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On self H-equivalences of an H-space with respect to any multiplication

To the memory of Shichirô Oka

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Introduction

Let X be an H-space. Then a homotopy equivalence $h: X \to X$ is called a self H-equivalence of X with respect to a multiplication $m: X \times X \to X$ if $hm \sim m(h \times h): X \times X \to X$ (homotopic); and all the homotopy classes of such self H-equivalences form the group

$$\mathrm{HE}(X, m) \text{ (the notation } \mathscr{E}_H(X, m) \text{ is used in the recent papers)}$$

under the composition. In general, X has several multiplications and this group depends on m. For example, the complex conjugate $C: SU(n) \to SU(n)$ of the special unitary group is an H-map with respect to the usual multiplication, but not so to some one on $SU(n)$ for $n \geqq 3$, as is proved by Maruyama-Oka [9].

In this note, we consider the group

$$\mathrm{HE}(X) = \cap_m \mathrm{HE}(X, m) \text{ (} m \text{ ranges over all multiplications on } X\text{)}$$

formed by all self H-equivalences of X with respect to any multiplication, and study its basic properties. The main result is stated as follows:

THEOREM. *Let X be the unitary group $U(n)$ $(n \geqq 3)$, the special unitary group $SU(n)$ $(n \geqq 1)$ or the symplectic group $Sp(n)$ $(n \geqq 1)$. Then, any self H-equivalence $h \in \mathrm{HE}(X)$ with respect to any multiplication induces the identity map $h_* = \mathrm{id}$ on $\pi_*(X) \otimes Z_{(p)}$ for a large prime p; and* $\mathrm{HE}(X)$ *is a finite nilpotent group.*

We prove the basic equality on $\mathrm{HE}(X)$ in Proposition 1.4, and study it in case that X is a product H-space in Theorem 2.4. Furthermore, by using the fact that the localization $X_{(p)}$ of $X = SU(n)$ or $Sp(n)$ at a large prime p is homotopy equivalent to the product space of the localizations of some odd spheres, we study $\mathrm{HE}(X_{(p)})$ in Corollary 3.4; and the main result is proved in Theorem 4.1 and Corollary 4.2 by a similar method to that used in [9].

§1. Basic equality on HE(X)

Throughout this note, we assume that all spaces, maps and homotopies are based and spaces have homotopy types of CW-complexes. A map $f: X \to Y$ and its homotopy class f in the homotopy set $[X, Y]$ are always denoted by a same letter.

When $X = (X, m)$ is an *H-space*, i.e., X admits a *multiplication* $m: X \times X \to X$ such that $m|X \vee X = \nabla$ (the folding map) in $[X \vee X, X]$, we consider the set

(1.1.1) $\mathrm{M}(X)(\subset [X \times X, X])$ of all homotopy classes of multiplications on X.

Then, using the sum $+$ on $[\ , X]$ induced by m, we have easily a bijection

(1.1.2) $[X \wedge X, X] \cong \mathrm{M}(X)$ by sending

$$\alpha \in [X \wedge X, X] \text{ to } m_\alpha = m + \alpha\pi \in \mathrm{M}(X),$$

where $\pi: X \times X \to X \times X / X \vee X = X \wedge X$ is the collapsing map (cf., e.g., [11; Th. 2.3]).

When $Y = (Y, m')$ is also an H-space, $f: (X, m) \to (Y, m')$ is an *H-map* if $fm = m'(f \times f)$ in $[X \times X, Y]$, and such H-maps form the subset

(1.1.3) $$[X, m; Y, m']_{\mathrm{H}} \subset [X, Y] \qquad (m \in \mathrm{M}(X),\ m' \in \mathrm{M}(Y)).$$

By taking their intersection, we have also the subsets

(1.1.4) $[X, Y]_{\mathrm{H}} = \bigcap_{m \in \mathrm{M}(X), m' \in \mathrm{M}(Y)} [X, m; Y, m']_{\mathrm{H}}$ of $[X, Y]$, and

$$\mathrm{HMap}(X) = \bigcap_{m \in \mathrm{M}(X)} [X, m; X, m]_{\mathrm{H}} \supset [X, X]_{\mathrm{H}} \text{ of } [X, X].$$

LEMMA 1.2. (i) $[X, Y]_{\mathrm{H}} = [X, m; Y, m']_{\mathrm{H}} \cap \mathrm{O}[X, Y]$ *for any* $m \in \mathrm{M}(X)$ *and* $m' \in \mathrm{M}(Y)$, *where* $\mathrm{O}[X, Y]$ *consists of all* $f \in [X, Y]$ *satisfying*

(1.2.1) $$f_* = 0: [X \wedge X, X] \longrightarrow [X \wedge X, Y] \text{ and}$$
$$(f \wedge f)^* = 0: [Y \wedge Y, Y] \longrightarrow [X \wedge X, Y].$$

(ii) $\mathrm{HMap}(X) = [X, m; X, m]_{\mathrm{H}} \cap \mathrm{I}(X)$ *for any* $m \in \mathrm{M}(X)$, *where*

(1.2.2) $\mathrm{I}(X) = \{f \in [X, X] \mid f_* = (f \wedge f)^*: [X \wedge X, X] \longrightarrow [X \wedge X, X]\}$.

PROOF. Take $f \in [X, m; Y, m']_{\mathrm{H}}$. Then, for $m_\alpha = m + \alpha\pi \in \mathrm{M}(X) (\alpha \in [X \wedge X, X])$ and $m'_\beta = m' +' \beta\pi \in \mathrm{M}(Y) (\beta \in [Y \wedge Y, Y])$ in (1.1.2), the equality $fm = m'(f \times f)$ implies the ones

$$fm_\alpha = f(m + \alpha\pi) = fm +' f\alpha\pi,$$
$$m'_\beta(f \times f) = m'(f \times f) +' \beta\pi(f \times f) = fm +' \beta(f \wedge f)\pi$$

in $[X \times X, Y]$; and $f \in [X, m_\alpha; Y, m'_\beta]_{\mathrm{H}}$ means that these are equal to each other. Therefore, [6; Th. 1.1] and the injectivity of $\pi^*: [X \wedge X, Y] \to [X \times X, Y]$ imply that

(1.2.3) $\quad f \in [X, m_\alpha; Y, m'_\beta]_{\mathrm{H}}$ *if and only if* $f\alpha \doteq \beta(f \wedge f)$ *in* $[X \wedge X, Y]$.

This shows the lemma by definition. q.e.d.

Now, for an H-space X, consider the group

(1.3.1) $\quad \mathrm{E}(X) = \{h \mid h: X \to X \text{ is a homotopy equivalence}\} \; (\subset [X, X])$,

with group-multiplication given by the composition, and its subgroups

(1.3.2) $\quad \mathrm{HE}(X, m) = \mathrm{E}(X) \cap [X, m; X, m]_{\mathrm{H}}$ for each $m \in \mathrm{M}(X)$, and

(1.3.3) $\quad \mathrm{HE}(X) = \bigcap_{m \in \mathrm{M}(X)} \mathrm{HE}(X, m) = \mathrm{E}(X) \cap \mathrm{HMap}(X)$.

Furthermore, consider the action of $\mathrm{E}(X)$ on $[X \wedge X, X]$ given by

(1.3.4) $\quad h * \alpha = h^{-1}\alpha(h \wedge h) \in [X \wedge X, X]$ for $h \in \mathrm{E}(X)$ and $\alpha \in [X \wedge X, X]$.

Then, we have the isotropy subgroup and their intersection

(1.3.5) $\quad \mathrm{E}(X)_\alpha = \{h \in \mathrm{E}(X) \mid h * \alpha = \alpha\}$ at $\alpha \in [X \wedge X, X]$ and

$\mathrm{IE}(X) = \bigcap_{\alpha \in [X \wedge X, X]} \mathrm{E}(X)_\alpha = \mathrm{E}(X) \cap \mathrm{I}(X)$ (see (1.2.2)),

where $\mathrm{IE}(X)$ is a normal subgroup of $\mathrm{E}(X)$.

The following equalities play a basic role in our study.

Proposition 1.4. *For any H-space X, $\mathrm{HE}(X)$ is a normal subgroup of $\mathrm{E}(X)$; and for each multiplications m and $m_\alpha \in \mathrm{M}(X)$ ($\alpha \in [X \wedge X, X]$, see (1.1.2)), we have*

(1.4.1) $\quad \mathrm{HE}(X, m) \cap \mathrm{HE}(X, m_\alpha) = \mathrm{HE}(X, m) \cap \mathrm{E}(X)_\alpha$,

(1.4.2) $\quad \mathrm{HE}(X) = \mathrm{HE}(X, m) \cap \mathrm{IE}(X)$.

Proof. If $h \in \mathrm{E}(X)$, then $m' = h^{-1}m(h \times h) \in \mathrm{M}(X)$ and $h^{-1}\,\mathrm{HE}(X, m)h = \mathrm{HE}(X, m')$. Thus, we see the first half. (1.2.3) for $Y = X$, $m' = m$ and $\beta = \alpha$ means (1.4.1), and (1.4.2) follows from (1.4.1) and (1.3.5). q.e.d.

Example 1.5 ([12; Th. 4.1]). *If X is S^n ($n = 3, 7$) or the Eilenberg-MacLane space $K(\pi, n)$ for an abelian group π, then $\mathrm{HE}(X) = \mathrm{HE}(X, m)$ for any $m \in \mathrm{M}(X)$ and*

$$\mathrm{HE}(S^n) = 1, \quad \mathrm{HE}(K(\pi, n)) = \mathrm{aut}\,\pi.$$

Now, let p be a prime $\geqq 3$ and consider

(1.6.1) the localization $S = S^n_{(p)}$ of the n-sphere S^n $(n \geqq 1)$ at p,

(1.6.2) the subring $Z_{(p)} = \{s/t \mid s, t \in Z, t > 0, (t, p) = 1\}$

of the rational field Q, and

(1.6.3) the multiplicative group $Z^*_{(p)}$ consisting of all units in $Z_{(p)}$.

