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Oscillation of nonlinear parabolic equations with functional arguments

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1. Introduction

Recently there has been an increasing interest in studying parabolic equations with functional arguments. It seems, however, that very little is known about the oscillatory behavior of solutions for such equations. The only results that the author knows of in this connection are oscillation theorems obtained by Bykov and Kultaev [1] and Kreith and Ladas [5].

The purpose of this paper is to establish oscillation criteria for the nonlinear parabolic equations

$$(E_-) \qquad u_t = a(t)\Delta u - q(x, t)f(u(x, \sigma(t))), \quad (x, t) \in \Omega \times R_+,$$

$$(E_+) \qquad u_t = a(t)\Delta u + q(x, t)f(u(x, \tau(t))), \quad (x, t) \in \Omega \times R_+,$$

where Δ is the Laplacian in R^n, $R_+=[0, \infty)$ and Ω is a bounded domain in R^n with piecewise smooth boundary $\partial\Omega$. We assume throughout this paper that:

(A–I) $a(t)$ is a nonnegative continuous function in R_+ and $q(x, t)$ is a nonnegative continuous function in $\bar{\Omega} \times R_+$;

(A–II) $f(s)$ is continuous in R^1, $f(s)$ is positive and convex in $(0, \infty)$, and $f(-s)=-f(s)$ for $s \geqq 0$;

(A–III) $\sigma(t)$ and $\tau(t)$ are continuous functions in R_+ such that $\lim_{t\to\infty}\sigma(t)=\lim_{t\to\infty}\tau(t)=\infty$.

In Sections 2 and 3 we consider the equations (E_-) and (E_+), respectively. In each section we give conditions which imply that every (classical) solution u of (E_-) [or (E_+)] satisfying the first or the third boundary condition is oscillatory in the sense that u has a zero in $\Omega \times [\tau, \infty)$ for any $\tau > 0$. Our approach is to reduce the multi-dimensional problem under study to a one-dimensional oscillation problem for ordinary differential equations or inequalities.

2. Oscillation criteria for the equation (E_-)

We consider two kinds of boundary conditions:

$$(1) \qquad u = 0 \quad \text{on} \quad \partial\Omega \times R_+,$$

(2) $$\frac{\partial u}{\partial \nu} + \mu u = 0 \quad \text{on} \quad \partial\Omega \times R_+,$$

where ν is the unit exterior normal vector to $\partial\Omega$ and μ is a nonnegative continuous function on $\partial\Omega \times R_+$.

It is well known that the first eigenvalue λ_1 of the eigenvalue problem

$$\Delta v + \lambda v = 0 \quad \text{in} \quad \Omega,$$

$$v = 0 \quad \text{on} \quad \partial\Omega$$

is positive and the corresponding eigenfunction $\Phi(x)$ is positive in Ω.

Associated with a function $u \in C(\bar{\Omega} \times R_+)$, we define

(*) $$U(t) \equiv \int_\Omega u(x, t)\Phi(x)dx \Big/ \int_\Omega \Phi(x)dx,$$

(**) $$\tilde{U}(t) \equiv \frac{1}{|\Omega|}\int_\Omega u(x, t)dx,$$

where $|\Omega|$ denotes the volume of Ω, i.e. $|\Omega| = \int_\Omega dx$.

The following notation will be used:

$$A(t) = \int_0^t a(s)ds, \quad Q(t) = \min\{q(x, t)\colon x \in \bar{\Omega}\}.$$

THEOREM 1. *Assume that* (A–I)–(A–III) *hold, and that*:

(A–IV) $f(s_1 s_2) \geqq f_1(s_1) f_2(s_2)$ *for* $s_1 > 0$, $s_2 > 0$, *where* $f_1(s_1) \geqq 0$, $f_2(s_2) > 0$ *and* $f_2(s_2)$ *is nondecreasing for* $s_2 > 0$.

If every eventually positive solution $y(t)$ *of the first order ordinary differential inequality*

(3) $$y'(t) + Q(t)\exp(\lambda_1 A(t)) f_1(\exp(-\lambda_1 A(\sigma(t)))) f_2(y(\sigma(t))) \leqq 0$$

satisfies $\lim_{t\to\infty} y(t) = 0$, *then every solution* u *of the problem* (E_-), (1) *is oscillatory in* $\Omega \times R_+$, *or satisfies*

(4) $$\lim_{t\to\infty} \exp(\lambda_1 A(t)) U(t) = 0,$$

where $U(t)$ *is given by* (*).

