HIROSHIMA MATH. J.
16 (1986), 361–366

Positive entire solutions of superlinear elliptic equations

Takaŝi KUSANO and Manabu NAITO

(Received August 20, 1985)

1. Introduction

In this paper we consider the semilinear elliptic equation

$$\Delta u + p(|x|)u^{\gamma} = 0, \qquad x \in R^n, \tag{1}$$

where $n \geqq 3$, Δ is the n-dimensional Laplace operator, and $|x|$ denotes the Euclidean length of $x \in R^n$. It is assumed throughout that

(a) $\gamma > 1$ (namely, (1) is superlinear);

(b) p is continuous on $[0, \infty)$, differentiable on $(0, \infty)$ and $p(t) > 0$ for $t > 0$.

Our main concern is to study the existence and nonexistence of entire solutions of (1) which are radially symmetric and positive in R^n. Here, by an entire solution of (1) we mean a function $u \in C^2(R^n)$ which satisfies (1) at every point of R^n, and the radial symmetry of a function means that it depends only on $|x|$.

The principal results of this paper are as follows:

THEOREM 1 (*Existence*). *Suppose that*

$$\frac{d}{dt}\left(t^{[n+2-\gamma(n-2)]/2}p(t)\right) \leqq 0 \qquad \text{for} \quad t > 0. \tag{2}$$

Then, for any $\alpha > 0$, *equation* (1) *has a radially symmetric positive entire solution* u *such that* $u(0) = \alpha$.

THEOREM 2 (*Nonexistence*). *Suppose that*

$$\frac{d}{dt}\left(t^{[n+2-\gamma(n-2)]/2}p(t)\right) \geqq 0 \qquad \text{for} \quad t > 0 \tag{3}$$

and

$$t^{[n+2-\gamma(n-2)]/2}p(t) \longrightarrow \infty \quad \text{as} \quad t \longrightarrow \infty. \tag{4}$$

Then, equation (1) *has no radially symmetric positive entire solutions.*

Since our attention is restricted to radially symmetric solutions, the problem for (1) under consideration reduces to the one-dimensional initial value problem

$$(t^{n-1}y')' + t^{n-1}p(t)y^{\gamma} = 0, \qquad t > 0, \tag{5}$$

$$(6) \qquad y(0) = \alpha, \qquad y'(0) = 0,$$

where $\alpha > 0$ is a constant, and the above theorems are proved through an analysis of the problem (5)–(6) with the aid of Liapunov-like functions which can be constructed explicitly. We note here that in certain cases all positive entire solutions of (1) happen to be radially symmetric; see Gidas, Ni and Nirenberg [2]. The proofs of Theorems 1 and 2 are given in Section 2. Examples illustrating the main results and remarks on the asymptotic behavior at infinity of entire solutions of (1) are presented in Section 3.

The problem of existence and nonexistence of entire solutions for semilinear elliptic equations of the form $\Delta u + f(x, u) = 0$ has been the subject of intensive investigations in recent years; see, for example, Berestycki, Lions and Peletier [1], Gidas and Spruck [3], Joseph and Lundgren [4], Kawano [5], Kusano and Oharu [7], Ni [8] and Toland [9]. However, our results cannot be covered by any of the previous works including these papers.

2. Proofs of main theorems

We observe that Theorems 1 and 2 are equivalent, respectively, to Theorems 1′ and 2′ stated below.

THEOREM 1′. *Suppose that* (2) *holds. Then, for any* $\alpha > 0$, *the problem* (5)–(6) *has a unique solution which is positive on the whole interval* $[0, \infty)$.

THEOREM 2′. *Suppose that* (3) *and* (4) *hold. Then, for any* $\alpha > 0$, *the solution of the problem* (5)–(6) *has a zero at some finite point of* $(0, \infty)$.