Then, we can identify as follows (cf. D. Sullivan [14; 4.9, Cor.1]):

(1.6.4) $\pi_n(S) = Z_{(p)}$, $[S, S] = \mathrm{Hom}(\pi_n(S), \pi_n(S)) = Z_{(p)}$ *as rings, and*

$$\mathrm{E}(S) = Z^*_{(p)}.$$

Furthermore, J. F. Adams [1] proved the following

(1.6.5) $S = S^n_{(p)}$ *(n: odd) is an H-space with a homotopy commutative multiplication m.*

In this case, for any s/t in $Z_{(p)} = [S, S]$, s and t are H-maps in $[S, m; S, m]_{\mathrm{H}}$, and so is s/t since $(t, p) = 1$. Thus, we see the following

(1.6.6) *In case of* (1.6.5), $[S, m; S, m]_{\mathrm{H}} = [S, S] = Z_{(p)}$ *and*

$$\mathrm{HE}(S, m) = \mathrm{E}(S) = Z^*_{(p)}.$$

Also, we denote the p-component of $\pi_i(X)$ by $\pi_i(X; p)$, and consider the subgroup

(1.6.7) $U_{p^r} = 1 + p^r Z_{(p)}$ when $r \geqq 1$ or $U_1 = Z^*_{(p)}$ when $r = 0$

of $Z^*_{(p)}$ in (1.6.3).

PROPOSITION 1.7. *For a prime $p \geqq 3$ and an odd integer $n \geqq 1$, let p^r be the largest order of elements in $\pi_{2n}(S^n; p)$. Then,* $\mathrm{HE}(S) = U_{p^r}$ *for the H-space $S = S^n_{(p)}$ in* (1.6.5).

PROOF. Let $S' = S^{n'}_{(p)}$ $(n' \geqq n)$. Then, we can identify as follows:

(1.7.1) $[S', S] = \pi_{n'}(S^n) \otimes Z_{(p)} = \pi_{n'}(S^n; p)$ $(n' > n)$, $= Z_{(p)}$ $(n' = n)$.

Here the group structure is given by the suspended space S' of $S^{n'-1}_{(p)}$, and is also induced from $m \in \mathrm{M}(S)$, and we see that $t\alpha s = s\alpha t$ $(s, t \in Z)$ and so

(1.7.2) $\alpha q = q\alpha = q \cdot \alpha$

for any $q = s/t \in \mathrm{E}(S') = \mathrm{E}(S) = Z^*_{(p)}$ *and* $\alpha \in [S', S]$.

By (1.3.5) and (1.7.2), $q \in \mathrm{E}(S) = Z^*_{(p)}$ is in $\mathrm{IE}(S)$ if and only if

$$\alpha = q^{-1}\alpha(q \wedge q) = q \cdot \alpha \quad \text{for any } \alpha \in [S \wedge S, S] = \pi_{2n}(S^n; p),$$

which is equivalent to $q \in U_{p^r}$ by the definition of p^r and U_{p^r}. Thus, $\mathrm{HE}(S) = U_{p^r}$ by Proposition 1.4 and (1.6.6). q.e.d.

§2. Product H-spaces

In this section, we consider

(2.1.1) H-spaces (X_k, m_k) and their product H-space $X = \prod_{k=1}^n X_k$ with

$$m_X = (\prod m_k)T\colon X \times X \approx \prod (X_k \times X_k) \to X \text{ as multiplication}$$

(T: the permuting homeomorphism).

Also, for any H-space (Y, m), we consider the n-fold product H-space

(2.1.2) $(Y^n, m^n) = (\prod Y_k, (\prod m_k)T)$ with $(Y_k, m_k) = (Y, m)$ for $1 \leqq k \leqq n$,

the iterated multiplication

(2.1.3) $\bar{m}\colon Y^n \to Y$, given inductively by $\bar{m} = m$ when $n = 2$ and

$$\bar{m} = m(\bar{m} \times 1),$$

the obstruction $h(\bar{m})$ for $\bar{m}$ to be an H-map $(Y^n, m^n) \to (Y, m)$, i.e.,

(2.1.4) $h(\bar{m}) \in [Y^n \wedge Y^n, Y]$ with $m(\bar{m} \times \bar{m}) = \bar{m}m^n + h(\bar{m})\pi$ in $[Y^n \times Y^n, Y]$,

and the one $c(m)$ or $a(m)$ for m to be homotopy commutative or homotopy associative, i.e.,

(2.1.5) $c(m) \in [Y \wedge Y, Y]$ with

$$mT = m + c(m)\pi (= m_{c(m)} \text{ in } (1.1.2)) \text{ in } [Y \times Y, Y],$$

(2.1.6) $a(m) \in [Y \wedge Y \wedge Y, Y]$ with

$$m(m \times 1) = m(1 \times m) + a(m)\pi \text{ in } [Y \times Y \times Y, Y].$$

By the k-th inclusion and projection $X_k \xrightarrow{i_k} X \xrightarrow{p_k} X_k$, we define the maps

(2.1.7) $[X, Y] \xrightarrow{i^*} \prod_{k=1}^n [X_k, Y] \xrightarrow{\theta_m} [X, Y]$ by $i^*f = (fi_1, \dots, fi_n)(f \in [X, Y])$,

$$\theta_m(f_1, \dots, f_n) = \bar{m}\tilde{f},\ \tilde{f} = \prod f_k \in [X, Y^n],\ (f_k \in [X_k, Y],\ 1 \leqq k \leqq n);$$

and consider the following subsets of $\prod [X_k, Y]$, where $\mathrm{I}_a(m)$ is given only when $n = 2$:

(2.1.8) $\mathrm{H}\{m_k; m\} = \{(f_1,\dots,f_n) \mid f_k \in [X_k, m_k; Y, m]_{\mathrm{H}}$ are H-maps$\}$,

$\mathrm{I}_h(m) = \{(f_1,\dots,f_n) \mid h(\bar{m})(\bar{f} \wedge \bar{f}) = 0 \quad \text{in } [X \wedge X, Y]\}$,

$\mathrm{I}_c(m) = \{(f_1,\dots,f_n) \mid c(m)(f_k \wedge f_l) = 0 \quad \text{in } [X_k \wedge X_l, Y]\ (k<l)\}$,

$\mathrm{I}_a(m) = \{(f_1, f_2) \mid$

$a(m)(m(f_1 \times f_k) \wedge f_l \wedge f_2) = 0$ in $[(X_1 \times X_k) \wedge X_l \wedge X_2, Y]$ and

$a(m)(f_1 \wedge f_k \wedge f_l) = 0$ in $[X_1 \wedge X_k \wedge X_l, Y]$, for $k \neq l\}$.

LEMMA 2.2. (i) *$\theta_m i^* 1 = 1$ and $i^* 1 \in \mathrm{H}\{m_k; m\} \cap \mathrm{I}_b(m)$ $(b=h, c, a)$ if $(Y, m) = (X, m_X)$.*

(ii) *$i^*\theta_m = \mathrm{id}$, and by restricting i^* and θ_m, we have the bijection*

$$i^*\colon [X, m_X; Y, m]_{\mathrm{H}} \cong \mathrm{H}\{m_k; m\} \cap \mathrm{I}_h(m) \quad \textit{with } \theta_m i^* = \mathrm{id}.$$

(iii) *$\mathrm{H}\{m_k; m\} \cap \mathrm{I}_h(m)$ is contained in $\mathrm{I}_c(m)$, and coincides with*

$\mathrm{H}\{m_k; m\} \cap \mathrm{I}_c(m)$ if m is homotopy associative, and
$\mathrm{H}\{m_k; m\} \cap \mathrm{I}_c(m) \cap \mathrm{I}_a(m)$ if $n=2$.

PROOF. Let $Y=X$ and $m=m_X$. Then $\theta_m i^* 1 = \bar{m}\bar{\imath} = 1$ $(\bar{\imath} = \prod i_k)$, $m(\bar{m} \times \bar{m}) \cdot (\bar{\imath} \times \bar{\imath}) = m = \bar{m}\bar{\imath}(\prod m_k)T = \bar{m}(\prod m(i_k \times i_k))T = \bar{m}m^n(\bar{\imath} \times \bar{\imath})$, and $mTi' = mi'$, $\bar{m}i'' = m(m \times m)i''$, etc. for $i' = i_k \times i_l$, $i'' = i_1 \times i' \times i_2$ $(k \neq l)$. Thus we see (i).

Now, for any (Y, m), (i) shows that

(2.2.1) *if $f \in [X, m_X; Y, m]_{\mathrm{H}}$, then*

$$\theta_m i^* f = f \textit{ and } i^* f \in \mathrm{H}\{m_k; m\} \cap \mathrm{I}_b(m)(b=h, c, a),$$

because $\bar{m}f^n = f\bar{m}_X$ and f commutes with the obstructions in (2.1.4–6), e.g., $h(\bar{m})(f^n \wedge f^n) = fh(\bar{m}_X)$. Conversely, let $(f_k) = (f_1,\dots,f_n) \in \mathrm{H}\{m_k; m\}$. Then $m^n(\bar{f} \times \bar{f}) = (\prod m(f_k \times f_k))T = (\prod f_k m_k)T = \bar{f}m_X$. Therefore, if $(f_k) \in \mathrm{I}_h(m)$ in addition, then

(2.2.2) $m(\bar{m} \times \bar{m})(\bar{f} \times \bar{f}) = \bar{m}m^n(\bar{f} \times \bar{f}) = \bar{m}\bar{f}m_X$ and so

$$\theta_m(f_k) = \bar{m}\bar{f} \in [X, m_X; Y, m]_{\mathrm{H}}.$$

Thus we see (ii). If $(f_k) \in \mathrm{I}_c(m)$, then $m(f_k \times f_l) = m(f_l \times f_k)T$ for $k \neq l$. Therefore, by the definition of m^n, we can certify the first equality in (2.2.2) when m is homotopy associative, i.e., $m(\bar{m} \times 1) = m(1 \times \bar{m})$ in (2.1.3), or when so are several compositions of product maps of f_k's and those of $\bar{m}$'s, e.g., when $n=2$ and $(f_1, f_2) \in \mathrm{I}_a(m)$. Thus we see (iii). q.e.d.