PROOF. Suppose that there is a nonoscillatory solution u which does not satisfy (4). Without loss of generality we may assume that $u > 0$ in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Since $\lim_{t\to\infty} \sigma(t) = \infty$, $u(x, \sigma(t)) > 0$ in $\Omega \times [t_1, \infty)$ for some $t_1 \geqq t_0$. Multiplying (E_-) by $\Phi(x)$ and integrating over Ω, we obtain

(5) $$\frac{d}{dt}\left(\int_\Omega u\Phi(x)dx\right) = a(t)\int_\Omega (\Delta u)\Phi(x)dx - \int_\Omega q(x, t)f(u(x, \sigma(t)))\Phi(x)dx.$$

From Green's formula it follows that

(6) $$\int_\Omega (\Delta u)\Phi(x)dx = \int_\Omega u\Delta\Phi(x)dx = -\lambda_1\int_\Omega u\Phi(x)dx.$$

An application of Jensen's inequality [6, p. 160] shows that

(7) $$\int_\Omega q(x, t)f(u(x, \sigma(t)))\Phi(x)dx \geqq Q(t)\int_\Omega f(u(x, \sigma(t)))\Phi(x)dx$$
$$\geqq Q(t)f\left(\int_\Omega u(x, \sigma(t))\Phi(x)dx \Big/ \int_\Omega \Phi(x)dx\right)\int_\Omega \Phi(x)dx.$$

Combining (5)–(7) yields

$$U'(t) \leqq -\lambda_1 a(t)U(t) - Q(t)f(U(\sigma(t))),$$

which is equivalent to

(8) $$\frac{d}{dt}(\exp(\lambda_1 A(t))U(t)) + Q(t)\exp(\lambda_1 A(t))f(U(\sigma(t))) \leqq 0.$$

By assumption (A–IV) we get

(9) $$f(U(\sigma(t))) = f(\exp(-\lambda_1 A(\sigma(t)))\cdot\exp(\lambda_1 A(\sigma(t)))U(\sigma(t)))$$
$$\geqq f_1(\exp(-\lambda_1 A(\sigma(t))))f_2(\exp(\lambda_1 A(\sigma(t)))U(\sigma(t))).$$

In view of (9) we see that (8) can be rewritten as

$$\frac{d}{dt}(\exp(\lambda_1 A(t))U(t))$$
$$+ Q(t)f_1(\exp(-\lambda_1 A(\sigma(t))))f_2(\exp(\lambda_1 A(\sigma(t)))U(\sigma(t))) \leqq 0.$$

Hence, $\exp(\lambda_1 A(t))U(t)$ is a positive solution of (3) in (t_1, ∞) such that $\lim_{t\to\infty}\exp(\lambda_1 A(t))U(t) \neq 0$. This contradicts the hypothesis and completes the proof.

THEOREM 2. *Assume that* (A–I)–(A–III) *hold. If every eventually positive solution $y(t)$ of the first order ordinary differential inequality*

(10) $$y'(t) + Q(t)f(y(\sigma(t))) \leqq 0$$

satisfies $\lim_{t\to\infty} y(t) = 0$, *then every solution u of the problem* (E_-), (2) *is oscillatory in $\Omega \times R_+$, or satisfies*

(11) $$\lim_{t\to\infty}\int_\Omega u(x, t)dx = 0.$$

PROOF. Suppose to the contrary that there exists a solution u of the problem (E_-), (2) such that $u>0$ in $\Omega\times[t_0, \infty)$ for some $t_0>0$ and $\lim_{t\to\infty}\int_\Omega u(x, t)dx \neq 0$. By (A–III) we get $u(x, \sigma(t))>0$ in $\Omega\times[t_1, \infty)$ for some $t_1\geqq t_0$. Integrating (E_-) over Ω and using the divergence theorem, we obtain

(12) $$\frac{d}{dt}\int_\Omega u(x, t)\, dx = a(t)\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\, dS - \int_\Omega q(x, t)f(u(x, \sigma(t)))dx.$$

Since $f(s)$ is convex in $(0, \infty)$, we have

(13) $$\int_\Omega q(x, t)f(u(x, \sigma(t)))dx \geqq Q(t)\,|\Omega|f\Big(\frac{1}{|\Omega|}\int_\Omega u(x, \sigma(t))dx\Big).$$

Combining (12) and (13) yields

$$\frac{d}{dt}\Big(\frac{1}{|\Omega|}\int_\Omega u(x, t)\, dx\Big) + Q(t)f\Big(\frac{1}{|\Omega|}\int_\Omega u(x, \sigma(t))dx\Big)$$
$$\leqq (a(t)/|\Omega|)\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\, dS = -(a(t)/|\Omega|)\int_{\partial\Omega}\mu u dS \leqq 0, \quad t > t_1,$$

which shows that the function $\tilde{U}(t)$ defined by (**) is a positive solution of (10) satisfying $\lim_{t\to\infty}\tilde{U}(t)\neq 0$. This contradicts the hypothesis and completes the proof.