PROOF OF THEOREM 1′. For a given $\alpha > 0$, we denote by $y_\alpha(t)$ the unique solution of (5)–(6). It is clear that $y_\alpha(t)$ exists and is positive on some small interval $[0, \delta)$. Let $[0, t_\alpha)$ be the maximal interval on which $y_\alpha(t)$ is positive. We claim that $t_\alpha = \infty$ for all $\alpha > 0$. Suppose the contrary. Then there is an $\alpha > 0$ for which t_α is finite, and we have $y_\alpha(t_\alpha) = 0$ and $y_\alpha(t) > 0$ on $[0, t_\alpha)$. Define the function $V(t)$ on $[0, t_\alpha]$ by

$$(7) \qquad V(t) = t^{n-1} y_\alpha'(t) y_\alpha(t) + \frac{1}{n-2} t^n [y_\alpha'(t)]^2 + \frac{2}{(n-2)(\gamma+1)} t^n p(t) [y_\alpha(t)]^{\gamma+1}.$$

A straightforward computation with the use of (5) yields

$$(8) \qquad V'(t) = \frac{2}{(n-2)(\gamma+1)} t^{(n-2)(\gamma+1)/2} (t^{[n+2-\gamma(n-2)]/2} p(t))' [y_\alpha(t)]^{\gamma+1}$$

for $t \in (0, t_\alpha)$. From (8) and (2) we have $V'(t) \leqq 0$ for $t \in (0, t_\alpha)$; in particular,

$V(t_\alpha) \leqq V(0)$. Since $V(0)=0$ and $V(t_\alpha)=t_\alpha^n[y'_\alpha(t_\alpha)]^2/(n-2)$, we obtain $y'_\alpha(t_\alpha)=0$. The "initial condition" $y_\alpha(t_\alpha)=y'_\alpha(t_\alpha)=0$ clearly implies $y_\alpha(t)\equiv 0$ for $t\in[0, t_\alpha]$ by uniqueness. This, however, is a contradiction, and the proof is complete.

PROOF OF THEOREM 2′. We use the same notation as in the proof of Theorem 1′. Suppose that the conclusion of Theorem 2′ is false. Then, there is an $\alpha>0$ for which $t_\alpha=\infty$, that is, the solution $y_\alpha(t)$ of (5)–(6) exists and is positive on $[0, \infty)$. Note that $y_\alpha(t)$ satisfies

$$(9)\qquad (t^{3-n}(t^{n-2}y_\alpha(t))')' + tp(t)[y_\alpha(t)]^\gamma = 0$$

for $t>0$, and in particular for $t\geqq 1$. Integrating (9) over $[t, \tau]$, using the fact that

$$(10)\qquad (t^{n-2}y_\alpha(t))' \geqq 0, \qquad t\in[1, \infty),$$

(see e.g. [9]), and letting $\tau\to\infty$, we have

$$(11)\qquad (t^{n-2}y_\alpha(t))' \geqq t^{n-3}\int_t^\infty sp(s)[y_\alpha(s)]^\gamma ds, \qquad t\in[1, \infty).$$

Rewriting the function $sp(s)[y_\alpha(s)]^\gamma$ as

$$s^{1-\gamma(n-2)-[n+2-\gamma(n-2)]/2}\cdot s^{[n+2-\gamma(n-2)]/2}p(s)\cdot[s^{n-2}y_\alpha(s)]^\gamma$$

and using (3) and (10), we deduce from (11) that

$$(t^{n-2}y_\alpha(t))' \geqq t^{n-3}t^{[n+2-\gamma(n-2)]/2}p(t)[t^{n-2}y_\alpha(t)]^\gamma\int_t^\infty s^\lambda ds,$$

where $\lambda=1-\gamma(n-2)-[n+2-\gamma(n-2)]/2$, or

$$(12)\qquad (t^{n-2}y_\alpha(t))' \geqq \frac{2}{(n-2)(\gamma+1)}\, t^{n-1}p(t)[y_\alpha(t)]^\gamma, \quad t\in[1, \infty).$$