We now consider the set of matrices

(2.3.1) $\mathrm{M}\{m_k\} = \{(a_{jk}) \mid a_{jk} \in [X_k, X_j]\ (1 \leqq j, k \leqq n)\}$, with multiplication

$$(a_{jk})(b_{jk}) = (\bar{m}_j(\textstyle\prod_l a_{jl}b_{lk})\Delta) \quad (\Delta\colon X_k \to (X_k)^n \text{ is the diagonal map}),$$

and the following maps and subsets of $\mathrm{M}\{m_k\}$:

(2.3.2) $[X, X] \xrightarrow{\phi} \mathrm{M}\{m_k\} \xrightarrow{\theta} [X, X]$, given by $\phi(f) = (p_j f i_k)$ $(f \in [X, X])$,

$$p_j\theta(a_{jk}) = \bar{m}_j\tilde{a}_j,\ \tilde{a}_j = \textstyle\prod_k a_{jk} \in [X, (X_j)^n],\ (a_{jk} \in [X_k, X_j]);$$

(2.3.3) $\mathrm{HM}\{m_k\} = \{(a_{jk}) \mid a_{jk} \in [X_k, m_k; X_j, m_j]_{\mathrm{H}}$ i.e.

$$(a_{j1}, \dots, a_{jn}) \in \mathrm{H}\{m_k; m_j\}\},$$

$\mathrm{I}_b\mathrm{M}\{m_k\} = \{(a_{jk}) \mid (a_{j1}, \dots, a_{jn}) \in \mathrm{I}_b(m_j)\}$ $(b = h, c$; and $b = a$ for $n = 2)$.

Then, Lemma 2.2(ii) implies the following

(2.3.4) $\phi\theta = \mathrm{id}$, *and the restrictions of* ϕ *and* θ *give us the multiplicative bijection*

$$\phi\colon [X, m_X; X, m_X]_{\mathrm{H}} \cong \mathrm{HM}\{m_k\} \cap \mathrm{I}_h\mathrm{M}\{m_k\} \quad \textit{with } \phi^{-1} = \theta.$$

In fact, let $f \in [X, m_X; X, m_X]_{\mathrm{H}}$. Then $p_j f = p_j\theta\phi(f) = \bar{m}_j(\prod_l p_j f i_l)$,

$$p_j f g i_k = \bar{m}_j(\textstyle\prod_l p_j f i_l p_l g i_k)\Delta \text{ and so } \phi(fg) = \phi(f)\phi(g)\ (g \in [X, X])$$

by the definition of the multiplication in (2.3.1). Therefore, we have the isomorphism

(2.3.5) $\phi\colon \mathrm{HE}(X, m_X) \cong \mathrm{HGL}\{m_k\} \cap \mathrm{I}_h\mathrm{M}\{m_k\}$ of the group in (1.3.2), where

$$\mathrm{HGL}\{m_k\} = \{\text{invertible matrices in } \mathrm{HM}\{m_k\}\} \quad \text{(cf. [12; Th. 3.8])}.$$

By using the sets in (1.1.4), we define the following subsets of $\mathrm{HM}\{m_k\}$:

(2.3.6) $\mathrm{HM} = \{(a_{jk}) \mid a_{jk} \in [X_k, X_j]_{\mathrm{H}}\ (k \neq j),\ a_{kk} \in \mathrm{HMap}(X_k)\} = \cap\mathrm{HM}\{m'_k\}$

$$\supset \mathrm{HGL} = \mathrm{HGL}\{m_k\} \cap \mathrm{HM} = \{\text{invertible ones in HM}\} = \cap\mathrm{HGL}\{m'_k\},$$

where the intersections are taken over all multiplications $m'_k \in \mathrm{M}(X_k)$ $(1 \leq k \leq n)$. Then (2.3.5) and Lemma 2.2 imply the following theorem on the group

(2.3.7) $\quad \mathrm{HE}(X) = \mathrm{HE}(X, m_X) \cap \mathrm{IE}(X)$ (see Proposition 1.4):

THEOREM 2.4. *Let* $X = \prod_{k=1}^n X_k$ *be a product H-space in* (2.1.1) *of H-spaces* (X_k, m_k).

(i) *Then the restrictions of* ϕ *and* θ *in* (2.3.2) *give us the isomorphism*

$$\phi\colon \mathrm{HE}(X) \cong \mathrm{HGL}\{m_k\} \cap \mathrm{I}_h\mathrm{M}\{m_k\} \cap \phi\mathrm{IE}(X)$$
$$= \mathrm{HGL} \cap \mathrm{I}_h\mathrm{M}\{m_k\} \cap \phi\mathrm{IE}(X) \quad \textit{with } \phi^{-1} = \theta.$$

(ii) $\phi\mathrm{HE}(X) = \mathrm{HGL} \cap \phi\mathrm{IE}(X)$ *if each* m_k *is homotopy associative.*
(iii) $\phi\,\mathrm{HE}(X) = \mathrm{HGL} \cap \mathrm{I}_a\mathrm{M}\{m_k\} \cap \phi\mathrm{IE}(X)$ *if* $n = 2$.

PROOF. It is sufficient to show that if $h \in \mathrm{IE}(X)$, then $\phi(h)=(p_jhi_k) \in \mathrm{I}_c\mathrm{M}\{m_k\}$, which is shown by definition (1.3.4–5) as follows:

$$c(m_j)(p_jhi_k \wedge p_jhi_l) = p_jc(m_X)(h \wedge h)(i_k \wedge i_l) = p_jhc(m_X)(i_k \wedge i_l) = 0\ (k<l).$$

q.e.d.

EXAMPLE 2.5. (i) *Let* Y *be a 2-connected H-space. Then,*

$$\mathrm{HE}(S^1 \times Y) \cong \{(\varepsilon, h) \mid \varepsilon = \pm 1 \in Z_2 = \mathrm{HE}(S^1),\ h \in \mathrm{HE}(Y) \text{ with the following (2.5.1)}\}:$$

$$(2.5.1)\quad (\varepsilon \wedge h)^* = h_* \text{ on } [S^1 \wedge Y, Y],\quad (\varepsilon \wedge h \wedge h)^* = h_* \text{ on } [S^1 \wedge Y \wedge Y, Y],$$
$$(1 \wedge h)^* = h_* \text{ on } [S^2 \wedge Y, Y],\quad (1 \wedge h \wedge h)^* = h_* \text{ on } [S^2 \wedge Y \wedge Y, Y].$$

In particular, $\mathrm{HE}(S^1 \times S^n) = Z_2$ *for* $n=3, 7$ ([12; Th. 4.3]).

(ii) *For the Eilenberg-MacLane spaces* $K(G, k)$ *and* $K(H, l)$ *with* $k<l$ *and abelian groups* G *and* H,

$$\mathrm{HE}(K(G, k) \times K(H, l)) \cong PH^l(G, k; H) \times_s D \text{ (the semi-direct product)},$$

where $PH^l(G, k; H)$ *is the subgroup of all primitive elements in* $H^l(G, k; H)$,

$$D = \{(g, h) \in \mathrm{aut}\,G \times \mathrm{aut}\,H \mid (g \wedge g)^* = h_* \text{ on } H^l(K(G, k) \wedge K(G, k); H)\},$$

and s *is given by* $\alpha s(g, h) = h^{-1}\alpha g$ *for* $\alpha \in PH^l(G, k; H)$ *and* $(g, h) \in D$.

PROOF. (i) Since $[S^1, Y]=0=[Y, S^1]$ by assumption, we have $\mathrm{HGL} = \mathrm{HE}(S^1) \times \mathrm{HE}(Y)$ and $\mathrm{HE}(X) \cong \mathrm{HGL} \cap \phi\mathrm{IE}(X)$ for $X = S^1 \times Y$ by Theorem 2.4(iii). (2.5.1) for $(\varepsilon, h) \in \mathrm{HE}(S^1) \times \mathrm{HE}(Y)$ means $\varepsilon \times h \in \mathrm{IE}(X)$ by definition, since $[X \wedge X, X] = [X \wedge X, Y]$ can be identified with $\prod_\delta [\wedge_{\delta_i=1} Y_i, Y]$ ($\delta = (\delta_1,\dots,\delta_4) \in \{0, 1\}^4$, $\delta_1+\delta_2 \neq 0 \neq \delta_3+\delta_4$, $Y_1 = Y_3 = S^1$, $Y_2 = Y_4 = Y$).

(iii) Since $[K(H, l), K(G, k)]=0$ ($l>k$) and $[K(G, k), K(H, l)]_{\mathrm{H}} = PH^l(G, k; H)$, we have $\mathrm{HGL} = PH^l(G, k; H) \times_s (\mathrm{aut}\,G \times \mathrm{aut}\,H)$ and $\mathrm{HE}(X) \cong \mathrm{HGL} \cap \phi\mathrm{IE}(X)$ for $X = K(G, k) \times K(H, l)$. Since $[X \wedge X, X] = H^l(K(G, k) \wedge K(G, k); H)$, we see that $(x, (1,1)) \in \phi\mathrm{IE}(X)$ for any $x \in PH^l(G, k; H)$ and that $(0, (g, h)) \in \phi\mathrm{IE}(X)$ ($g \in \mathrm{aut}\,G$, $h \in \mathrm{aut}\,H$) if and only if $(g, h) \in D$. q.e.d.