In the linear case we can improve Theorems 1 and 2. By the same arguments as were used in Theorems 1 and 2, we obtain the following theorems.

THEOREM 3 (*linear case*). *Assume that* (A–I) *and* (A–III) *hold. If the ordinary differential inequality*

(14) $$y'(t) + Q(t)\exp(\lambda_1(A(t) - A(\sigma(t))))y(\sigma(t)) \leqq 0$$

has no eventually positive solution, then every solution u of the linear parabolic equation

(15) $$u_t = a(t)\Delta u - q(x, t)u(x, \sigma(t))$$

satisfying (1) *is oscillatory in* $\Omega\times R_+$.

THEOREM 4 (*linear case*). *Assume that* (A–I) *and* (A–III) *hold. If the ordinary differential inequality*

(16) $$y'(t) + Q(t)y(\sigma(t)) \leqq 0$$

has no eventually positive solution, then every solution u of (15) *satisfying* (2) *is oscillatory in* $\Omega \times R_+$.

Oscillation criteria for first order functional differential inequalities have been established by numerous authors, see, e.g. Fukagai and Kusano [2], Kitamura and Kusano [3], Koplatadze and Čanturija [4], Tomaras [7] and the references cited therein.

COROLLARY 1. *Under assumptions* (A–I)–(A–IV), *every solution u of the problem* (E_-), (1) *is oscillatory in* $\Omega \times R_+$, *or satisfies* (4), *if*

$$\int_{R[\sigma]} Q(t)\exp(\lambda_1 A(t)) f_1(\exp(-\lambda_1 A(\sigma(t))))dt = \infty, \tag{17}$$

where $R[\sigma]=\{t \in R_+ : 0 \leqq \sigma(t) \leqq t\}$.

PROOF. Condition (17) implies that every eventually positive solution $y(t)$ of (3) satisfies $\lim_{t\to\infty} y(t)=0$ (see [3, Theorem 2]). The conclusion follows from Theorem 1.

COROLLARY 2. *Under assumptions* (A–I)–(A–III) *and the following*
(A–V) $f(s)$ *is nondecreasing for* $s>0$,
every solution u of the problem (E_-), (2) *is oscillatory in* $\Omega \times R_+$, *or satisfies* (11), *if*

$$\int_{R[\sigma]} Q(t)dt = \infty.$$

The proof follows by using the same arguments as in Corollary 1 and will be omitted.

Applying a result of [4] to (14) and (16), we obtain the following.

COROLLARY 3 (*linear case*). *Under assumptions* (A–I), (A–III) *and the following*
(A–VI) $\sigma(t) \leqq t$ $(t \geqq t_0)$ *and* $\sigma(t)$ *is nondecreasing for* $t \geqq t_0$,
every solution u of the problem (1), (15) *is oscillatory in* $\Omega \times R_+$, *if*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)\exp(\lambda_1(A(s)-A(\sigma(s))))ds > 1/e. \tag{18}$$

COROLLARY 4 (*linear case*). *Under assumptions* (A–I), (A–III) *and* (A–VI), *every solution u of the problem* (2), (15) *is oscillatory in* $\Omega \times R_+$, *if*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)ds > 1/e. \tag{19}$$

REMARK 1. In case $\sigma(t) \leqq t$ $(t \in R_+)$, $R[\sigma]=[T, \infty)$ for some $T>0$.

A special case of the problem (E_-), (1) is

$$u_t = u_{xx} - q(x, t)(u(x, t-h))^{\gamma}, \quad (x, t) \in (0, L) \times R_+, \tag{20}$$

$$u(0, t) = u(L, t) = 0, \quad t > 0, \tag{21}$$

where h is a positive constant and γ ($\geqq 1$) is the quotient of odd integers.

COROLLARY 5. *If*

$$\int^{\infty} Q(t) \exp((\pi/L)^2(1-\gamma)t)dt = \infty, \tag{22}$$

then every solution u of the problem (20), (21) *is oscillatory in* $(0, L) \times R_+$, *or satisfies*

$$\lim_{t\to\infty} \exp((\pi/L)^2 t) \int_0^L u(x, t) \sin(\pi/L)x dx = 0. \tag{23}$$

PROOF. In the case where $\Omega = (0, L) \subset R^1$, we observe that $\lambda_1 = (\pi/L)^2$ and $\Phi(x) = \sin(\pi/L)x$. Since $f(s) = s^{\gamma}$, we may choose $f_1(s) = f_2(s) = s^{\gamma}$. It is easy to see that condition (22) implies

$$\int^{\infty} Q(t) \exp((\pi/L)^2 t)(\exp(-(\pi/L)^2(t-h)))^{\gamma} dt = \infty.$$

Hence, the conclusion follows from Corollary 1.