We multiply (12) by $[t^{n-2}y_\alpha(t)]^{-\gamma}$ and integrate over $[t, \tau]$. Letting $\tau\to\infty$ and taking (3) into account, we obtain

$$\begin{aligned} t^{n-2}y_\alpha(t) &\leqq \left(\frac{2(\gamma-1)}{(n-2)(\gamma+1)}\int_t^\infty s^{n-1-\gamma(n-2)}p(s)ds\right)^{1/(1-\gamma)} \\ &\leqq \left(\frac{2(\gamma-1)}{(n-2)(\gamma+1)}\, t^{[n+2-\gamma(n-2)]/2}p(t)\int_t^\infty s^\mu ds\right)^{1/(1-\gamma)}, \quad t\in[1, \infty), \end{aligned}$$

where $\mu=n-1-\gamma(n-2)-[n+2-\gamma(n-2)]/2$, which reduces to

$$(13)\quad t^{(n-2)/2}y_\alpha(t) \leqq \left(\frac{4}{(n-2)^2(\gamma+1)}\, t^{[n+2-\gamma(n-2)]/2}p(t)\right)^{1/(1-\gamma)}, \quad t\in[1, \infty).$$

From (13) and (4) it follows that

$$(14)\qquad \lim_{t\to\infty} t^{(n-2)/2}y_\alpha(t) = 0.$$

We now consider the function $V(t)$ defined by (7). In view of (12),

$$V(t) \leqq t^{n-1}y'_\alpha(t)y_\alpha(t) + \frac{1}{n-2}t^n[y'_\alpha(t)]^2 + ty_\alpha(t)(t^{n-2}y_\alpha(t))' \tag{15}$$
$$= \frac{t^{n-2}}{n-2}[t^{3-n}(t^{n-2}y_\alpha(t))']^2, \qquad t\in[1, \infty).$$

On the other hand, (8) holds for $t>0$, and (3) and (4) imply that $V'(t)\geqq 0$ and $V'(t)\not\equiv 0$ for $t>0$. Choose a $T\geqq 1$ such that $V(t)\geqq V(T)>V(0)=0$ for $t\geqq T$. Combining this inequality with (15) and noting (10), we obtain

$$(t^{n-2}y_\alpha(t))' \geqq ct^{(n/2)-2}, \qquad t\in[T, \infty), \tag{16}$$

where $c=[(n-2)V(T)]^{1/2}>0$. Integrating (16) yields

$$t^{n-2}y_\alpha(t) \geqq T^{n-2}y_\alpha(T) + \frac{2c}{n-2}t^{(n-2)/2} - \frac{2c}{n-2}T^{(n-2)/2}, \quad t\in[T, \infty),$$

which implies

$$\limsup_{t\to\infty} t^{(n-2)/2}y_\alpha(t) \geqq \frac{2c}{n-2} > 0. \tag{17}$$

Clearly (14) and (17) are contradictory. This completes the proof.

3. Remarks and examples

An important problem is to study the asymptotic behavior at infinity of positive entire solutions of (1) whose existence is guaranteed by Theorem 1.

Let $u(x)$ be an entire solution of (1) which is positive and radially symmetric. Then, $u(x)$ is decreasing in $|x|$ and has a nonnegative limit u_∞ as $|x|\to\infty$: $\lim_{|x|\to\infty} u(x)=u_\infty\geqq 0$. It is easily seen that if $u_\infty>0$ then

$$\int_0^\infty tp(t)dt < \infty, \tag{18}$$

so that the condition

$$\int_0^\infty tp(t)dt = \infty \tag{19}$$

implies that $u_\infty=0$ (see [6, Theorem 8]). This observation combined with Theorem 1 yields the following result.