THEOREM 2.6. *Let a simply connected CW-complex* X *be an H-space of rank 2. Then* $\mathrm{HE}(X)$ *is trivial unless* X *is homotopy equivalent to* $S^l \times S^l$ ($l=3, 7$), *and*

$$\mathrm{HE}(S^l \times S^l) \cong H = \{(a_{ij}) \mid a_{ij} \in Z\ (1 \leqq i, j \leqq 2),\ \det(a_{ij}) = 1,\ a_{ij} \equiv \delta_{ij} \bmod 2k_l\}$$

for $l=3, 7$ *by the isomorphism* ϕ *in Theorem* 2.4, *where* $k_3=12$ *and* $k_7=120$.

PROOF. In the first case, $\mathrm{HE}(X, m)=1$ for some or any m by [7], [8; Th. 4.1] and [13; Th. 5.8]. Let m be the usual multiplication on S^l ($l=3, 7$). Then, by [6; p. 176], [16], [13; p. 325] and [2; Prop. D], we have

(2.6.1) $\pi_{2l}(S^l) = Z_{k_l}$ *generated by* $c(m)$ *and* $k_l\pi_{rl}(S^l) = 0$ $(r=2, 3, 4)$,

(2.6.2) $$[S^l, m; S^l, m]_{\mathrm{H}} = \{n \in Z \mid n^2 \equiv n \bmod 2k_l\}.$$

Thus, for $X=S^l\times S^l$ and $Y=S^l$, $\mathrm{I}_c(m)=\{(n_1, n_2) \mid n_1n_2\equiv 0 \bmod k_l\}$, $\mathrm{H}\{m_k; m\} \cap \mathrm{I}_c(m) \subset \mathrm{I}_a(m) (m_1=m_2=m)$ in (2.1.8) by (2.6.1–2) and so $\mathrm{HE}(X, m_X)$ is isomorphic to

(2.6.3) $$\begin{aligned}&\{(a_{ij}) \mid \det(a_{ij}) = \pm 1,\ a_{ij}^2 \equiv a_{ij} \bmod 2k_l,\ a_{i1}a_{i2} \equiv 0 \bmod k_l\}\\ &= H \cup \{(a_{ij}) \mid \det(a_{ij}) = -1,\ a_{ij} \equiv 1 - \delta_{ij} \bmod 2k_l\}\end{aligned}$$

by (2.3.5) and Lemma 2.2(iii) (cf. [12; Ex. 3.10]). Also, $\mathrm{O}[S^l, S^l]=\{n \mid n\equiv 0 \bmod k_l\}$ and so $[S^l, S^l]_{\mathrm{H}}=\{n \mid n\equiv 0 \bmod 2k_l\}$ by Lemma 1.2(i). Therefore, we see that

(2.6.4) HGL *in* (2.3.6) *for* $X = S^l \times S^l$ *is contained in* H.

Now, we see ϕ: $\mathrm{HE}(X)\cong H$ by Theorem 2.4(i), (2.6.3–4) and the following

(2.6.5) $h = \theta(a_{ij})$ *for* $(a_{ij})\in H$ *satisfies*

$$(h \wedge h)^* = \mathrm{id} = h_*\colon [X\wedge X, X] \longrightarrow [X\wedge X, X],$$

because this shows $h\in \mathrm{IE}(X)$. To prove (2.6.5), consider the exact sequence

$$0 \longrightarrow [S^{2l}\wedge X, S^l] \xrightarrow{(\pi\wedge 1)^*} [X\wedge X, S^l] \longrightarrow [(S^l\vee S^l)\wedge X, S^l] \longrightarrow 0$$

and take any $\alpha\in[X\wedge X, S^l]$. Then, by the sum $+$ induced by $\bar{m}$, we have

(2.6.6) $\alpha = \alpha(i_1p_1\wedge 1) + \alpha(i_2p_2\wedge 1) + \omega(\pi\wedge 1)$ *for some* $\omega\in[S^{2l}\wedge X, S^l]$.

Consider $h=\theta(a_{ij})$ with $p_jh=m\tilde{a}_j$, $\tilde{a}_j=a_{j1}\times a_{j2}$, for $(a_{ij})\in H$. Then, by (2.6.6) and (2.6.2),

$$\alpha(\tilde{a}_j\wedge 1) = \alpha(i_1a_{j1}p_1\wedge 1) + \alpha(i_2a_{j2}p_2\wedge 1) + \omega((a_{j1}\wedge a_{j2})\pi\wedge 1) = \alpha(i_jp_j\wedge 1),$$

since $a_{ij}\equiv\delta_{ij} \bmod 2k_l$. Hence $\beta(p_jh\wedge 1)=\beta(mi_jp_j\wedge 1)=\beta(p_j\wedge 1)$ $(\beta\in[S^l\wedge X, S^l])$. Also $\pi h=\pi$ on $[X, S^{2l}]$ since $\det(a_{ij})=1$. Therefore, by (2.6.6), we have

$$\alpha(h\wedge 1) = \alpha(i_1p_1h\wedge 1) + \alpha(i_2p_2h\wedge 1) + \omega(\pi h\wedge 1) = \alpha, \text{ i.e.,}$$

$$(h\wedge 1)^* = \mathrm{id} \text{ on } [X\wedge X, S^l].$$

Similarly $(1 \wedge h)^* = \mathrm{id}$ and so $(h \wedge h)^* = \mathrm{id}$ on $[X \wedge X, X]$. We can prove that $h_* = \mathrm{id}$ by a similar way, considering $[\ , X]$ in addition to $[\ , S^l]$ and noticing that h is an H-map with respect to m_X by (2.6.3). Thus we see (2.6.5). q.e.d.

§ 3. Localizations of $SU(n)$ and $Sp(n)$

The rest of this note is based on the following classical result due to J.-P. Serre:

(3.1.1) *$\pi_{n+k}(S^n; p)$ (n: odd $\geqq 3$, p: odd prime) is 0 if $0<k<2p-3$ and Z_{p^r} ($r \geqq 1$) if $k=2p-3$.*

We consider the case that X_k in Theorem 2.4 is the one in (1.6.5) stated as follows:

(3.1.2) Let p be a prime$\geqq 5$ and $N=(n_1,\dots, n_l)$ be a sequence of odd integers with $1 \leqq n_1 < \cdots < n_l$ and

$$\pi_{n_j}(S^{n_i}; p) = 0 \text{ (e.g. } n_j - n_i < 2p-3 \text{ by (3.1.1)) for any } i<j \text{ with } n_i > 1,$$

and consider the localizations and their product H-space

$$S_i = S^{n_i}_{(p)} \text{ in (1.6.5) and } S = S(N) = \textstyle\prod_{i=1}^l S_i \text{ in (2.1.1)}$$

with multiplications $m_i \in \mathrm{M}(S_i)$ and $m = (\prod m_i)T \in \mathrm{M}(S)$, respectively, where

(3.1.3) m_i is taken to be homotopy commutative and homotopy associative by [1].

Then $[S_j, S_i] = 0$ ($i \neq j$), and (2.3.2–5) and Theorem 2.4(ii) imply the following

$$\text{(3.1.4)} \quad \begin{aligned} &\mathrm{HE}(S, m) \cong \mathrm{HGL}\{m_i\} = \textstyle\prod_{i=1}^l \mathrm{HE}(S_i, m_i) = (Z^*_{(p)})^l \quad \text{(see (1.6.6))}, \\ &\mathrm{HE}(S) \cong \mathrm{I}(N) = \mathrm{HGL}\{m_i\} \cap \phi\mathrm{IE}(S) \ (S = S(N)), \end{aligned}$$

and $a = (a_1,\dots, a_l)$ ($a_i \in \mathrm{HE}(S_i, m_i) = Z^*_{(p)}$) belongs to $\phi\mathrm{IE}(S)$ if and only if

(3.1.5) $(\theta(a) \wedge \theta(a))^* = \theta(a)_*$ *on* $[S \wedge S, S]$ *for* $\theta(a) = \prod_i a_i \in \mathrm{E}(S)$.

Here, by (3.1.3), we can identify $[S \wedge S, S]$ with the direct sum of

$$\text{(3.1.6)} \quad \begin{aligned} &[S_\delta, S_i] = \pi_{N(\delta)}(S^{n_i}) \otimes Z_{(p)} \quad \text{for } 1 \leqq i \leqq l \text{ and} \\ &\delta = (\delta_1,\dots, \delta_{2l}) \in \{0, 1\}^{2l} \quad \text{with } \textstyle\sum_{j=1}^l \delta_j \neq 0 \neq \sum_{j=1}^l \delta_{l+j}, \end{aligned}$$

where $S_\delta = \wedge_{\delta_j=1} S_j = S^{N(\delta)}_{(p)}$ ($S_{l+j} = S_j$) and $N(\delta) = \sum_{j=1}^l \varepsilon_j n_j$ ($\varepsilon_j = \delta_j + \delta_{l+j}$); and by (1.7.2), we can identify $(\theta(a) \wedge \theta(a))^*$ (resp. $\theta(a)_*$) with the multiplication by the element

(3.1.7) $a(\delta) = \prod_{i=1}^l a_i^{\varepsilon_i}$ (resp. a_i) in $Z^*_{(p)}$ on each summand $\pi_{N(\delta)}(S^{n_i}) \otimes Z_{(p)}$.

Thus, we see the following theorem, where (ii) follows from (i) and $\pi_n(S^n)\otimes Z_{(p)}=Z_{(p)}$.