EXAMPLE 1. We consider the parabolic equation

$$u_t = u_{xx} - \exp(-h((\pi/L)^2 + 1))u(x, t-h), \quad (x, t) \in (0, L) \times R_+. \tag{24}$$

Here $n=1$, $a(t)=1$, $\sigma(t)=t-h$ and $\gamma=1$. Corollary 3 is not applicable to (24), since

$$\int_{t-h}^{t} \exp(-h((\pi/L)^2+1)) \exp((\pi/L)^2 h) ds = he^{-h} \leqq 1/e \quad (h>0).$$

Since $\int^{\infty} \exp(-h((\pi/L)^2+1))dt = \infty$, from Corollary 5 it follows that every nonoscillatory solution u of (21), (24) satisfies (23). In fact,

$$u(x, t) = \exp(-((\pi/L)^2+1)t) \sin(\pi/L)x$$

is a nonoscillatory solution of (24) satisfying (23).

EXAMPLE 2. Consider the boundary value problem

$$u_t = a(t)u_{xx} - e^{2t-3}(u(x, t-1))^3, \quad (x, t) \in (0, L) \times R_+, \tag{25}$$

$$-u_x(0, t) + \mu_1(t)u(0, t) = u_x(L, t) + \mu_2(t)u(L, t) = 0, \quad t > 0, \tag{26}$$

where $\mu_i(t)$ $(i=1, 2)$ are nonnegative continuous functions in R_+. Here $n=1$, $\gamma=3$, $h=1$ and $Q(t)=e^{2t-3}$. Since $\int^{\infty} e^{2t-3}dt=\infty$, Corollary 2 implies that every nonoscillatory solution u of (25), (26) satisfies

$$\lim_{t\to\infty}\int_0^L u(x, t)dx = 0. \tag{27}$$

In fact, one such solution is $u(x, t)=e^{-t}$.

EXAMPLE 3. Consider the parabolic equation

$$u_t = u_{xx} - e^{-4}u(x, t-2), \quad (x, t)\in(0, \pi) \times R_+. \tag{28}$$

Here $a(t)=1$, $L=\pi$, $\gamma=1$ and $Q(t)=e^{-4}$. Since

$$\int_{t-2}^{t} e^{-4}ds = 2e^{-4} \leqq 1/e, \quad \int^{\infty} e^{-4}ds = \infty,$$

Corollary 4 does not apply but Corollary 2 does, and we conclude that every solution u of (26), (28) is oscillatory in $(0, \pi)\times R_+$, or satisfies (27). There exists an oscillatory solution $u(x, t)=e^{-2t}\cos x$ of (26), (28) with the property that $\lim_{t\to\infty}\int_0^{\pi} u(x, t)dx=0$.

REMARK 2. If $u=0$ on $\partial\Omega\times R_+$ and $u>0$ in $\Omega\times[t_0, \infty)$ for some $t_0>0$, we find that $\frac{\partial u}{\partial \nu} \leqq 0$ on $\partial\Omega\times[t_0, \infty)$. Hence, Theorems 2 and 4, Corollaries 2 and 4 remain true if (2) is replaced by (1).

REMARK 3. Our results in this section can easily be generalized to the parabolic equation

$$u_t = a(t)\Delta u - c(x, t, u(x, t), u(x, \sigma(t))),$$

where $c(x, t, \xi, \eta)$ satisfies

(i) $c(x, t, -\xi, -\eta) = -c(x, t, \xi, \eta)$ for $(x, t)\in\Omega \times R_+$, $\xi \geqq 0, \eta \geqq 0$,

(ii) $c(x, t, \xi, \eta) \geqq q(x, t)f(\eta)$ for $(x, t)\in\Omega \times R_+$, $\xi \geqq 0$, $\eta \geqq 0$.

3. Oscillation criteria for the equation (E_+)

In this section we shall derive oscillation criteria for (E_+). Boundary conditions to be considered are (1) and the following:

$$\frac{\partial u}{\partial \nu} - \mu u = 0 \quad \text{on} \quad \partial\Omega \times R_+, \tag{29}$$

where $\mu\geqq 0$ on $\partial\Omega\times R_+$.

