mathrm{R}\mathfrak{F}$ but $L \notin \text{Min-}\lhd\mathfrak{A}$. If $L \in \text{wmin-}\lhd^2\mathfrak{A}$, then $X \in \mathfrak{A} \cap \text{wmin} \leq \mathfrak{F}$, and so $L \notin \text{wmin-}\lhd^2\mathfrak{A}$. By the way, as $L = \langle x_0, \sigma\rangle$ we have $L \notin \mathrm{L}\mathfrak{F}$. So $L \notin \mathrm{LE}(\lhd)\mathfrak{F}_1$ from the fact that $\mathrm{LE}(\lhd)\mathfrak{F}_1 \leq \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ ([7, Proposition 3.1]). Thus we have

$$\text{Min-}\lhd < \text{wmin-}\lhd, \quad \text{Min-}\lhd\mathfrak{A} < \text{wmin-}\lhd\mathfrak{A},$$

$$\text{wmin-}\lhd^2 < \text{wmin-}\lhd, \quad \text{wmin-}\lhd^2\mathfrak{A} < \text{wmin-}\lhd\mathfrak{A},$$

$$\mathrm{E}\mathfrak{A} \cap \mathrm{R}\mathfrak{F} \cap \text{Max-}\lhd \cap \text{wmin-}\lhd \nleq \mathfrak{F}, \quad \mathrm{LE}(\lhd)\mathfrak{F}_1 < (\mathrm{L}\mathfrak{N})\mathfrak{A}.$$

EXAMPLE 6.2. Let X be a vector space with basis $\{x_0, x_1, x_2, \dots\}$ and let σ be the downward shift on X, that is, $x_0\sigma = 0$ and $x_i\sigma = x_{i-1}$ for all $i > 0$. Think of X as an abelian Lie algebra and form the split extension $L = X \dotplus \langle\sigma\rangle$. It is evident that the proper ideals of L are X and $\zeta_n(L) = \langle x_0, \dots, x_{n-1}\rangle$ $(n = 1, 2, \dots)$. Therefore $L \in \text{Min-}\lhd \cap \text{wmax-}\lhd$ but $L \notin \text{Max-}\lhd\mathfrak{A}$. If $L \in \text{wmax-}\lhd^2\mathfrak{A}$ then $X \in \mathfrak{A} \cap \text{wmax} \leq \mathfrak{F}$, and so $L \notin \text{wmax-}\lhd^2\mathfrak{A}$. Thus we have

$$\text{Max-}\lhd < \text{wmax-}\lhd, \quad \text{Max-}\lhd\mathfrak{A} < \text{wmax-}\lhd\mathfrak{A},$$

$$\text{wmax-}\lhd^2 < \text{wmax-}\lhd, \quad \text{wmax-}\lhd^2\mathfrak{A} < \text{wmax-}\lhd\mathfrak{A}.$$

On the other hand, L is hypercentral and $\langle\sigma\rangle$ is not a subideal of L ([2, p. 119]), and hence

$$L \in \mathfrak{Z} \leq \acute{\mathrm{E}}(\lhd)\mathfrak{A} \cap \mathrm{L}\mathfrak{N} \leq \acute{\mathrm{E}}(\mathrm{si})\mathfrak{A} \cap \mathrm{L}\mathfrak{N} \quad \text{but} \quad L \notin \mathfrak{B}.$$

Thus $\mathfrak{B} < \acute{\mathrm{E}}(\mathrm{si})\mathfrak{A} \cap \mathrm{L}\mathfrak{N}$.

EXAMPLE 6.3. Let c be an infinite cardinal with successor c^+ and let c_i $(i = 1, 2, \dots)$ be infinite cardinals such that

$$c_1 < c_2 < \cdots \quad \text{and} \quad c_i < c \qquad \text{for all} \quad i \geq 1.$$

Select a vector space V of dimension c. For any infinite cardinal $d \leq c^+$, $L(c, d)$

is defined to be the Lie algebra of all linear transformations α: $V \to V$ such that the image of α has dimension $< d$. From [12, Theorem A] it follows that $L = L(c, c^+) \in \text{Min-}\lhd$, $I_i = L(c, c_i) \lhd L$ $(i = 1, 2,\dots)$, $I_1 < I_2 < \cdots$ and $\dim I_{i+1}/I_i = \infty$ $(i = 1, 2,\dots)$. Therefore we have $L \not\in \text{wmax-}\lhd$.