THEOREM 3.2. *Let $S(N)$ be a product H-space in* (3.1.2). *Then*:

(i) $\mathrm{HE}(S(N))\cong \mathrm{I}(N)\subset (Z^*_{(p)})^l$ ($Z^*_{(p)}$ *is the group given in* (1.6.3)), *and the subgroup* $\mathrm{I}(N)$ *consists of all* $a=(a_1,\dots,a_l)\in(Z^*_{(p)})^l$ *satisfying*

(3.2.1) $a(\varepsilon)\cdot\alpha = a_i\cdot\alpha$ *in* $\pi(\varepsilon, i)=\pi_{N(\varepsilon)}(S^{n_i})\otimes Z_{(p)}$ *for any* $\alpha\in\pi(\varepsilon, i)$,

for each $1\leqq i\leqq l$ *and each* $\varepsilon=(\varepsilon_1,\dots,\varepsilon_l)\in\{0, 1, 2\}^l$, *where*

(3.2.2) $$a(\varepsilon)=\prod_{j=1}^l a_j^{\varepsilon_j}\in Z^*_{(p)} \quad and \quad N(\varepsilon)=\sum_{j=1}^l \varepsilon_j n_j.$$

(ii) *If* $\pi(\varepsilon, i)=\pi_{N(\varepsilon)}(S^{n_i}; p)=0$ *for any i and ε with* $N(\varepsilon)>n_i$, *e.g., if* $2\sum_j n_j<n_i+2p-3$ *for* $n_i>1$ *by* (3.1.1), *in addition, then*

(3.2.3) $$\mathrm{I}(N)=\{(a_1,\dots,a_l)\in(Z^*_{(p)})^l \mid a_i=a(\varepsilon) \text{ in } Z^*_{(p)} \text{ if } n_i=N(\varepsilon)\}.$$

Now, we consider the special unitary group or the symplectic group by

(3.3.1) putting $(G(l), g)=(SU(l+1), 1)$ or $(Sp(l), 2)$
and taking a prime $p>\max\{gl, 4\}$.

Then, the localization $G(l)_{(p)}$ of $G(l)$ at p is homotopy equivalent ($\simeq$) to $SU(l)_{(p)}\times S^{2l+1}_{(p)}\simeq\prod_{i=1}^l S^{2i+1}_{(p)}$ or $Sp(l-1)_{(p)}\times S^{4l-1}_{(p)}\simeq\prod_{i=1}^l S^{4i-1}_{(p)}$, respectively, (cf. Lemma 4.3 below); and so Theorem 3.2 implies that

(3.3.2) $$\mathrm{HE}(G(l)_{(p)})\cong\mathrm{HE}(S(N_l))\cong\mathrm{I}(N_l)$$
for $N_l=(n_1,\dots,n_l)$ *with* $n_i=2gi-(-1)^g$ $(1\leqq i\leqq l)$,

since $n_l-n_1=2g(l-1)<2p-3$ by (3.3.1).

COROLLARY 3.4. (i) $\mathrm{HE}(SU(l+1)_{(p)})\subset\mathrm{HE}(SU(l)_{(p)})\subset\mathrm{HE}(SU(5)_{(p)})\subset(Z^*_{(p)})^4$ *if* $p>l\geqq 5$. *If* $p>l(l+2)$, *then* $\mathrm{HE}(SU(l+1)_{(p)})$ *is isomorphic to*

$$Z^*_{(p)}\ (l\geqq 8),\quad (Z^*_{(p)})^{9-l}\ (7\geqq l\geqq 5),\quad (Z^*_{(p)})^l\ (4\geqq l\geqq 1).$$

(ii) $\mathrm{HE}(Sp(l)_{(p)})\subset\mathrm{HE}(Sp(l-1)_{(p)})\subset\mathrm{HE}(Sp(7)_{(p)})\subset(Z^*_{(p)})^7$ *if* $p/2>l\geqq 8$. *If* $p>l(2l+1)$, *then* $\mathrm{HE}(Sp(l)_{(p)})$ *is isomorphic to*

$$Z^*_{(p)}\ (l\geqq 13),\quad (Z^*_{(p)})^{14-l}\ (12\geqq l\geqq 10),\quad (Z^*_{(p)})^{15-l}\ (l=9, 8),\quad (Z^*_{(p)})^l\ (7\geqq l\geqq 1).$$

PROOF. Take $a=(a_1,\dots,a_l)\in\mathrm{I}(N_l)$ for N_l in (3.3.2). Then, since $n_l=2\cdot 3+2l-5$ $(g=1)$, $=2(3+7)+4l-21$ $(g=2)$, the definition of $\mathrm{I}(N)$ in (3.2.1–2) shows that

$$a_l = a_1^2 a_{l-3}\ (g=1,\ l\geqq 5),\ = a_1^2 a_2^2 a_{l-5}\ (g=2,\ l\geqq 8),$$
$$\text{and}\quad a' = (a_1,\dots, a_{l-1}) \in \mathrm{I}(N_{l-1});$$

and so $\mathrm{I}(N_l) \subset \mathrm{I}(N_{l-1})$ by sending a to a'. Thus, the first halves in (i) and (ii) hold.

Assume that $p > l(gl+g-(-1)^g) = \sum_i n_i$. Then $2\sum_i n_i < n_1+2p-3$ and so

(3.4.1) $\mathrm{I}(N_l)$ *is given by* (3.2.3) *for* $N=N_l$, *and* $(q^{n_1},\dots, q^{n_l}) \in \mathrm{I}(N_l)$ *for any* $q \in Z^*_{(p)}$,

by Theorem 3.2(ii). This shows the second halves arithmetically as follows.

(i) Let $g=1$ and $n_i=2i+1$. Then the conditions for $(a_1,\dots, a_l) \in \mathrm{I}(N_l)$ in (3.2.3) are nothing when $l \leqq 4$, and so $\mathrm{I}(N_l) = (Z^*_{(p)})^l$. They consist of $a_5 = a_1^2 a_2$ when $l=5$, and

$$a_i = a_1^2 a_{i-3}\ (5 \leqq i \leqq l) \quad\text{and}\quad a_i = a_1 a_2 a_{i-4}\ (6 \leqq i \leqq l) \quad \text{when } l = 6, 7;$$

and so $\mathrm{I}(N_5) \cong \{(a_1, a_2, a_3, a_4)\}$, $\mathrm{I}(N_6) \cong \{(a_1, a_2, a_4)\}$ and $\mathrm{I}(N_7) \cong \{(a_1, a_2)\}$. Also, they contain $a_8 = a_1^2 a_5 = a_1 a_2 a_4$ when $l=8$, and so $\mathrm{I}(N_8) \subset \{a_1\}$, which shows $\mathrm{I}(N_l) \cong Z^*_{(p)}$ for $l \geqq 8$ by the second half of (3.4.1) and the first half.

(ii) Let $g=2$ and $n_i=4i-1$. Then the conditions for $(a_1,\dots, a_l) \in \mathrm{I}(N_l)$ are nothing when $l \leqq 7$, and so $\mathrm{I}(N_l) = (Z^*_{(p)})^l$. They consist of $a_8 = a_1^2 a_2^2 a_3$ when $l=8$, and

$$a_i = a_1^2 a_2^2 a_{i-5}\ (8 \leqq i \leqq l),\ = a_1^2 a_2 a_3 a_{i-6}\ (9 \leqq i \leqq l),\ = a_1 a_2^2 a_3 a_{i-7}\ (10 \leqq i \leqq l),$$
$$= a_1^2 a_2 a_4 a_{i-7} = a_1^2 a_3^2 a_{i-7}\ (11 \leqq i \leqq l),\ = a_1 a_2^2 a_4^2 = a_1 a_2 a_3^2 a_4\ (i=l=12)$$

when $9 \leqq l \leqq 12$; and so $\mathrm{I}(N_8) \cong \{(a_1,\dots, a_7)\}$, $\mathrm{I}(N_9) \cong \{(a_1, a_2, a_3, a_5, a_6, a_7)\}$, $\mathrm{I}(N_{10}) \cong \{(a_1, a_2, a_6, a_7)\}$, $\mathrm{I}(N_{11}) \cong \{(a_1, a_2, a_7)\}$ and $\mathrm{I}(N_{12}) \cong \{(a_1, a_2)\}$. Also they contain $a_{13} = a_1^2 a_2^2 a_8 = a_1^2 a_2 a_3 a_7$ when $l=13$, and so $\mathrm{I}(N_{13}) \subset \{a_1\}$ which shows $\mathrm{I}(N_l) \cong Z^*_{(p)}$ for $l \geqq 13$ by (3.4.1) and the first half. q. e. d.

Here, we remark on the rationalization $X_{(0)}$ of X. For the n-sphere S^n (n: odd), we have

$$S^n_{(0)} = K(Q, n),\ \mathrm{E}(S^n_{(0)}) = \mathrm{aut}\, Q = Q^*\ (= Q - \{0\}\text{: the group of all units of } Q);$$

and $[S^{n'}_{(0)}, S^n_{(0)}] = \pi_{n'}(S^n_{(0)}) = 0$ if $n \neq n'$, $= Q$ if $n = n'$. Thus, in the same way as Theorem 3.2 and Corollary 3.4, we see the following

PROPOSITION 3.5. (i) *For a sequence* $1 \leqq n_1 < \cdots < n_l$ *of odd integers,*

$$\mathrm{HE}\left(\textstyle\prod_{i=1}^l K(Q, n_i)\right) \cong \mathrm{I}(N) \subset (Q^*)^l,$$

where $\mathrm{I}(N)$ $(N=(n_1,\dots, n_l))$ *is given by* (3.2.3) *using* Q^* *instead of* $Z^*_{(p)}$.

(ii) *For the rationalizations* $SU(l+1)_{(0)}$ *and* $Sp(l)_{(0)}$, *the conclusions of Corollary* 3.4 *also hold by putting* $p=0$ *and* $Z^*_{(0)}=Q^*$.