THEOREM 3. *Suppose that* (2) *and* (19) *hold. Then, for every* $\alpha>0$, *there exists a radially symmetric positive entire solution* $u(x)$ *of* (1) *which satisfies* $u(0)=\alpha$ *and*

$$u(x) \to 0 \qquad as \quad |x| \to \infty. \tag{20}$$

A question arises as to what will happen in case (2) and (18) hold. In this case, as shown by Kawano [5], equation (1) possesses (infinitely many) radially symmetric positive entire solutions which tend to positive constants as $|x| \to \infty$. However, it is not known whether (2) and (18) ensure the existence of a positive entire solution $u(x)$ satisfying (20), nor is there any result characterizing the coexistence of entire solutions decaying to zero at infinity and those tending to nonzero constants as $|x| \to \infty$.

We conclude with two examples illustrating our main results.

EXAMPLE 1. Consider the equation

$$\Delta u + |x|^{\beta} u^{\gamma} = 0 \qquad \text{in} \quad R^n, \tag{21}$$

where $\beta \geqq 0$ and $\gamma > 1$ are constants.

(i) If $\gamma \geqq (n+2+2\beta)/(n-2)$, then $p(t) = t^{\beta}$ satisfies both (2) and (19), and so from Theorem 3 it follows that (21) has (infinitely many) radially symmetric positive entire solutions, all of which decay to zero as $|x| \to \infty$. The case $\beta = 0$ is relevant to some results in [1, Proposition III.1] and [8, Theorem 4.5].

(ii) If $1 < \gamma < (n+2+2\beta)/(n-2)$, then applying Theorem 2, we see that (21) has no radially symmetric positive entire solutions. This result has recently been obtained in [9, Corollary]. Note that the case $\beta = 0$ was discussed in [4] and [3, Theorem 1.1].

EXAMPLE 2. Consider the equation

$$\Delta u + (1+|x|)^{\beta} u^{\gamma} = 0 \qquad \text{in} \quad R^n, \tag{22}$$

where $\gamma > 1$ and β is a constant which is allowed to be negative.

(i) Suppose that $\gamma \geqq \max\{(n+2)/(n-2),\ (n+2+2\beta)/(n-2)\}$. Then, condition (2) is satisfied, so that by Theorem 1 equation (22) has (infinitely many) radially symmetric positive entire solutions. If in addition $\beta \geqq -2$, then all of these solutions tend to zero as $|x| \to \infty$ (Theorem 3), while if $\beta < -2$, then some (or all) of them have positive limits as $|x| \to \infty$.

(ii) Suppose that $1 < \gamma < \min\{(n+2)/(n-2),\ (n+2+2\beta)/(n-2)\}$. Then, Theorem 2 is applicable, and there are no positive entire solutions of (22) which are radially symmetric.

References

[1] H. Berestycki, P. L. Lions and L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in R^N, Indiana Univ. Math. J. **30** (1981), 141–157.

[2] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear

elliptic equations in R^n; In: Mathematical analysis and applications, Part A, pp. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.
[3] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. **34** (1981), 525–598.
[4] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rational Mech. Anal. **49** (1973), 241–269.
[5] N. Kawano, On bounded entire solutions of semilinear elliptic equations, Hiroshima Math. J. **14** (1984), 125–158.
[6] N. Kawano, T. Kusano and M. Naito, Nonoscillatory solutions of forced differential equations of the second order, J. Math. Anal. Appl. **90** (1982), 323–342.
[7] T. Kusano and S. Oharu, Bounded entire solutions of second order semilinear elliptic equations with application to a parabolic initial value problem, Indiana Univ. Math. J. **34** (1985), 85–95.
[8] W.-M. Ni, On the elliptic equation $\Delta u+K(x)u^{(n+2)/(n-2)}=0$, its generalizations, and applications in geometry, Indiana Univ. Math. J. **31** (1982), 493–529.
[9] J. F. Toland, On positive solutions of $-\Delta u=F(x, u)$, Math. Z. **182** (1983), 351–357.

Department of Mathematics,
Faculty of Science,
Hiroshima University
and
Department of Mathematics,
Faculty of Education,
Tokushima University)*

*) The present address of the second author is as follows: Department of Mathematics, Faculty of Science, Hiroshima University.