EXAMPLE 6.4. Let X be a vector space with basis $\{x_{ij}: i, j \in N\}$ and think of X as an abelian Lie algebra. We define two derivations f and g of X as follows:

$$x_{ij}f = x_{i,j+1} \quad \text{and} \quad x_{ij}g = x_{i+1,j} \qquad \text{for all} \quad i, j \geq 0.$$

It is easy to see that $[f, g] = 0$. We form the split extension $L = X \dotplus \langle f, g \rangle$. Then $L = \langle x_{00}, f, g \rangle \in \mathfrak{A}^2 \cap \mathfrak{G}$. From [2, Corollary 11.1.8] we deduce that $L \in \text{Max-}\lhd$. Put $I_n = \langle x_{ij}: i \geq n, j \in N \rangle$ for each $n \geq 1$. Then I_n is an ideal of L. Furthermore we have

$$I_1 > I_2 > \cdots \quad \text{and} \quad \dim I_n/I_{n+1} = \infty \qquad \text{for all} \quad n \geq 1,$$

which implies that $L \not\in \text{wmin-}\lhd$.

THEOREM 6.5. (1) *There exists a Lie algebra satisfying the weak minimal and the weak maximal conditions on ideals but neither the minimal nor the maximal condition on ideals.*

(2) *There exists a Lie algebra satisfying the weak minimal condition on ideals but neither the weak maximal nor the minimal condition on ideals.*

(3) *There exists a Lie algebra satisfying the weak maximal condition on ideals but neither the weak minimal nor the maximal condition on ideals.*

PROOF. Let L_i be the Lie algebra in Example 6.i for $i = 1, 2, 3, 4$.

(1) Set $L = L_1 \oplus L_2$. Then

$$L/L_2 \cong L_1 \in \text{wmin-}\lhd \cap \text{Max-}\lhd \quad \text{and} \quad L_2 \in \text{Min-}\lhd \cap \text{wmax-}\lhd.$$

By using Lemma 1.1 we have $L \in \text{wmin-}\lhd \cap \text{wmax-}\lhd$. But

$$L/L_2 \cong L_1 \not\in \text{Min-}\lhd \quad \text{and} \quad L/L_1 \cong L_2 \not\in \text{Max-}\lhd.$$

Therefore we have $L \not\in \text{Min-}\lhd \cup \text{Max-}\lhd$.

(2) Set $L = L_1 \oplus L_3$.

(3) Set $L = L_2 \oplus L_4$.

EXAMPLE 6.6. Let W be the Witt algebra, that is, W be a Lie algebra over a field of characteristic zero with basis $\{w_1, w_2,\dots\}$ and multiplication $[w_i, w_j] = (i-j)w_{i+j}$. From $W \in \text{Max}$ ([8, Theorem]) we can deduce that every ascendant subalgebra of W is a subideal of W. Since every subideal of W is of finite codimension ([2, Theorem 8.7.1]) we see that $W \in \text{wmin-asc}$. However, we notice that $W \not\in$ wmin. In fact, let H_i be the subspace of W spanned by all $w_{2^i k}$ with $k \geq 1$ ($i=1, 2,\dots$). Then clearly H_i is a subalgebra of W, and we see that

$$H_1 > H_2 > \cdots \quad \text{and} \quad \dim H_i/H_{i+1} = \infty \qquad \text{for all} \quad i \geq 1.$$

Furthermore we have $W^{(1)} > W^{(2)} > \cdots$, and so W does not satisfy the minimal condition on non-abelian ideals. Therefore using [2, Theorem 8.1.4] and [14, Theorem] we see that over fields of characteristic zero

$$\text{wmin} < \text{wmin-asc}, \quad \text{Min-asc} = \text{Min-si} = \text{Min-}\lhd^2 < \text{wmin-asc},$$

$$\text{wmin-}\lhd^2 \not\leq \text{Min-}\lhd, \quad \text{wmin-}\lhd^2\mathfrak{A} \not\leq \text{Min-}\lhd\mathfrak{A},$$

where wmin-$\lhd^2\mathfrak{A}$ (resp. Min-$\lhd\mathfrak{A}$) is the class of Lie algebras satisfying the weak minimal (resp. minimal) condition on non-abelian 2-step subideals (resp. non-abelian ideals).