In connection with Corollary 3.4, we note furthermore the following

EXAMPLE 3.6. (i) $\mathrm{HE}(SU(5)_{(p)})$ *is isomorphic to* $(Z^*_{(p)})^4$ *if* $p>23$,

$\{a_1^2a_2a_3^2a_4^2\equiv 1 \bmod 23\}$ *if* $p=23$, $\{a_1a_3\equiv a_2^2$ *and* $a_1\equiv a_2^4a_4^2 \bmod 19\}$ *if* $p=19$,

$\{a_i\equiv q^{7+i} \bmod 17\ (1\leqq i\leqq 4)$ *for some* $q\in Z^*_{(17)}\}$ *if* $p=17$, $(U_p)^4$ *if* $13\geqq p\geqq 7$,

$\{a_i\in U_5\ (1\leqq i\leqq 4),\ a_1a_3\equiv a_2^2\equiv a_4$ *and* $a_4^2\equiv 1 \bmod 25\}$ *if* $p=5$,

where $\{\ \}$ *consists of all* $(a_1,\dots,a_4)\in(Z^*_{(p)})^4$ *with the relations contained in* $\{\ \}$,

(3.6.1) $U_p=1+pZ_{(p)}=\{q\in Z^*_{(p)}|q\equiv 1 \bmod p\}$ *is the group in* (1.6.7), *and*

(3.6.2) $q_1\equiv q_2 \bmod p^r\ (q_k=s_k/t_k\in Z^*_{(p)})$ *means* $s_1t_2\equiv s_2t_1 \bmod p^r\ (r\geqq 1)$.

(ii) $\mathrm{HE}(Sp(7)_{(p)})$ *is isomorphic to* $(Z^*_{(p)})^7$ *if* $p>103$,

$\{a_j\equiv \prod_{i=1}^7 a_i^2 \bmod p\}\ (2j=107-p)$ *if* $103\geqq p\geqq 97$,

$\{a_i\equiv a_1^{2-i}a_2^{i-1}\ (1\leqq i\leqq 5)$ *and* $a_1^{10}\equiv a_2^{15}a_6^2a_7^2 \bmod 89\}$ *if* $p=89$,

$\{a_i\equiv a_1^{2-i}a_2^{i-1}\ (1\leqq i\leqq 7)$ *and* $a_1^k\equiv a_2^{k+9} \bmod p\}\ (2k=p-33)$ *if* $83\geqq p\geqq 71$,

and $(U_p)^7$ *if* $67\geqq p\geqq 17$, *where* $\{\ \}$ *consists of all* $(a_1,\dots,a_7)\in(Z^*_{(p)})^7$ *with the relations in* $\{\ \}$.

PROOF. (i) for $p>23$ and (ii) for $p>103$ are in Corollary 3.4.
Consider (3.2.1) for $N=N_l=(n_1,\dots,n_l)$ and a prime p with

$$l=4,\ n_i=2i+1 \quad\text{and}\quad 5\leqq p\leqq 23, \quad\text{or} \tag{3.6.3}$$
$$l=7,\ n_i=4i-1 \quad\text{and}\quad 17\leqq p\leqq 103.$$

Then $N_l(\varepsilon)=\sum_j\varepsilon_jn_j\leqq 2\sum_j n_j=48$ or 210 for any $\varepsilon=(\varepsilon_1,\dots,\varepsilon_l)\in\{0,1,2\}^l$ and $N_l(\varepsilon)\neq n_i$ except for the trivial case $\varepsilon_i=1$ and $\varepsilon_j=0\ (j\neq i)$. Also, by Toda [16; Th. 13.4],

(3.6.4) $\pi(\varepsilon,i)=\pi_{N_l(\varepsilon)}(S^{n_i};p)\ (0<N_l(\varepsilon)-n_i<2p(p-1)-2)$ *is* 0 *except for*

(3.6.5) Z_p *if* $N_l(\varepsilon)-n_i=2k(p-1)-1\ (1\leqq k<p)$ *or* $2k(p-1)-2\ (n_i/2<k<p)$.

Further (3.6.2) is equivalent to hold $q_1\cdot\alpha=q_2\cdot\alpha$ in Z_{p^r} for any $\alpha\in Z_{p^r}$. Thus,

(3.6.6) $\mathrm{I}(N_l)=\{(a_1,\dots,a_l)\in(Z^*_{(p)})^l|$

$a_i\equiv\prod_{j=1}^l a_j^{\varepsilon_j} \bmod p$ *if* (3.6.5) *holds*$\}$ *for* $p\geqq 7$.

This and (3.3.2) imply the results for $p \geqq 7$ as follows, where $\equiv$ denotes $\equiv \bmod p$.

(i) The case $l=4$ and $n_j=2j+1$: Let $p=23$, 19 or 17. Then (3.6.5) holds when and only when $N_4(\varepsilon)=48$ and $i=2$, $N_4(\varepsilon)=35+n_i$ and $i \leqq 3$, or $N_4(\varepsilon)=31+n_i$, respectively; and so the condition in (3.6.6) consists of $a_2 \equiv \tilde{a}\ (=\prod_{j=1}^4 a_j^2)$ if $p=23$,

$$a_i \equiv \tilde{a}(a_1 a_{4-i})^{-1}\ (1 \leqq i \leqq 3) \text{ and } a_1 \equiv \tilde{a} a_2^{-2}$$
$$(\Leftrightarrow a_1 a_3 \equiv a_2^2 \text{ and } a_1 \equiv a_2^4 a_4^2) \quad \text{if } p=19,$$
$$a_i \equiv \tilde{a}(a_1 a_{6-i})^{-1}\ (2 \leqq i \leqq 4),\ a_i \equiv \tilde{a}(a_2 a_{5-i})^{-1}\ (1 \leqq i \leqq 3) \text{ and } a_1 \equiv \tilde{a} a_3^{-2}$$

if $p=17$. The relations for $p=17$ are equivalent to

$$\text{(3.6.7)} \quad a_i \equiv a_{i-1} q \equiv a_1 q^{i-1}\ (2 \leqq i \leqq l) \quad \text{for some } q \in Z_{(p)}^*,$$

and the last one $a_1^5 q^8 \equiv 1$, which implies $a_1 \equiv q^8$ by Fermat's theorem $q^{p-1} \equiv 1$.

Let $p=13$ or 7. Then we can take $(N_4(\varepsilon), i)=(26, 1)$ or $(32, 4)$ in (3.6.5), and so

$$a_1 \equiv a_1 a_2 a_4^2 \equiv a_1 a_3^2 a_4 \equiv a_2^2 a_3 a_4, \text{ which imply (3.6.7), and } a_4 \equiv \tilde{a}(a_3 a_4)^{-1}$$

are in the condition in (3.6.6). These imply $a_1^3 q^7 \equiv 1 \equiv a_1^5 q^4$ and so $q \equiv a_i \equiv 1$ $(1 \leqq i \leqq 4)$ since $q^{12} \equiv 1$. If $p=11$, then for $(N_4(\varepsilon), i)=(26, 3)$ or $(42, 1)$, we have similarly

$$a_3 \equiv a_1 a_2 a_4^2, \text{ (3.6.7) and } a_1 \equiv \tilde{a} a_1^{-2}, \quad \text{which imply}$$
$$a_1^3 q^5 \equiv 1 \equiv a_1^5 q^{12} \quad \text{and so } a_i \equiv 1\ (1 \leqq i \leqq 4).$$

Thus $\mathrm{I}(N_4)=(U_p)^4$ for $13 \geqq p \geqq 7$, since $(U_p)^4 \subset \mathrm{I}(N_4)$ is clear by (3.6.6).

(ii) The case $l=7$ and $n_j=4j-1$: If $p=103$, 101 or 97, (3.6.5) holds when and only when $N_7(\varepsilon)=210$ and $i=(107-p)/2$, and so the condition in (3.6.6) consists of $a_i \equiv \tilde{a}\ (=\prod_{j=1}^7 a_j^2)$. If $p=89$, then we have $N_7(\varepsilon)=175+n_i=210-(2n_1+n_2+n_{6-i})$ for $i \leqq 4$ in (3.6.5) and so

$$\tilde{a} a_i^{-1} \equiv a_1^2 a_2 a_{6-i}\ (1 \leqq i \leqq 4),\ \equiv a_1^2 a_3 a_{5-i}\ (i=1, 2),\ \equiv a_1 a_2 a_3^2\ (i=1) \text{ in (3.6.6)}.$$

These are equivalent to $a_i \equiv a_1 q^{i-1}$ $(1 \leqq i \leqq 5)$ for some q and $a_6^2 a_7^2 a_1^5 q^{15} \equiv 1$. If $p=83$, 79, 73 or 71, then we have $N_7(\varepsilon)=2p-3+n_i=210-(n_1+\tilde{n}+n_{j+1-i})$ for $i \leqq j$ ($j=6$, 7, 7 or 7, and $\tilde{n}=n_2+n_3$, n_2+n_4, n_3+n_6 or n_4+n_6, respectively) so that the condition in (3.6.6) is equivalent to (3.6.7) and $a_1^{13} q^{42} \equiv a_1^4 q^n$ $(2n=99-p)$ $(\Leftrightarrow a_1^9 q^{k+9} \equiv 1\ (2k=p-33))$ similarly.

If $67 \geqq p \geqq 59$, then for $N_7(\varepsilon)=134$ and $i=j(=(69-p)/2)$, we have $\tilde{a} a_j^{-1} \equiv a_1 a_5 a_7^2 \equiv a_1^2 a_2^2 a_3^2 a_4 a_5$ and (3.6.7), which imply $a_1^9 q^{27-j} \equiv 1$, $a_1^4 \equiv q^3$ and $a_i \equiv 1$ $(1 \leqq i \leqq 7)$. If $53 \geqq p \geqq 17$, then we see $a_i \equiv 1$ $(1 \leqq i \leqq 7)$ by taking $(N_7(\varepsilon), i)$ to be

$(106, j)$, $(110, j+1)$ $(2j=55-p)$ for $53 \geqq p \geqq 43$;

$(86, 2)$, $(170, 3)$ for $p=41$; $(86, 4)$, $(146, 1)$ for $p=37$;

$(86, 7)$, $(121, 1)$ for $p=31$; $(58, 1)$, $(113, 1)$ for $p=29$;

and $(58, j)$, $(54, j-1)$ $(2j=31-p)$ for $23 \geqq p \geqq 17$. Thus $\mathrm{I}(N_7)=(U_p)^7$ for $67 \geqq p \geqq 17$.