THEOREM 5. *Assume that* (A–I)–(A–IV) *hold. If the ordinary differential inequality*

$$(30) \qquad y'(t) - Q(t)\exp(\lambda_1 A(t)) f_1(\exp(-\lambda_1 A(\tau(t)))) f_2(y(\tau(t))) \geqq 0$$

has no eventually positive solution, then every solution u of the problem (E_+), (1) *is oscillatory in* $\Omega \times R_+$.

PROOF. Suppose to the contrary that there exists a nonoscillatory solution u of (E_+), (1). We may assume that $u>0$ in $\Omega \times [t_0, \infty)$ for some $t_0>0$. Proceeding as in the proof of Theorem 1, we find that $\exp(\lambda_1 A(t))U(t)$ is a positive solution of (30) in (t_1, ∞) for some $t_1 \geqq t_0$. This contradicts the hypothesis and the proof is complete.

THEOREM 6. *Assume that* (A–I)–(A–III) *hold. If the ordinary differential inequality*

$$(31) \qquad y'(t) - Q(t) f(y(\tau(t))) \geqq 0$$

has no eventually positive solution, then every solution u of the problem (E_+), (29) *is oscillatory in* $\Omega \times R_+$.

PROOF. Suppose that there is a solution u of (E_+), (29) such that $u>0$ in $\Omega \times [t_0, \infty)$ for some $t_0>0$. Arguing as in the proof of Theorem 2, we observe that $\tilde{U}(t)$ is a positive solution of (31) in (t_1, ∞) for some $t_1 \geqq t_0$. The contradiction establishes the theorem.

Using a result of Kitamura and Kusano [3, Theorem 1], we obtain the following results.

COROLLARY 6. *Under assumptions* (A–I)–(A–IV), *every solution u of the problem* (E_+), (1) *is oscillatory in* $\Omega \times R_+$, *if*

$$\int_M^\infty \frac{ds}{f_2(s)} < \infty \qquad \text{for any} \quad M > 0,$$

$$\int_{A[\tau]} Q(t)\exp(\lambda_1 A(t)) f_1(\exp(-\lambda_1 A(\tau(t))))dt = \infty,$$

where $A[\tau] = \{t \in R_+ : \tau(t) \geqq t\}$.

COROLLARY 7. *Under assumptions* (A–I)–(A–III) *and* (A–V), *every solution u of the problem* (E_+), (29) *is oscillatory in* $\Omega \times R_+$, *if*

$$\int_M^\infty \frac{ds}{f(s)} < \infty \qquad \text{for any} \quad M > 0, \qquad \int_{A[\tau]} Q(t)dt = \infty.$$

Applying Corollary 6 to the equation

(32) $$u_t = u_{xx} + q(x, t)(u(x, t+h))^\gamma, \quad (x, t) \in (0, L) \times R_+,$$

where h is a positive constant and γ (>1) is the ratio of odd integers, we have the following.

COROLLARY 8. *If condition* (22) *is satisfied, then every solution* u *of the problem* (21), (32) *is oscillatory in* $(0, L) \times R_+$.

REMARK 4. It is easy to extend oscillation results in this section to the more general equation

$$u_t = a(t)\Delta u + c(x, t, u(x, t), u(x, \tau(t))),$$

where $c(x, t, \xi, \eta)$ satisfies conditions (i) and (ii) of Remark 3.

EXAMPLE 4. It is known that Duffing's equation $g'' + 4g + 4g^3 = 0$ has a periodic solution $g(x)$. We see that the periodic solution $g(x)$ is oscillatory at $x = \infty$. There exists an interval (a, b) such that $g(a) = g(b) = 0$ and $g(x) > 0$ in (a, b). We consider the parabolic equation

(33) $$u_t = u_{xx} + 2u + e^{-2t-3}(u(x, t+1))^3, \quad (x, t) \in (0, L) \times R_+,$$

where $L = 2(b-a)$. It is clear that

$$2u + e^{-2t-3}(u(x, t+1))^3 \geqq e^{-2t-3}(u(x, t+1))^3 \quad \text{if} \quad u \geqq 0.$$

We combine Corollary 8 and Remark 4 to conclude that every solution u of (21), (33) is oscillatory in $(0, L) \times R_+$, if (22) is satisfied. Since

$$\int^\infty e^{-2t-3} \exp(-2(\pi/L)^2 t) dt < \infty,$$

condition (22) is violated. In this case, there is a nonoscillatory solution $u(x, t) = e^t g(x/2 + a)$ of (21), (33).

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