Example 6.7. Let $U = U(L)$ be the universal enveloping algebra of a Lie algebra L and let $V = V(L)$ be the (associative) ideal in U which is generated by L. Suppose that L is nilpotent of class $n-1$ ($n \geq 2$). By means of the lower central series we develop a totally ordered basis $\{u_i : i \in I\}$ in L. The weight $s = s(u_i)$ of an element u_i is the positive integer s with $u_i \in L^s \backslash L^{s+1}$. The weight of a standard monomial $\prod_{i \in I} u_i^{m_i}$ is $\sum_{i \in I} m_i s_i$, where s_i is the weight of u_i. The weight of an arbitrary element in V is the minimum of the weights of the standard monomials which occur in the linear combination of the element. The set of elements of V with weight $> n$ forms an ideal in V, which we designate $S = S(L)$. Then $N = N(L) = V/S$ is an associative nilpotent algebra of class n and can be considered as a right L-module in the usual way. Thus we form the split extension $E = E(L) = N \dotplus L$. Since

$$E^i = (\cdots((N \cdot \underbrace{L) \cdot L) \cdots) \cdot L}_{i-1 \text{ times}} \dotplus L^i \qquad \text{for} \quad 1 \leq i \leq n+1,$$

we see that E is nilpotent of class n.

Let L_1 be a one-dimensional Lie algebra. By defining $L_2 = E(L_1)$, $L_3 = E(L_2)$, and so on, we obtain an ascending chain of nilpotent Lie algebras

$$L_1 < L_2 < \cdots < L_n < L_{n+1} < \cdots.$$

Set $L = \bigcup_{n=1}^{\infty} L_n$ (the direct limit of $\{L_n\}$). Then by [11, Theorem 4] we know that $L \in \mathrm{L}\mathfrak{N} \cap \acute{\mathrm{E}}\mathfrak{A}$ and that L has no non-zero bounded left Engel elements. Assume that $L \in \acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$. Then there exists a non-zero abelian subideal A of L. If we take a non-zero element z in A, then $\langle z \rangle$ is a subideal of L. Hence z is a bounded left Engel element, a contradiction. Thus we have $L \not\in \acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$. From this we deduce that $\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A} \cap \mathrm{L}\mathfrak{N} < \mathfrak{Gr}$.

We put $N_0=L_1$ and $N_i=N(L_i)$ for all $i\geq 1$. Then

$$L_{i+1} = N_i \dotplus (N_{i-1} \dotplus (\cdots \dotplus N_0)\cdots).$$

So $L=\sum_{n=0}^{\infty} N_n$. It is easy to see that $L^i=\bigcup_{n=0}^{\infty} L_{n+1}^i$. Since

$$L_{n+1}^i = N_n \cdot \underbrace{L_n \cdot \cdots \cdot L_n}_{i-1} \dotplus L_n^i ,$$

we have

$$L^i = \textstyle\sum_{n=0}^{\infty} (N_n \cdot \underbrace{L_n \cdot \cdots \cdot L_n}_{i-1}) \qquad \text{for any} \quad i \geq 1.$$

Therefore $L^i/L^{i+1} \notin \mathfrak{F}$ and so $L \notin \mathrm{wmin}\text{-}\lhd$. From the paragraph above L has no non-zero soluble ideals. Therefore $L \in \mathrm{Min}\text{-}\lhd \mathrm{E}\mathfrak{A} \cap \mathrm{Max}\text{-}\lhd \mathrm{E}\mathfrak{A}$. So we have $\mathrm{wmin}\text{-}\lhd < \mathrm{wmin}\text{-}\lhd \mathrm{E}\mathfrak{A}$.

REMARKS. (1) The two Lie algebras constructed in [5, Theorems 1 and 2] show that

$$\mathrm{wmin}\text{-}\lhd \mathrm{E}\mathfrak{A} < \mathrm{wmin}\text{-}\lhd \mathfrak{N} \quad \text{and} \quad \mathrm{wmax}\text{-}\lhd \mathrm{E}\mathfrak{A} < \mathrm{wmax}\text{-}\lhd \mathfrak{N} < \mathrm{wmax}\text{-}\lhd \mathfrak{A}.$$

(2) The Lie algebra constructed in [10, Theorem] shows that over the rational number field

$$\mathrm{wmin}\text{-}\lhd \mathfrak{N} < \mathrm{wmin}\text{-}\lhd \mathfrak{A}.$$

(3) The Lie algebra constructed in [2, pp. 167–170] shows that over a field of characteristic $p>0$

$$\mathrm{wmin}\text{-}\lhd^3 < \mathrm{wmin}\text{-}\lhd^2, \quad \mathrm{wmin}\text{-}\lhd^3 \mathfrak{A} < \mathrm{wmin}\text{-}\lhd^2 \mathfrak{A},$$

$$\mathrm{wmax}\text{-}\lhd^3 < \mathrm{wmax}\text{-}\lhd^2, \quad \mathrm{wmax}\text{-}\lhd^3 \mathfrak{A} < \mathrm{wmax}\text{-}\lhd^2 \mathfrak{A}.$$

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