Finally, let $p=5$, $l=4$ and $n_i=2i+1$. Then, in the same way as the case $p=13$ or 7 in (i), we see that the relations in (3.2.1) for $\pi(\varepsilon, i)=Z_5$ are equivalent to $a_i \equiv 1$ $(1 \leqq i \leqq 4)$. On the other hand, by Toda [17; Th. 7.1–2],

(3.6.8) $\pi(\varepsilon, i)=\pi_{N_4(\varepsilon)}(S^{n_i}; 5)$ $(n_i<N_4(\varepsilon) \leqq 48)$ *is* Z_{25} *if* $(N_4(\varepsilon), n_i)=(43, 5)$, $(45, 7)$ *or* $(48, 9)$, *and* Z_5 *or* 0 *otherwise.*

Therefore, the relations in (3.2.1) for $\pi(\varepsilon, i)=Z_{25}$ consist of $a_2 \equiv \tilde{a} a_2^{-1}$, $a_3 \equiv \tilde{a} a_1^{-1}$ and $a_4 \equiv \tilde{a} \bmod 25$, which are equivalent to $a_1 a_3 \equiv a_2^2 \equiv a_4$ and $a_4^2 \equiv 1 \bmod 25$ since $q^{20} \equiv 1 \bmod 25$. q. e. d.

§4. HE(G) for $G=U(n)$, $SU(n)$, $Sp(n)$

In this section, we prove the following

THEOREM 4.1. *Let G be the (special) unitary group $U(n)$ $(n \geqq 3)$, $SU(n)$ $(n \geqq 1)$ or the symplectic group $Sp(n)$ $(n \geqq 1)$. Then, any $h \in \mathrm{HE}(G)$ satisfies the following* (1) *and* (2):

(1) *The localization $h_{(p)}: G_{(p)} \to G_{(p)}$ of h at a prime $p \geqq gn$ is homotopic to the identity map, where $g=1$ when $G=U(n)$ or $SU(n)$ and $g=2$ when $G=Sp(n)$.*

(2) *$h^*=\mathrm{id}$ on the integral cohomology group $H^*(G; Z)$.*

For example, when $n \geqq 3$, the complex conjugate C on $U(n)$ or $SU(n)$ satisfies $C^* \neq \mathrm{id}$, and so C is not an H-map with respect to some multiplication on $U(n)$ or $SU(n)$.

COROLLARY 4.2. *The group* HE(G) *for G in Theorem* 4.1 *is finite and nilpotent.*

PROOF. If X is the k-skeleton of G for any k, then the group $[X, G]$ induced by the usual multiplication $\bar{m}$ on G is nilpotent by [3]. Furthermore, we see by induction on k that this group is finitely generated, since so are the homotopy groups of G; and especially $[G, G]$ satisfies the maximal condition for subgroups. If $h \in \mathrm{HE}(G)$, then $1-h$ is of finite order in $[G, G]$ by [5; Cor. 6.5], because $(1-h)_{(p)}=0$ for any $p \geqq gn$ by (1) of Theorem 4.1. Thus $\{1-h \mid h \in \mathrm{HE}(G)\}$ is contained in a finite subgroup of $[G, G]$; and so HE(G) is finite. (More generally, so is $\mathrm{HE}(G, \bar{m})$ by [2; Th. C].) On the other hand, the kernel of the natural homomorphism $\mathrm{E}(G)\to \mathrm{aut}\, H_*(G; Z)$ sending h to h_* is nilpotent by [18; Cor. 9.10] and [15]; and so is $\mathrm{HE}(G)$ by (2) of Theorem 4.1. q.e.d.

To prove Theorem 4.1, we use the following notations as in (3.3.1–2) and (3.1.2):

(4.3.1) $G(l)=SU(l+1)$ or $Sp(l)$, with usual multiplication $\bar{m}$, $g=1$ or 2, $N_l=(n_1,\dots, n_l)$ with $n_i=2gi-(-1)^g$, p: a prime$>\max\{gl, 4\}$, $S_i=S^{n_i}_{(p)}$, $m_i\in \mathrm{M}(S_i)$ with (3.1.3), $S(N_l)=\prod S_i$, $m=(\prod m_i)T\in \mathrm{M}(S(N_l))$.

LEMMA 4.3. *There exist a multiplication $\tilde{m}$ on $G(l)$ and a homotopy equivalence*

(4.3.2) e: $S(N_l)\simeq G(l)_{(p)}$ *which is an H-map with respect to m and $\tilde{m}_{(p)}$,*

where $\tilde{m}_{(p)}$ is the multiplication on $G(l)_{(p)}$ induced from $\tilde{m}$.

PROOF. The characteristic map $S^{n_l-1}\to G(l-1)$ of the principal bundle $G(l-1)\xrightarrow{j}G(l)\xrightarrow{q}S^{n_l}$ is proved by [4] to be of order ρ, where

$$\rho = l!\ (g=1), \quad =(2l-1)!\ (g=2,\ l \text{ is odd}), \quad =2((2l-1)!)\ (g=2,\ l \text{ is even}).$$

Thus, we have the bundle map

$$\tilde{\rho}\colon G(l-1)\times S^{n_l} \longrightarrow G(l) \quad \text{which covers } \rho=\rho\iota_{n_l}\colon S^{n_l}\longrightarrow S^{n_l},$$

and these are p-equivalences since $p>gl$ and so $(\rho, p)=1$. Hence, a homotopy equivalence e in (4.3.2) can be defined inductively by

(4.3.3) $$e = \tilde{\rho}_{(p)}(e\times \rho^{-1}_{(p)})\colon S(N_l) = S(N_{l-1})\times S_l \to G(l-1)_{(p)}\times S_l \to G(l)_{(p)},$$

and we have the homotopy commutative diagram

(4.3.4)
$$\begin{array}{ccccc}
S(N_{l-1}) & \xrightarrow{i} & S(N_l)=S(N_{l-1})\times S_l & \xrightarrow{p} & S_l \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \| \\
G(l-1)_{(p)} & \xrightarrow{j'} & G(l)_{(p)} \quad \xrightarrow{q'} \quad S^{n_l}_{(p)} & = & S_l,
\end{array}$$

where i is the inclusion, p is the projection and $f'=f_{(p)}$ for $f=j$ or q.

On the other hand, for the generator $s_l\in H^{n_l}(S^{n_l}; Z)$, we have

(4.3.5) $H^*(G(l); Z)=\Lambda(x_1,\dots, x_l)$ *with* $j^*x_i=x_i$ $(i<l)$, $x_l=q^*s_l$ *and* x_i $(1\leqq i\leqq l)$ *are primitive with respect to the usual multiplication* $\bar{m}$;

and by taking the localization $x'\in H^*(X_{(p)}; Z_{(p)})$ of $x\in H^*(X; Z)$, (4.3.6) $H^*(G(l)_{(p)}) = \Lambda(x'_1,\dots, x'_l)$ *with* $j'^*x'_i = x'_i$ $(i<l)$, $x'_l = q'^*s'_l$,

$H^*(S(N_l)) = \Lambda(y_1,\dots, y_l)$ *with* $i^*y_i = y_i$ $(i<l)$, $y_l = p^*s'_l$,

and x'_i *and* y_i $(1\leqq i\leqq l)$ *are primitive with respect to* $\overline{m}_{(p)}$ *and* m *in* (4.3.1), *respectively, where the coefficient ring is* $Z_{(p)}$.

Then $y_i=e^*x'_i$ $(1\leqq i\leqq l)$ by (4.3.4), and $x'_i=e^{*-1}y_i$ are also primitive with respect to $m'=em(e^{-1}\times e^{-1})$. Thus, by taking the rationalization $X_{(0)}=(X_{(p)})_{(0)}$, the multiplications $\overline{m}_{(0)}$ and $m'_{(0)}$ on $G(l)_{(0)}$, induced from $\overline{m}_{(p)}$ and m', respectively, give us the same Hopf algebra structure on $H^*(G(l)_{(0)}; Q)$; and so $\overline{m}_{(0)}=m'_{(0)}$.

Now, by [5; Cor. 5.13], we see immediately the following

(4.3.7) *For a prime* p, *let* $\bar{p}$ *denote the set of all primes* $\neq p$, *and consider also the localization* $X_{\bar{p}}$ *at* $\bar{p}$. *For a simple finite CW-complex* X, *assume that* $X_{(p)}$ *and* $X_{\bar{p}}$ *are H-spaces with multiplications* m *and* m', *respectively, and they induce the same one* $m_{(0)}=m'_{(0)}$ *on* $X_{(0)}=(X_{(p)})_{(0)}=(X_{\bar{p}})_{(0)}$. *Then,* X *is an H-space with a multiplication* $\tilde{m}$ *with* $\tilde{m}_{(p)}=m$ *on* $X_{(p)}$ *and* $\tilde{m}_{\bar{p}}=m'$ *on* $X_{\bar{p}}$.

Apply this for m' and $\overline{m}_{\bar{p}}$ of above with $m'_{(0)}=\overline{m}_{(0)}=(\overline{m}_{\bar{p}})_{(0)}$. Then

(4.3.8) $em(e^{-1}\times e^{-1}) = m' = \tilde{m}_{(p)}$ *and* $\overline{m}_{\bar{p}} = \tilde{m}_{\bar{p}}$ *for some* $\tilde{m}\in \mathrm{M}(G(l))$.

The first equality means that e is an H-map with respect to m and $\tilde{m}_{(p)}$. q. e. d.

PROOF OF THEOREM 4.1. In the first place, we prove the theorem in case that $G=SU(n)$ or $Sp(n)$. If $G=S^3$, $SU(3)$ or $Sp(2)$, then $\mathrm{HE}(G)=1$ by Example 1.5 and Theorem 2.6, and so the theorem is trivial. Therefore, we consider the group

(4.4.1) $G(l)$ in (4.3.1) for $l\geqq 3$ by using the notations given in (4.3.1).

Take any $h\in \mathrm{HE}(G(l))$. Then $h\in \mathrm{HE}(G(l), \overline{m})$ and so h^*x_i's are also primitive with respect to $\overline{m}$ in (4.3.5), and we have

(4.4.2) $h^*x_i=\eta_i x_i$ *in* $H^*(G(l); Z)$ *for some* $\eta_i = \pm 1$ $(1\leqq i\leqq l)$.

Take a prime $p>gl$, and consider the localization $h'=h_{(p)}\in \mathrm{HE}(G(l)_{(p)}, \tilde{m}_{(p)})$ of $h\in \mathrm{HE}(G(l), \tilde{m})$ at p and $e^{-1}h'e\in \mathrm{HE}(S(N_l), m)$ by $\tilde{m}$ and e in (4.3.2). Then

(4.4.3) *there is* $a=(a_1,\dots,a_l)\in (Z^*_{(p)})^l=\prod_{i=1}^l \mathrm{HE}(S_i, m_i)\cong \mathrm{HE}(S(N_l), m)$

with $e^{-1}h'e = \theta(a) = \prod_{i=1}^l a_i$ *in* $\mathrm{HE}(S(N_l), m)$,

by (3.1.2–4) since $n_l-n_1<2p-3$. These together with (4.3.6) imply

(4.4.4) $a_i=\eta_i = \pm 1$ *in* $Z^*_{(p)} = \mathrm{HE}(S_i, m_i)$ *for* $1\leqq i\leqq l$, since $a_i \cdot y_i = \theta(a)^* y_i = (e^{-1}h'e)^* y_i = e^* h'^* x'_i = e^*(\eta_i x'_i) = \eta_i y_i$ in $H^*(S(N_l); Z_{(p)})$.

We now fix any i $(1 \leqq i \leqq l)$ and any prime $p > gl$ (and so $p \geqq 5$), and

(4.4.5) put $n = n_i + 2p - 3$ and take $\alpha \in \pi_n(S^{n_i}; p) = [S^n_{(p)}, S_i]$ of order p,

by (3.1.1). Furthermore, consider the multiplication

(4.4.6) $m_\delta = m + i\alpha\pi_\delta$ on $S(N_l)$ for each $\delta = (\delta_1, \dots, \delta_{2l}) \in \{0, 1\}^{2l}$

with $n = N_l(\delta)$ and $\sum_{j=1}^{l} \delta_j \neq 0 \neq \sum_{j=1}^{l} \delta_{l+j}$,

where $N_l(\delta) = \sum_{j=1}^{l} \varepsilon_j n_j$ $(\varepsilon_j = \delta_j + \delta_{l+j})$, i: $S_i \subset S(N_l)$ is the inclusion and

$$\pi_\delta\colon S(N_l) \times S(N_l) \longrightarrow S_\delta = \wedge_{\delta_j = 1} S_j = S^n_{(p)} \ (S_{l+j} = S_j)$$

is the projection. Then the assumption that α is of order p and (4.3.8) imply

$$\begin{aligned}(em_\delta(e^{-1} \times e^{-1}))_{(0)} &= (em(e^{-1} \times e^{-1}))_{(0)} \\ &= (\tilde{m}_{(p)})_{(0)} = \tilde{m}_{(0)} = (\tilde{m}_{\bar{p}})_{(0)} \quad \text{on } G(l)_{(0)},\end{aligned}$$

and so (4.3.7) implies that $em_\delta(e^{-1} \times e^{-1}) = (\tilde{m}_\delta)_{(p)}$ for some $\tilde{m}_\delta \in \mathrm{M}\,(G(l))$. Thus, $h \in \mathrm{HE}\,(G(l)) \subset \mathrm{HE}\,(G(l), \tilde{m}_\delta)$, $h' = h_{(p)} \in \mathrm{HE}\,(G(l)_{(p)}, em_\delta(e^{-1} \times e^{-1}))$ and

$$\textstyle\prod_{j=1}^{l} a_j = \theta(a) = e^{-1}h'e \in \mathrm{HE}\,(S(N_l), m_\delta) \text{ (cf. (4.4.3))}.$$

For $a_\delta = \wedge_{\delta_j = 1} a_j\colon S_\delta \to S_\delta$ $(a_{l+j} = a_j)$, this together with (4.4.6) and (1.4.1) shows that

$$ia_i\alpha\pi = \theta(a)i\alpha\pi = i\alpha\pi(\theta(a) \wedge \theta(a)) = i\alpha a_\delta\pi \quad \text{in } [S(N_l) \wedge S(N_l), S(N_l)]\,;$$

and so the injectivities of i_* and π^* imply that

(4.4.7) $a_i\alpha = \alpha a_\delta$ *in* $[S_\delta, S_i] = \pi_n(S^{n_i}; p)$.

Furthermore, by (4.4.5) and (1.7.2), this means the following

(4.4.8) *If* $(\varepsilon_1, \dots, \varepsilon_l) \in \{0, 1, 2\}^l$ *satisfies* $n_i + 2p - 3 = \sum_{j=1}^{l} \varepsilon_j n_j$, *then* $\eta_i = \prod_{j=1}^{l} \eta_j^{\varepsilon_j}$ *where* $\eta_i = a_i = \pm 1$ *in* (4.4.2–4).

Now, this implies $\eta_i = 1$ for $1 \leqq i \leqq l$ as follows, by noticing that η_i's are independent of a prime $p > gl$; and we see the theorem by (4.4.2–4) and (4.3.2).

(i) The case $g = 1$, $n_j = 2j + 1$ and $l \geqq 3$: We can choose suitably a prime $p = 2q + 1 > l$ with $p \leqq 2l + 1$ (i.e. $q \leqq l$) by the classical result due to Čebyšev and $(\varepsilon_1, \dots, \varepsilon_l) \in \{0, 1, 2\}^l$ with $n_i + 2p - 3 = \sum_{j=1}^{l} \varepsilon_j n_j$ and even ε_j for $j \geqq i$ so that (4.4.8) shows the following equalities, which imply $\eta_i = 1$ inductively since $\eta_i = \pm 1$:

$$\eta_1 = \eta_q^2; \quad \eta_2 = \eta_1^2\eta_{q-1}^2 \text{ taking } p \geqq 7;$$
$$\eta_3 = \eta_3^2 \text{ and } \eta_4 = \eta_1^2\eta_2^2 \text{ taking } p = 5 \text{ for } l \leqq 4;$$
$$\eta_i = \eta_1\eta_{i-1}\eta_{q-1}^2 \text{ if } 3 \leqq i \neq q \text{ and } \eta_i = \eta_1\eta_{i-3}\eta_q^2 \text{ if } i = q$$
$$\text{taking } p \geqq 11 \text{ for } l \geqq 5.$$

(ii) The case $g=2$, $n_j=4j-1$ and $l\geqq 3$: In the same way, by taking a prime $p=4q+r>2l$ $(r=\pm 1)$ suitably with $r=-1$ for $l\leqq 9$ and with $p<4l$ (i.e., $q\leqq l$, and $q<l$ if $r=1$) for $l\geqq 10$, (4.4.8) shows the following equalities, which imply $\eta_i=1$ inductively:

$$\eta_1 = \eta_q^2 \text{ taking } p = 11\ (l\leqq 5),\ = 23\ (l\geqq 6), \quad \eta_2 = \eta_1^2\eta_2^2\eta_3^2 \text{ taking } p = 19,$$
$$\eta_3 = \eta_{q+1}^2 \text{ taking } p = 7\ (l=3),\ = 11\ (l=4, 5),\ = 19\ (l\geqq 6), \qquad \text{for } l \leqq 9;$$
$$\eta_1 = \eta_q^2, \quad \eta_2 = \eta_1^2\eta_2^2\eta_{q-2}^2, \quad \eta_3 = \eta_1\eta_2\eta_3^2\eta_{q-3}^2 \text{ taking } p > 23 \text{ if } r=-1,$$
$$\eta_1 = \eta_1^2\eta_2^2\eta_{q-2}^2, \quad \eta_2 = \eta_{q+1}^2, \quad \eta_3 = \eta_1^2\eta_2^2\eta_{q-1}^2 \text{ if } r = 1, \qquad \text{for } l \geqq 10;$$
$$\eta_i = \eta_1^2\eta_2\eta_{i-2}\eta_{q-1}^2\ (i\leqq q),\ =\eta_1^2\eta_2\eta_{q-2}\eta_{q-1}\eta_{i-1}\ (q<i) \text{ taking } p>11 \text{ if } r=-1,$$
$$\eta_i = \eta_1^2\eta_2\eta_{i-1}\eta_{q-1}^2\ (i\neq q),\ = \eta_1^2\eta_2\eta_{i-3}\eta_q^2\ (i=q) \text{ taking } p > 17 \text{ if } r = 1,$$
$$\text{for } 4 \leqq i \leqq l.$$

Finally, we prove the theorem when $G=U(n)=S^1\times SU(n)$ $(n\geqq 3)$. Take any

$$(\varepsilon, h)\in \mathrm{HE}(U(n)) \quad \text{with} \quad \varepsilon = \pm 1,\ h\in \mathrm{HE}(SU(n)) \text{ and } (2.5.1)$$

for $Y=SU(n)$ by Example 2.5(i). Then $(\varepsilon\wedge h\wedge h)^*=h_*$ on $[S^1\wedge Y\wedge Y, Y]$ by (2.5.1) and $h_{(p)}\sim 1$ by the theorem for $Y=SU(n)$, where $p\geqq 5$ is a prime with $n\leqq p<2n$. Therefore $(\varepsilon\wedge 1\wedge 1)^*=\mathrm{id}$ on $[(S^1\wedge Y\wedge Y)_{(p)}, Y_{(p)}]$ and so on $[(S^1\wedge S^3\wedge S^{p-2}\wedge S^p)_{(p)}, S^5_{(p)}]=\pi_{2p+2}(S^5; p)=Z_p$. Thus $\varepsilon=1$ and the theorem for $G=U(n)$ is proved. q. e. d.

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